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ON THE LINEAR CODES OVER THE RING $Z_4 + v_1 Z_4 + ... + v_t Z_4$

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Abstract. Some results on linear codes over the ring $Z_4 + uZ_4 + vZ_4, u^2 = u, v^2 = v, uv = vu = 0$ in [6,7] are generalized to the ring $D_t = Z_4 + v_1 Z_4 + ... + v_t Z_4, v_i^2 = v_i, v_i v_j = v_j v_i = 0$ for $i \neq j, 1 \leq i, j \leq t$. A Gray map Φ_t from D_t^n to $Z_4^{(t+1)n}$ is defined. The Gray images of the cyclic, constacyclic and quasi-cyclic codes over D_t are determined. The cyclic DNA codes over D_t are introduced. The binary images of them are determined. The nontrivial automorphism on D_i for $i = 2, 3, ..., t$ is given. The skew cyclic, skew constacyclic and skew quasi-cyclic codes over D_t are introduced. The Gray images of them are determined. The skew cyclic DNA codes over D_t are introduced. Moreover, some properties of MDS codes over D_t are discussed.

1. Introduction

The certain type of codes over many finite rings were studied [2,4,5,8,9,13,15,16,20, 21,22]. Many of good codes were obtained from them.

Some special error correcting codes over some finite fields and finite rings with 4^n elements where $n \in N$ were used for DNA computing applications. The construction of DNA codes were by several authors in [1,6,12,14,18].

Optimal codes attain maximum minimum distances. So their class is very important class of codes. Optimal codes over finite rings were studied by several authors in [3,10,11,17,19].

In [6], the finite ring $D = Z_4 + uZ_4 + vZ_4, u^2 = u, v^2 = v, uv = vu = 0$ was introduced, firstly. Some results on linear codes over D were obtained. Moreover, in [7], the MacWilliams identities and optimal codes over D were studied. In this paper, we generalize some results to the linear codes over D_t.

This paper is organized as follows. In section 2, a Gray map from D_t to $Z_4^{(t+1)}$ is defined. The Gray images of cyclic, constacyclic, and quasi-cyclic codes over D_t are determined. A linear code C over D_t is represented by means of $(t+1)$ codes over Z_4. In section 3, the constacyclic codes over D_t are investigated. In section

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4, the cyclic codes of odd length over D_t satisfy reverse and reverse complement properties are studied. In section 5, the binary images of cyclic DNA codes over D_t are determined. In section 6, the nontrivial automorphism on D_i for $i = 2, 3, ..., t$ is determined. By introducing the skew cyclic, skew constacyclic and skew quasi-cyclic codes over D_t, the Gray images of them are found in section 7. In section 8, we investigated skew cyclic DNA codes over D_t. In section 9, some properties of optimal codes over D_t are determined.

2. Preliminaries

Let $D_t = Z_4 + v_1 Z_4 + ... + v_t Z_4$,where $v_i^2 = v_i, v_i v_j = v_j v_i = 0$ for $i \neq j$, $1 \leq i$, $j \leq t$. The ring D_t can be also viewed as the quotient ring

$$Z_4[v_1, v_2, ..., v_t] / \left\langle v_i^2 - v_i, v_i v_j = v_j v_i \right\rangle.$$

Let d be any element of D_t, which can be expressed uniquely as $d = d_0 + v_1 d_1 + ... + v_t d_t$.

A code of length n over D_t is a subset of D_t^n. C is a linear iff C is an D_t-submodule of D_t^n.The elements of the code (linear code) are called codewords.

Let $\sigma, \sigma_\lambda, \zeta$ be maps from D_t^n to D_t^n given by

$$\begin{aligned} \sigma(\alpha_0, ..., \alpha_{n-1}) &= (\alpha_{n-1}, \alpha_0, ..., \alpha_{n-2}) \\ \sigma_\lambda(\alpha_0, ..., \alpha_{n-1}) &= (\lambda\alpha_{n-1}, \alpha_0, ..., \alpha_{n-2}) \\ \zeta(\alpha_0, ..., \alpha_{n-1}) &= (-\alpha_{n-1}, \alpha_0 ..., \alpha_{n-2}) \end{aligned}$$

where λ is a unit in D_t. Let C be a linear code of length n over D_t. Then C is said to be cyclic if $\sigma(C) = C$, λ-constacyclic if $\sigma_\lambda(C) = C$, negacyclic, if $\zeta(C) = C$.

