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FIXED POINT RESULTS FOR F-EXPANSIVE MAPPINGS IN ORDERED METRIC SPACES

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Abstract. In this article, we prove some existence and uniqueness fixed point results for F-expansive mappings in partially ordered metric spaces. We support the usability of our results adopting suitable examples.

1. Introduction

In what follows, M stands for a non-empty set whereas I_M denotes the identity mapping on M. As usual, $\mathbb{R}_+ = (0, \infty)$ while "$\leq$" refers the usual order on $\mathbb{R}$. Generally, all other involved symbols are used in their standard sense. For the sake of brevity, we write Su instead of writing $S(u)$ whereas for all n, we mean for all $n \in \mathbb{N}$. Throughout, $Fix(S)$ stands for the set of all fixed points of the mapping S

In order to generalize Banach contraction principle, Wardowski [1] employed a new type of auxiliary functions as under:

Definition 1. *[1] Let $\mathcal{F}$ be the family of all functions $F : \mathbb{R}_+ \to \mathbb{R}$ satisfying the following conditions:*

(F1) F is strictly increasing,

(F2) for every sequence $\{s_n\}$ of positive real numbers,

$$\lim_{n\to\infty} s_n = 0 \Leftrightarrow \lim_{n\to\infty} F(s_n) = -\infty, \text{ and}$$

(F3) there exists $k \in (0, 1)$ such that $\lim_{s\to 0^+} s^k F(s) = 0$.

Utilizing above auxiliary functions, Wardowski [1] proved the following:

Theorem 1. *[1] Let (M, d) be a complete metric space and $S : M \to M$. If there exists $\tau > 0$ and $F \in \mathcal{F}$ such that*

$$d(Su, Sv) > 0 \Rightarrow \tau + F(d(Su, Sv)) \leq F(d(u, v)), \tag{1.1}$$

for all $u, v \in M$, then S possesses a unique fixed point.

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Recall that, every self-mapping satisfying 1.1 is called F-contraction and on varying the elements of $\mathcal{F}$ suitably, a variety of contractions can be derived.

In recent years, the concept of F-contractions has attracted the attention of several researchers and by now there exists a considerable literature on and around this concept.

Very recently, Górnicki [2] undertaken the expansive analogues of Theorem 1. To accomplish this object, the author considered the following definition followed by an auxiliary result:

Definition 2. *[2] A self-mapping S on a metric space (M, d) is said to be* F-expansive *if there exists a function $F \in \mathcal{F}$ and $\tau > 0$ such that*

$$d(u, v) > 0 \Rightarrow F(d(Su, Sv) \geq F(d(u, v)) + \tau, \tag{1.2}$$

for all $u, v \in M$.

Lemma 1. *[2] Let (M, d) be a metric space and $S : M \to M$ a surjective mapping. Then S has a right inverse mapping i.e., a mapping $S^* : M \to M$ such that $S \circ S^* = I_M$.*

The main result of [2] is the following one:

Theorem 2. *[2] Every surjective F-expansive self-mapping on a complete metric space possesses a unique fixed point.*

A metric space (M, d) endowed with a partial order "$\preceq$" is called *an ordered metric space* and often denoted by $(M, d, \preceq)$. Further, M is said to be *regular* if for every increasing sequence $\{u_n\}$ in M with $u_n \to u$, we have $u_n \preceq u$ for all n. For arbitrary elements u, v of M, we say that u and v are *comparable* if either $u \preceq v$ or $v \preceq u$. The mapping $S : M \to M$ is called $\preceq$*-increasing* if $Su \preceq Sv$ whenever $u \preceq v$.

In 2004, Ran and Reurings [3] presented the analogous of Banach contraction principle in partially ordered metric spaces. Thereafter, proving order-theoretic analogues of metric-theoretical fixed point results becomes increasingly active (e.g., [4–12]). Concerning F-contractions in order metric spaces, one can be refereed to [13–17] and references therein. Here, it can be pointed out that in the setting of ordered metric spaces, the contraction inequality (e.g., (1.1) and (1.2)) needs to hold merely for comparable pairs of elements.