Let $a \in Z_4^{(t+1)n}$ with $a = \left(a_0, a_1, ..., a_{(t+1)n-1}\right) = \left(a^{(0)} | a^{(1)} | ... | a^{(t)}\right)$, $a^{(i)} \in Z_4^n$ for $i = 0, 1, ..., t$. Let φ be a map from $Z_4^{(t+1)n}$ to $Z_4^{(t+1)n}$ given by $\varphi(a) = \left(\sigma\left(a^{(0)}\right) \middle| \sigma\left(a^{(1)}\right) \middle| ... | \sigma\left(a^{(t)}\right)\right)$, where σ is a cyclic shift from Z_4^n to Z_4^n given by $\sigma\left(a^{(i)}\right) = ((a^{(i,n-1)}), (a^{(i,0)}), ..., (a^{(i,n-2)}))$ for every $a^{(i)} = (a^{(i,0)}, ..., a^{(i,n-1)})$, where $a^{(i,j)} \in Z_4$, $j = 0, 1, ..., n-1$. A code of length $(t+1)n$ over Z_4 is said to be a quasi-cyclic code of index $t+1$ if $\varphi(C) = C$.

We define the Gray map as follows

$$\begin{aligned} \Phi_t \quad &: \quad D_t \longrightarrow Z_4^{t+1} \\ d_0 + v_1 d_1 + ... + v_t d_t \quad &\longmapsto \quad (d_0, d_0 + d_1, ..., d_0 + d_t) \end{aligned}$$

This map is extended componentwise to

$$\begin{aligned} \Phi_t \quad &: \quad D_t^n \longrightarrow Z_4^{(t+1)n} \\ (\alpha_1, ..., \alpha_n) \quad &= \quad (d_0^1, d_0^2, ..., d_0^n, ..., d_0^1 + d_t^1, ..., d_0^n + d_t^n) \end{aligned}$$

where $\alpha_i = d_0^i + v_1 d_1^i + ... + v_t d_t^i$ with $i = 1, 2, ..., n$.

Φ_t is a Z_4-module isomorphism.

The Lee weights of $0, 1, 2, 3 \in Z_4$ are defined by $w_L(0) = 0, w_L(1) = w_L(3) = 1, w_L(2) = 2$.

Let $d=d_0+v_1d_1+...+v_td_t$ be an element of D_t, then Lee weight of d is defined as $w_L(d)=w_L(d_0,d_0+d_1,...,d_0+d_t)$, where $d_0,d_1,...,d_t\in Z_4$. The Lee weight of a vector $c=(c_0,...,c_{n-1})\in D_t^n$ to be the sum of Lee weights its components. For any elements $c_1,c_2\in D_t^n$, the Lee distance between c_1 and c_2 is given by $d_L(c_1,c_2)=w_L(c_1-c_2)$. The minimum Lee distance of C is defined as $d_L(C)=\min d_L(c,\acute{c})$, where for any $\acute{c}\in C, c\neq\acute{c}$.

For any $x=(x_0,...,x_{n-1}), y=(y_0,...,y_{n-1})$ the inner product is defined as

$$xy=\sum_{i=0}^{n-1}x_iy_i$$

If $xy=0$, then x and y are said to be orthogonal. Let C be a linear code of length n over D_t, the dual of C

$$C^{\perp}=\{x:\forall y\in C, xy=0\}$$

which is also a linear code over D_t of length. A code C is self orthogonal, if $C\subset C^{\perp}$ and self dual, if $C=C^{\perp}$.

Theorem 1. *The Gray map Φ_t is distance preserving map from (D_t^n,Lee distance) to ($Z_4^{(t+1)n}$,Lee distance).*

Proof. Let $z_1=(z_{1,0},...,z_{1,n-1}), z_2=(z_{2,0},...,z_{2,n-1})$ be the elements of D_t^n, where $z_{1,i}=d_{1,i}^0+v_1d_{1,i}^1+...+v_td_{1,i}^t$ and $z_{2,i}=d_{2,i}^0+v_1d_{2,i}^1+...+v_td_{2,i}^t$, $i=0,1,...,n-1$. Then $z_1-z_2=(z_{1,0}-z_{2,0},...,z_{1,n-1}-z_{2,n-1})$ and $\Phi_t(z_1-z_2)=\Phi_t(z_1)-\Phi_t(z_2)$. So, $d_L(z_1,z_2)=w_L(z_1-z_2)=w_L(\Phi_t(z_1-z_2))=w_L(\Phi_t(z_1)-\Phi_t(z_2))=d_L(\Phi_t(z_1),\Phi_t(z_2))$. □

Theorem 2. *If C is self orthogonal, so is $\Phi_t(C)$.*

Proof. Let $x_1=d_0^1+v_1d_1^1+...+v_td_t^1, x_2=d_0^2+v_1d_1^2+...+v_td_t^2\in D_t$. From $x_1x_2=d_0^1d_0^2+v_1(d_0^1d_1^2+d_1^1d_0^2+d_1^1d_1^2)+...+v_t(d_0^1d_t^2+d_t^1d_0^2+d_t^1d_t^2)$. If C is self orthogonal, so we have $d_0^1d_0^2=0, d_0^1d_1^2+d_1^1d_0^2+d_1^1d_1^2=0,...,d_0^1d_t^2+d_t^1d_0^2+d_t^1d_t^2=0$. From this, we have $\Phi_t(x_1)\Phi_t(x_2)=(d_0^1,d_0^1+d_1^1,...,d_0^1+d_t^1)(d_0^2,d_0^2+d_1^2,...,d_0^2+d_t^2)=0$. Therefore $\Phi_t(C)$ is self orthogonal. □