In this article, we prove some existence and uniqueness fixed point results for an F-expansive mappings in ordered metric spaces. Concretely speaking, we prove an order-theoretic analogue of Theorem 2.

2. Main Results

We begin this section by the following observations and auxiliary results which are needed in our subsequent discussions:

Proposition 1. *Let $(M, d, \preceq)$ be an ordered metric space and $S : M \to M$ a surjective mapping having a right inverse mapping S^*. If the mapping S^* is $\preceq$-increasing, then S need not be so (see Example 1 to be given later).*

Lemma 2. *Let (M, d) be a metric space and $S : M \to M$ a surjective mapping having a right inverse mapping S^*. If $u \in M$ is a fixed point of S^*, then u remains a fixed point of S.*

Proof. Let u be a fixed point of S^*. Then, $Su = S\big(S^*u\big) = u$. □

Lemma 3. *The converse of Lemma 2 is true if S is injective mapping.*

Proof. Let u be a fixed point of S. Then, $Su = u = S(S^*u)$. The injectiveness of S implies that $u = S^*u$ which concludes the proof.

To disprove the converse implication of Lemma 2, consider $S : [0, \infty) \to [0, \infty)$ given by

$$Su = \begin{cases} \sqrt{u}, & \text{for } u < 1, \\ 1, & \text{for } 1 \le u \le 2, \\ 2u - 3, & \text{Otherwise.} \end{cases}$$

Here, 1 remains fixed under S but not under its inverse function S^* as $S^*u = u^2$ for $u < 1$ and $S^*u = \frac{u+3}{2}$ elsewhere. □

Remark 1. *Observe that, neither S nor S^* is required to be continuous in Lemma 3.*

To support this claim, consider $M = [0, \infty)$ endowed with the usual metric and the partial order: $u \preceq v$ if and only if either $\{u = v \mid u, v \in \mathbb{N}\}$ or $\{u \le v \mid u, v \notin \mathbb{N}\}$. Define $S : M \to M$ by:

$$Su = \begin{cases} u + 1, & \text{if } u \text{ is odd number}, \\ u - 1, & \text{if } u \text{ is even number}, \\ u, & \text{Otherwise .} \end{cases}$$

Obviously, $S^* = S$. Further, S is bijective and $\preceq$-increasing but not continuous.

The following definition remains an order-theoretic analogue of Definition 2:

Definition 3. *Let $(M, d, \preceq)$ be an ordered metric space. A mapping $S : M \to M$ is said to be* F-expansive *if there exists $F \in \mathcal{F}$ and $\tau > 0$ such that for all comparable elements u and v in M, we have*

$$d(u, v) > 0 \Rightarrow F(d(Su, Sv) \ge F(d(u, v)) + \tau. \tag{2.1}$$

Now, we are equipped to prove our main result as follows:

Theorem 3. *Let $(M, \preceq, d)$ be an ordered complete metric space, $S : M \to M$ a surjective F-expansive mapping and S^* the right inverse of S. Then S has a fixed point if the following conditions hold:*

(a) there exists $u_0 \in M$ such that $u_0 \preceq S^*u_0$,

(b) S^* *is* $\preceq$*-increasing,*
(c) either S *is continuous or* M *is regular.*

Proof. Let $u_0 \in M$ be such that $u_0 \preceq S^*u_0$. Define a sequence $\{u_n\}$ in M by $u_n =: S^*u_{n-1}$ (for all n). Notice that, the construction of the sequence implies $u_n = Su_{n+1}$ for all n. In case, $u_n = u_{n+1}$ for some n, then u_n is the required fixed point and we are done. Therefore, we may assume that such equality does not occur for any n. Since $u_0 \preceq S^*u_0$ and S^* is $\preceq$-increasing, we have $u_n \preceq u_{n+1}$ for all n. On setting $u = u_n$ and $v = u_{n+1}$ in (2.1), we have

$$\begin{aligned} F(d(u_n, u_{n+1})) &= F(d(Su_{n+1}, Su_{n+2})) \\ &\geq F(d(u_{n+1}, u_{n+2})) + \tau. \end{aligned}$$