Proposition 3. *Let Φ_t be Gray map from D_t^n to $Z_4^{(t+1)n}$, let σ be the cyclic shift and let φ be a map as above. Then $\Phi_t\sigma=\varphi\Phi_t$.*

Proof. Let $a=(a_0,...,a_{n-1})\in D_t^n$. Let $a_i=d_i^0+v_1d_i^1+...+v_td_i^t$ where $d_i^0,d_i^1,...,d_i^t\in Z_4$, for $i=0,1,...,n-1$. From definition Φ_t, we have $\Phi_t(a)=(d_0^0,d_1^0,...,d_{n-1}^0,d_0^0+d_0^1,...,d_0^0+d_{n-1}^t,...,d_{n-1}^0+d_{n-1}^1,...,d_{n-1}^0+d_{n-1}^t)$. By applying φ,we have $\varphi(\Phi_t(a))=(d_{n-1}^0,d_0^0,...,d_{n-2}^0,d_0^0+d_{n-1}^t,...,d_0^0+d_{n-2}^t,...,d_{n-1}^0+d_{n-1}^t,...,d_{n-2}^0+d_{n-2}^t)$.

On the other hand, $\sigma(a)=(a_{n-1},a_0,...,a_{n-2})$. If we apply Φ_t, we have $\Phi_t(\sigma(a))=(d_{n-1}^0,d_0^0,...,d_{n-2}^0,d_0^0+d_{n-1}^t,...,d_0^0+d_{n-2}^t,...,d_{n-1}^0+d_{n-1}^t,...,d_{n-2}^0+d_{n-2}^t)$ □

Theorem 4. *Let σ and φ be as in section 2. A code C of length n over D_t is a cyclic code iff $\Phi_t(C)$ is a quasi-cyclic code of index $t+1$ over Z_4 with length $(t+1)n$.*

Proof. Let C be a cyclic code. Then $\sigma(C) = C$. If we apply Φ_t, we have $\Phi_t(\sigma(C)) = \Phi_t(C)$. By using Proposition 3, $\Phi_t(\sigma(C)) = \varphi(\Phi_t(C)) = \Phi_t(C)$. Hence, $\Phi_t(C)$ is a quasi- cyclic code of index $t+1$.

For the other part, if $\Phi_t(C)$ is a quasi-cyclic code of index $t+1$, then we have $\varphi(\Phi_t(C)) = \Phi_t(C)$. By using Proposition 3, we have $\varphi(\Phi_t(C)) = \Phi_t(\sigma(C)) = \Phi_t(C)$. Since Φ_t is injective, we have $\sigma(C) = C$. □

Let $A_1, A_2, ..., A_{t+1}$ be linear codes.

$$A_1 \otimes A_2 \otimes ... \otimes A_{t+1} = \{(a_1, a_2, ..., a_{t+1}) : a_i \in A_i, i = 1, 2, ..., t+1\}$$

and

$$A_1 \oplus A_2 \oplus ... \oplus A_{t+1} = \{a_1 + a_2 + ... + a_{t+1} : a_i \in A_i, i = 1, 2, ..., t+1\}$$

Definition 5. *Let $C^{(t)}$ be a linear code of length n over D_t. Define*

$$\begin{aligned}
C_1^{(t)} &= \{d_0 : \exists\, d_1, ..., d_t \in Z_4^n, d_0 + v_1 d_1 + ... + v_t d_t \in C^{(t)}\} \\
C_2^{(t)} &= \{d_0 + d_1 : \exists\, d_2, ..., d_t \in Z_4^n, d_0 + v_1 d_1 + ... + v_t d_t \in C^{(t)}\} \\
C_3^{(t)} &= \{d_0 + d_2 : \exists\, d_1, d_3, ..., d_t \in Z_4^n, d_0 + v_1 d_1 + ... + v_t d_t \in C^{(t)}\} \\
&\vdots \\
C_{t+1}^{(t)} &= \{d_0 + d_t : \exists\, d_1, d_2, ..., d_{t-1} \in Z_4^n, d_0 + v_1 d_1 + ... + v_t d_t \in C^{(t)}\}
\end{aligned}$$

where $C_1^{(t)}, C_2^{(t)}, ..., C_{t+1}^{(t)}$ are linear codes over Z_4 of length n.

Theorem 6. *Let $C^{(t)}$ be a linear code of length n over D_t. Then $\Phi_t(C^{(t)}) = C_1^{(t)} \otimes C_2^{(t)} \otimes \cdots \otimes C_{t+1}^{(t)}$ and $\left|C^{(t)}\right| = \left|C_1^{(t)}\right| \left|C_2^{(t)}\right| \cdots \left|C_{t+1}^{(t)}\right|$.*

Corollary 7. *If $\Phi_t(C^{(t)}) = C_1^{(t)} \otimes C_2^{(t)} \otimes \cdots \otimes C_{t+1}^{(t)}$, then $C^{(t)} = (1 - v_1 - \cdots - v_t) C_1^{(t)} \oplus v_1 C_2^{(t)} \oplus \cdots \oplus v_t C_{t+1}^{(t)}$.*

Theorem 8. *Let $C^{(t)} = (1 - v_1 - \cdots - v_t) C_1^{(t)} \oplus v_1 C_2^{(t)} \oplus \cdots \oplus v_t C_{t+1}^{(t)}$ be a linear code of any length n over D_t. Then $C^{(t)}$ is a cyclic code over D_t if and only if $C_1^{(t)}, C_2^{(t)}, ..., C_{t+1}^{(t)}$ are all cyclic codes over Z_4.*

Proof. It is proved that as in proof of Proposition 15, in [8]. □

Lemma 9. *(17) Let n be an odd positive integer and $x^n - 1 = \prod_{i=1}^{r} f_i(x)$ be the unique factorization of $x^n - 1$, where $f_1(x), ..., f_r(x)$ are basic irreducible polynomials over Z_4*

Theorem 10. *(17) Let C be a cyclic code of odd length n over Z_4, then*

$$C=(f_0(x),2f_1(x))=(f_0(x)+2f_1(x))$$

where $f_0(x)$ and $f_1(x)$ are monic factors of x^n-1 and $f_1(x)|f_0(x)$.

If C is a linear code of any length n over Z_4, then there exist monic polynomials $f(x),g(x),p(x)\in Z_4$ such that

$$C=(f(x)+2p(x),2g(x))$$

where $g(x)|f(x)|x^n-1$, $g(x)|p(x)[x^n-1/f(x)]$ and $|C|=2^{2n-\deg f(x)-\deg g(x)}$.

Theorem 11. *Let $C^{(t)}=(1-v_1-\cdots-v_t)\,C_1^{(t)}\oplus v_1C_2^{(t)}\oplus\cdots\oplus v_tC_{t+1}^{(t)}$ be a cyclic code of any length n over D_t. If there exist $f_i^1(x)$, $f_i^2(x)$, $f_i^3(x)\in Z_4[x]$ for $i=1,\cdots,t+1$ such that $C_i^{(t)}=\left(f_i^1(x)+2f_i^2(x),\ 2f_i^3(x)\right)$, then*

$$\begin{aligned}C^{(t)} \;=\; \Big(&(1-v_1-\cdots-v_t)\,f_1^1(x)+\cdots+v_tf_{t+1}^1(x)+2[(1-v_1-\cdots-v_t)\,f_1^2(x)\\&+\cdots+v_tf_{t+1}^2(x)],\ \ 2[(1-v_1-\cdots-v_t)\,f_1^3(x)+\cdots+v_tf_{t+1}^3(x)]\Big).\end{aligned}$$

If n is odd, then $C^{(t)}=((1-v_1-\cdots-v_t)\,(f_1^1(x)+2f_1^2(x))+\cdots+v_t(f_{t+1}^1(x)+2f_{t+1}^2(x)))$.

Proof. It is proved that as in proof of Theorem 10, in [17]. □

Definition 12. *A subset C of D_t^n is called a quasi-cyclic code of length $n=sl$ if C is satisfies the following conditions*

i) C is a submodule of D_t^n

ii) if $e=(e_{0,0},...,e_{0,l-1},e_{1,0},...,e_{1,l-1},...,e_{s-1,0},...,e_{s-1,l-1})\in C$, then $T_{s,l}\,(e)=(e_{s-1,0,...,}e_{s-1,l-1},e_{0,0},...,e_{0,l-1},...,e_{s-2,0},...,e_{s-2,l-1})\in C$.

Definition 13. *Let $a\in Z_4^{(t+1)n}$ with $a=\left(a_0,a_1,...,a_{(t+1)n-1}\right)=\left(a^{(0)}\left|a^{(1)}\right|...|a^{(t)}\right)$, $a^{(i)}\in Z_4^n$, for $i=0,1,...,t$. Let Γ be a map from $Z_4^{(t+1)n}$ to $Z_4^{(t+1)n}$ given by*

$$\Gamma\left(a\right)=\left(\mu\left(a^{(0)}\right)\left|\mu\left(a^{(1)}\right)\right|...|\mu\left(a^{(t)}\right)\right)$$

where μ is the map from Z_4^n to Z_4^n given by

$$\mu\left(a^{(i)}\right)=((a^{(i,s-1)}),(a^{(i,0)}),...,(a^{(i,s-2)}))$$

for every $a^{(i)}=\left(a^{(i,0)},...,a^{(i,s-1)}\right)$ where $a^{(i,j)}\in Z_4^l$, $j=0,1,...,s-1$ and $n=sl$. A code of length $(t+1)n$ over Z_4 is said to be l-quasi cyclic code of index $t+1$ if $\Gamma\left(C\right)=C$.