Put $t_n = d(u_n, u_{n+1})$. Now, it follows that

$$F(t_{n+1}) \leq F(t_n) - \tau \leq F(t_{n-1}) - 2\tau \leq ... \leq F(t_1) - n\tau, \tag{2.2}$$

implying therapy $\lim_{n\to\infty} F(t_n) = -\infty$ which in view of (F2) gives rise

$$\lim_{n\to\infty} t_n = 0. \tag{2.3}$$

Owing to (F3), there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} t_n^k F(t_n) = 0. \tag{2.4}$$

Now, from (2.2), we have

$$t_n^k\big(F(t_n) - F(t_1)\big) \leq -(n-1)\tau t_n^k \leq 0. \tag{2.5}$$

On using (2.3), (2.4) and letting $n \to \infty$ in (2.5), we get

$$\lim_{n\to\infty} nt_n^k = 0.$$

Hence, there exists $n_0 \in \mathbb{N}$ such that $nt_n^k \leq 1$ for all $n \geq n_0$, so that

$$t_n \leq \frac{1}{n^{\frac{1}{k}}}, \text{ for all } n \geq n_0. \tag{2.6}$$

We assert that $\{u_n\}$ is a Cauchy sequence. Conceder $p, q \in \mathbb{N}$ with $q > p \geq n_0$. Using the triangle inequality and (2.6), we have

$$d(u_p, u_q) \leq \sum_{i=p}^{q-1} t_i \leq \sum_{i=p}^{\infty} t_i \leq \sum_{i=p}^{\infty} \frac{1}{i^{\frac{1}{k}}}.$$

The convergence of the series $\sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{k}}}$ implies that

$$\lim_{p,q\to\infty} d(u_p, u_q) = 0$$

so that the intended assertion is established. Since M is complete, there exists some $u \in M$ such that

$$\lim_{n\to\infty} u_n = u. \tag{2.7}$$

If S is continuous, then $u = \lim_{n\to\infty} u_n = S(\lim_{n\to\infty} u_{n+1}) = Su$ and thus we are through. Otherwise, let M be regular so that $u_n \preceq u$ for all n. In view of Lemma 2, it is enough to show that $S^*u = u$. Here, we distinguish two cases, if there exits $m \in \mathbb{N}$ such that $u_m = u$, then $S^*u = S^*u_m = u_{m+1} \preceq u$. Also, $u = u_m \preceq u_{m+1} = S^*u_m = S^*u$ so that $S^*u = u$. In case, $u_n \neq u$ for all n. We assert that $d(u, Su) = 0$. On a contrary, assume that $d(u, Su) > 0$ so that

$$\begin{aligned} F(d(u_n, u)) &= F(d(Su_{n+1}, SS^*u)) \\ &\leq F(d(u_{n+1}, S^*u)) + \tau \\ &> F(d(u_{n+1}, S^*u)), \end{aligned}$$

it follows that $d(u_n, u) > d(u_{n+1}, S^*u)$ which on making $n \to \infty$ (on both the sides) gives rise $S^*u = u$. This concludes the proof. □

To prove a uniqueness result corresponding to Theorem 3, we have the following.