Proposition 14. *Let $T_{s,l}$ be the quasi-cyclic shift on D_t. Then $\Phi_tT_{s,l}=\Gamma\Phi_t$, where Γ is as above.*

Theorem 15. *The Gray image of a quasi-cyclic code over D_t of length n with index l is a l-quasi-cyclic code of index $t+1$ over Z_4 with length $(t+1)n$.*

3. Constacyclic codes over D_t

We investigate λ_t-constacyclic codes over D_t, where λ_t is unit.

For any element $\lambda_i = d_0 + v_1 d_1 + ... + v_i d_i \in D_i^*$ for $i = 1, 2, ..., t, \lambda_i$ is a unit if and only if $d_0 \neq 0, d_0 + d_1 \neq 0, ..., d_0 + d_i \neq 0$ for $i = 1, 2, ..., t$.

In [13], it was shown that the units are $1, 3, 1 + 2v_1, 3 + 2v_1$, for $D_1 = Z_4 + v_1 Z_4, v_1^2 = v_1$. In [6], it was shown that the units are $1, 3, 1 + 2v_1, 1 + 2v_2, 3 + 2v_1, 3 + 2v_2, 1 + 2v_1 + 2v_2, 3 + 2v_1 + 2v_2$ for $D_2 = Z_4 + v_1 Z_4 + v_2 Z_4, v_1^2 = v_1, v_2^2 = v_2, v_1 v_2 = v_2 v_1 = 0$.

Moreover, one can verify that if λ_i is a unit of D_i for $i = 1, 2, ..., t$, then $\lambda_i^2 = 1$, for $i = 1, 2, ..., t$.

Theorem 16. *Let $C^{(t)} = (1 - v_1 - \cdots - v_t)\, C_1^{(t)} \oplus v_1 C_2^{(t)} \oplus \cdots \oplus v_t C_{t+1}^{(t)}$ be a linear code of length n over D_t. Then $C^{(t)}$ is λ_t-constacyclic code over D_t if and only if $C_1^{(t)}$ is a d_0-constacyclic, $C_2^{(t)}$ is $d_0{+}d_1$-constacyclic,...,$C_{t+1}^{(t)}$ is a $d_0{+}d_t$-constacyclic codes of length n over Z_4.*

4. The reverse and reverse complement codes over D_t

In this section, we study cyclic codes of odd length over D_t satisfy reverse and reverse complement properties.

The elements $0, 1, 2, 3$ of Z_4 are in one to one correspondence with the nucleotide DNA bases A, T, C, G such that $0 \longrightarrow A, 1 \longrightarrow T, 2 \longrightarrow C$ and $3 \longrightarrow G$. The Watson Crick Complement is given by $\overline{A} = T, \overline{T} = A, \overline{G} = C, \overline{C} = G$.

Since the ring D_t is cardinality 4^{t+1}, then we give a one to one correspondence between the elements of D_t and the 4^{t+1} codons over the alphabet $\{A, T, G, C\}^{t+1}$ by using the Gray map. For example

Elements	Gray image	Codons
0	$\underbrace{(0,0,...,0)}_{t+1 \text{ times}}$	$\underbrace{AA...A}_{t+1 \text{ times}}$
1	$\underbrace{(1,1,...,1)}_{t+1 \text{ times}}$	$\underbrace{TT...T}_{t+1 \text{ times}}$
2	$\underbrace{(2,2,...,2)}_{t+1 \text{ times}}$	$\underbrace{CC...C}_{t+1 \text{ times}}$
3	$\underbrace{(3,3,...,3)}_{t+1 \text{ times}}$	$\underbrace{GG...G}_{t+1 \text{ times}}$
v_1	$\underbrace{(0,1,0,...,0)}_{t+1 \text{ times}}$	$\underbrace{ATA...A}_{t+1 \text{ times}}$
$1 + v_1$	$\underbrace{(1,2,1,...,1)}_{t+1 \text{ times}}$	$\underbrace{TCT...T}_{t+1 \text{ times}}$
$\vdots$	$\vdots$	$\vdots$

The codons satisfy the Watson Crick Complement.

Definition 17. *For $x=(x_0,x_1,...,x_{n-1})\in D_t^n$, the vector $(x_{n-1},x_{n-2},...,x_1,x_0)$ is called the reverse of x and is denoted by x^r. A linear code $C^{(t)}$ of length n over D_t, is said to be reversible if $x^r\in C^{(t)}$ for every $x\in C^{(t)}$.*

For $x=(x_0,x_1,...,x_{n-1})\in D_t^n$, the vector $(\overline{x}_0,\overline{x}_1,...,\overline{x}_{n-1})$ is called the complement of x and is denoted by x^c. A linear code $C^{(t)}$ of length n over D_t, is said to be complement if $x^c\in C^{(t)}$ for every $x\in C^{(t)}$.

For $x=(x_0,x_1,...,x_{n-1})\in D_t^n$, the vector $(\overline{x}_{n-1},\overline{x}_{n-2},...,\overline{x}_1,\overline{x}_0)$ is called the reversible complement of x and is denoted by x^{rc}. A linear code $C^{(t)}$ of length n over D_t, is said to be reversible complement if $x^{rc}\in C^{(t)}$ for every $x\in C^{(t)}$.

Definition 18. *Let $f(x)=a_0+a_1x+...+a_rx^r$ with $a_r\neq 0$ be polynomial. The reciprocal of $f(x)$ is defined as $f^*(x)=x^rf(\frac{1}{x})$. It is easy to see that $\deg f^*(x)\leq \deg f(x)$ and if $a_0\neq 0$, then $\deg f^*(x)=\deg f(x)$. $f(x)$ is called a self reciprocal polynomial if there is a constant m such that $f^*(x)=mf(x)$.*

Lemma 19. *Let $f(x),g(x)$ be polynomials in $D_i[x], 1\leq i\leq t$. Suppose $\deg f(x)-\deg g(x)=m$ then,*

i) $(f(x)g(x))^*=f^*(x)g^*(x)$
ii) $(f(x)+g(x))^*=f^*(x)+x^mg^*(x)$

Theorem 20. *Let $C^{(t)}=(1-v_1-\cdots-v_t)\,C_1^{(t)}\oplus v_1C_2^{(t)}\oplus\cdots\oplus v_tC_{t+1}^{(t)}$ be a cyclic code of odd length over D_t. Then $C^{(t)}$ is reversible code over D_t if and only if $C_1^{(t)},C_2^{(t)},...,C_{t+1}^{(t)}$ are reversible codes over Z_4.*

Proof. Let $C_i^{(t)}$ be reversible codes, where $i=1,2,...,t+1$. For any $b\in C^{(t)}, b=(1-v_1-\cdots-v_t)\,b_1+v_1b_2+...+v_tb_{t+1}$, where $b_i\in C_i^{(t)}$, for $1\leq i\leq t+1$. Since $C_i^{(t)}$ are reversible codes for all i, $b_i^r\in C_i^{(t)}$,where $i=1,2,...,t+1$. So, $b^r=(1-v_1-\cdots-v_t)\,b_1^r+v_2b_2^r+...+v_tb_{t+1}^r\in C^{(t)}$. Hence $C^{(t)}$ is reversible code.

On the other hand, let $C^{(t)}$ be a reversible code over D_t. So for any

$$(1-v_1-\cdots-v_t)\,b_1+v_1b_2+...+v_tb_{t+1},$$

where $b_i\in C_i^{(t)}$, for $1\leq i\leq t+1$, we get $b^r=(1-v_1-\cdots-v_t)\,b_1^r+v_2b_2^r+...+v_tb_{t+1}^r\in C^{(t)}$. Let $b^r=(1-v_1-\cdots-v_t)\,b_1^r+v_2b_2^r+...+v_tb_{t+1}^r=(1-v_1-\cdots-v_t)\,s_1+v_1s_2+...+v_ts_{t+1}$, where $s_i\in C_i^{(t)}$, for $1\leq i\leq t+1$. Therefore $C_i^{(t)}$ are reversible codes over Z_4 for $i=1,2,...,t+1$. □

Lemma 21. *For any $c\in D_i$,where $i=1,2,...,t$, we have $c+\overline{c}=1$.*

Lemma 22. *For any $a\in D_i$,where $i=1,2,...,t$, we have $\overline{a}+3\overline{0}=3a$.*

Theorem 23. *Let $C^{(t)}=(1-v_1-\cdots-v_t)\,C_1^{(t)}\oplus v_1C_2^{(t)}\oplus\cdots\oplus v_tC_{t+1}^{(t)}$ be a cyclic code of odd length n over D_t. Then $C^{(t)}$ is reversible complement over D_t iff $C^{(t)}$ is reversible over D_t and $(\overline{0},\overline{0},...,\overline{0})\in C^{(t)}$.*

Proof. Since $C^{(t)}$ is reversible complement, for any $c = (c_0, c_1, ..., c_{n-1}) \in C^{(t)}$, $c^{rc} = (\overline{c}_{n-1}, \overline{c}_{n-2}, ..., \overline{c}_0) \in C^{(t)}$. Since $C^{(t)}$ is a linear code, so $(0, 0, ..., 0) \in C^{(t)}$. Since $C^{(t)}$ is reversible complement, so $(\overline{0}, \overline{0}, ..., \overline{0}) \in C^{(t)}$. By using Lemma 22, we get

$$3c^r = 3(c_{n-1}, c_{n-2}, ..., c_0) = (\overline{c}_{n-1}, \overline{c}_{n-2}, ..., \overline{c}_0) + 3(\overline{0}, \overline{0}, ..., \overline{0}) \in C^{(t)}$$

Hence for any $c \in C^{(t)}$, we have $c^r \in C^{(t)}$.