Theorem 4. *The mapping S in Theorem 3 has a unique fixed point if either $Fix(S)$ is totally ordered set or (S is injective and for every pair of elements u and v of M there exists $w \in M$ which is comparable to both u and v).*

Proof. Let u, v be two distinct and comparable fixed points of S. Then we have

$$\begin{aligned} F(d(u, v)) &= F(d(Su, Sv) \\ &\geq F(d(u, v)) + \tau, \end{aligned}$$

a contradiction. Otherwise, let $w \in M$ be comparable to both u and v. We may assume that $w \preceq u$. Since S^* is $\preceq$-increasing, we have

$$(S^*)^n w \preceq u, \quad \text{for all } n.$$

Let $(S^*)^n w = w_n$. We assert that $\lim_{n\to\infty} d(w_n, u) = 0$. If $w_m = u$ for some $m \in \mathbb{N}$, then the assertion is obvious. Otherwise, in view of (2.1), we have

$$\begin{aligned} F(d(w_n, u)) &= F(d(Sw_{n+1}, Su)) \\ &\geq F(d(w_{n+1}, u)) + \tau. \end{aligned}$$

Therefore, by induction on n, we deduce

$$F(d(w_{n+1}, u)) \leq F(d(S^*w, u)) - n\tau.$$

Therefore, $\lim_{n\to\infty} F(d(w_n, u)) = -\infty$ which, in view of (F2), implies that

$$\lim_{n\to\infty} d(w_n, u) = 0. \tag{2.8}$$

Similarly, we can prove that $\lim_{n\to\infty} d(w_n, v) = 0$. Hence,

$$d(u, v) \leq d(u, w_n) + d(w_n, v) \to 0 \text{ as } n \to \infty.$$

This concludes the proof. □

The following example creates a situation wherein Theorem 3 is applicable while Theorem 2 is not.

Example 1. *Let $M = [0,\infty)$ be endowed with the usual metric d. Define a mapping $S : M \to M$ by:*

$$Su = \begin{cases} 2u - 3n + 3, & \text{for } u \in [3n-3, 3n-2]; \\ \frac{u+3n}{2}, & \text{for } u \in [3n-2, 3n]. \end{cases} \quad n = \{1, 2, 3, ...\}$$

Then, S is a surjective and having a right inverse S^* given by:

$$S^*u = \begin{cases} \frac{u+3n-3}{2}, & \text{for } u \in [3n-3, 3n-1]; \\ 2u - 3n, & \text{for } u \in [3n-1, 3n]. \end{cases} \quad n = \{1, 2, 3, ...\}$$

Define a partial order on M as follows: $u \preceq v$ if and only if either $u = v$ or ($u \leq v$ where both u, v in $[3n-3, 3n-2], n \in \mathbb{N}$). Then, S^* is $\preceq$-increasing. Consider $F(t) = \ln t$ and τ with $1 < e^\tau \leq 2$. Then, for distinct elements u and v in $[3n-3, 3n-2], n \in \mathbb{N}$, we have

$$\ln[d(Su, Sv)] = \ln\big[2|u-v|\big] \geq \ln\big[e^\tau|u-v|\big] = \tau + \ln\big[|u-v|\big] = \tau + \ln\big[d(u,v)\big].$$

By a routine calculation one can show that S satisfies all other hypotheses of Theorem 3 ensuring the existence of some fixed point of S.

Observe that, Theorem 2 is not applicable in the context of Example 1 as the inequality (1.2) does not hold for $u = 1$ and $v = 3$. Furthermore, the uniqueness requirement of Theorem 4 is not satisfied. Observe that S has infinitely many fixed points.

The following corollary remains an order-theoretic analogue of a result due to Wang et al. [18].

Corollary 1. *Let $(M, \preceq, d)$ be an ordered complete metric space and $S : M \to M$ be a surjective mapping satisfying the following:*

$$d(Su, Sv) \geq \lambda d(u, v)$$

for all $u, v \in M$ such that $u \preceq v$ where $\lambda > 1$. Suppose that the following conditions hold:

(a) there exists $u_0 \in M$ such that $u_0 \preceq S^*u_0$,
(b) S^* is $\preceq$-increasing,
(c) either S is continuous or M is regular.

Then S has a fixed point. Further, this fixed point is unique if either the set $Fix(S)$ is totally ordered or for every pair of elements u, v of M, there exists $w \in M$ which is comparable to both u and v.

Proof. The result follows from Theorem 4 by setting $F(s) = \ln s$ and τ with $e^\tau = \lambda$. □

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