On the other hand, let $C^{(t)}$ be reversible code over D_t. So, for any $c = (c_0, c_1, ..., c_{n-1}) \in C^{(t)}$, then $c^r = (c_{n-1}, c_{n-2}, ..., c_0) \in C^{(t)}$. For any $c \in C^{(t)}$,

$$c^{rc} = (\overline{c}_{n-1}, \overline{c}_{n-2}, ..., \overline{c}_0) = 3(c_{n-1}, c_{n-2}, ..., c_0) + (\overline{0}, \overline{0}, ..., \overline{0}) \in C^{(t)}$$

So, $C^{(t)}$ is reversible complement code over D_t. □

Theorem 24. *Let S_1 and S_2 be two reversible complement cyclic codes of length n over D_i, where $i = 1, 2, ..., t$. Then $S_1 + S_2$ and $S_1 \cap S_2$ are reversible complement cyclic codes.*

Proof. It is shown that as in proof of Theorem 23, in [6]. □

5. Binary images of cyclic DNA codes over D_t

In this section, we will determine binary images of cyclic DNA codes over D_i, where $i = 1, 2, ..., t$.

The 2-adic expansion of $c \in Z_4$ is $c = \alpha(c) + 2\beta(c)$ such that $\alpha(c) + \beta(c) + \gamma(c) = 0$ for all $c \in Z_4$

c	$\alpha(c)$	$\beta(c)$	$\gamma(c)$
0	0	0	0
1	1	0	1
2	0	1	1
3	1	1	0

The Gray map is given by

$$\begin{aligned} \Psi \quad &: \quad Z_4 \longrightarrow Z_2^2 \\ c \quad &\longmapsto \quad \Psi(c) = (\beta(c), \gamma(c)) \end{aligned}$$

for all $c \in Z_4$ in [18]. We define

$$\begin{aligned} \breve{O}_t \quad &: \quad D_t \longrightarrow Z_2^{2(t+1)} \\ d_0 + v_1 d_1 + ... + v_t d_t \quad &\longmapsto \quad \breve{O}_t(d_0 + v_1 d_1 + ... + v_t d_t) = \Psi\left(\Phi_t\left(d_0 + v_1 d_1 + ... + v_t d_t\right)\right) \\ &= \quad \Psi(d_0, d_0 + d_1, ..., d_0 + d_t) \\ &= \quad (\beta(d_0), \gamma(d_0), \beta(d_0 + d_1), \gamma(d_0 + d_1), ..., \beta(d_0 + d_t), \gamma(d_0 + d_t)) \end{aligned}$$

where Φ_t is a Gray map from D_t to Z_4^{t+1}.

Let $d_0 + v_1 d_1 + ... + v_t d_t$ be any element of the ring D_t. The Lee weight w_L of the ring D_t is defined as follows

$$w_L(d_0 + v_1 d_1 + ... + v_t d_t) = w_L(d_0, d_0 + d_1, ..., d_0 + d_t)$$

where $w_L(d_0, d_0+d_1, ..., d_0+d_t)$ described the usual Lee weight on Z_4^{t+1}. For any $c_1, c_2 \in D_t$ the Lee distance d_L is given by $d_L(c_1,c_2) = w_L(c_1-c_2)$.

The Hamming distance $d_H(c_1,c_2)$ between two codewords c_1 and c_2 is the Hamming weight of the codewords c_1-c_2.

$$\begin{array}{ccc}
\underbrace{AA...A}_{t+1 \text{ times}} & \longrightarrow & \underbrace{(0,0,...,0)}_{2(t+1) \text{ times}} \\
\underbrace{TT...T}_{t+1 \text{ times}} & \longrightarrow & \underbrace{(0,1,0,1,...,0,1)}_{2(t+1) \text{ times}} \\
\underbrace{GG...G}_{t+1 \text{ times}} & \longrightarrow & \underbrace{(1,0,1,0,...,1,0)}_{2(t+1) \text{ times}} \\
\underbrace{CC...C}_{t+1 \text{ times}} & \longrightarrow & \underbrace{(1,1,...,1)}_{2(t+1) \text{ times}} \\
\vdots & \vdots & \vdots
\end{array}$$

Lemma 25. *The Gray map $\breve{O}_t$ is a distance preserving map from (D_t^n, Lee distance) to ($Z_2^{2(t+1)n}$, Hamming distance). It is also Z_2-linear.*

Proof. For $c_1, c_2 \in D_t^n$, we have $\breve{O}_t(c_1-c_2) = \breve{O}_t(c_1) - \breve{O}_t(c_2)$. So, $d_L(c_1,c_2) = w_L(c_1-c_2) = w_H(\breve{O}_t(c_1-c_2)) = w_H(\breve{O}_t(c_1)-\breve{O}_t(c_2)) = d_H(\breve{O}_t(c_1), \breve{O}_t(c_2))$. So, the Gray map $\breve{O}_t$ is distance preserving map. For Z_2-linear, it is easily seen that $\breve{O}_t(k_1c_1+k_2c_2) = k_1\breve{O}_t(c_1) + k_2\breve{O}_t(c_2)$, where $c_1, c_2 \in D_t^n, k_1, k_2 \in Z_2$. □

Proposition 26. *Let σ be the cyclic shift of D_t^n and η be the $2(t+1)$-quasi-cyclic shift of $Z_2^{2(t+1)n}$. Let $\breve{O}_t$ be the Gray map from D_t^n to $Z_2^{2(t+1)n}$. Then $\breve{O}_t\sigma = \eta\breve{O}_t$.*

Theorem 27. *If C is a cyclic DNA code of length n over D_t then $\breve{O}_t(C)$ is a binary quasi-cyclic DNA code of length $2(t+1)n$ with index $2(t+1)$.*

6. Skew codes over D_t

We are interested in studying skew codes over D_i for $i = 2, ..., t$, in this section. Firstly, we define a nontrivial automorphism θ_t on the ring D_t for $t \geq 2$, by $\theta_t(v_i) = v_{i+1(\text{mod } t)}$, where $i = 1, 2, ..., t$.

For example, for $t = 2$, a nontrivial automorphism θ_2 on the ring D_2 as follows

$$\begin{array}{ccc}
\theta_2 & : & D_2 \longrightarrow D_2 \\
d_0+v_1d_1+v_2d_2 & \longmapsto & d_0+v_1d_2+v_2d_1
\end{array}$$

where $d_0, d_1, d_2 \in Z_4$.

The ring $D_t[x,\theta_t] = \{a_0+a_1x+...+a_{n-1}x^{n-1} : a_i \in D_t, i = 0, ..., n-1, n \in N\}$ is called skew polynomial ring. The ring is a non-commutative ring. The addition in the ring $D_t[x,\theta_t]$ is the usual polynomial additional and multiplication is defined using the rule, $(ax^i)(bx^j) = a\theta_t^i(b)x^{i+j}$. The order of the automorphism θ_t is t.

Definition 28. *A subset $C^{(t)}$ of D_t^n is called a skew cyclic code of length n if $C^{(t)}$ satisfies the following conditions,*

i) $C^{(t)}$ *is a submodule of* D_t^n,

ii) If $c=(c_0,c_1,...,c_{n-1})\in C^{(t)}$, *then* $\sigma_{\theta_t}(c)=(\theta_t(c_{n-1}),\theta_t(c_0),...,\theta_t(c_{n-2}))\in C^{(t)}$

Let $f_t(x)+\langle x^n-1\rangle$ be an element in the set $S_{t,n}=D_t[x,\theta_t]/\langle x^n-1\rangle$ and let $r_t(x)\in D_t[x,\theta_t]$. Define multiplication from left as follows,

$$r_t(x)(f_t(x)+\langle x^n-1\rangle)=r_t(x)f_t(x)+\langle x^n-1\rangle$$

for any $r_t(x)\in D_t[x,\theta_t]$.

Theorem 29. *$S_{t,n}$ is a left $D_t[x,\theta_t]$-module where multiplication defined as in above.*

Theorem 30. *A code $C^{(t)}$ in $S_{t,n}$ of length n is a skew cyclic code if and only if $C^{(t)}$ is a left $D_t[x,\theta_t]$-submodule of the left $D_t[x,\theta_t]$-module $S_{t,n}$.*

Theorem 31. *Let $C^{(t)}$ be a skew cyclic code over D_t of length n and let $f_t(x)$ be a polynomial in $C^{(t)}$ of minimal degree. If $f_t(x)$ is monic polynomial, then $C^{(t)}=\langle f_t(x)\rangle$, where $f_t(x)$ is a right divisor of x^n-1.*

Definition 32. *A subset $C^{(t)}$ of D_t^n is called a skew quasi-cyclic code of length n if $C^{(t)}$ satisfies the following conditions,*

i) $C^{(t)}$ is a submodule of D_t^n,

ii) If $e=(e_{0,0},...,e_{0,l-1},e_{1,0},...,e_{1,l-1},...,e_{s-1,0},..,e_{s-1,l-1})\in C^{(t)}$, then $\tau_{\theta_t,s,l}(e)=(\theta_t(e_{s-1,0}),...,\theta_t(e_{s-1,l-1}),\theta_t(e_{0,0}),...,\theta_t(e_{0,l-1}),...,\theta_t(e_{s-2,0}),...,\theta_t(e_{s-2,l-1}))\in C^{(t)}$.

We note that x^s-1 is a two sided ideal in $D_t[x,\theta_t]$ if $t|s$ where t is the order of θ_t. So $D_t[x,\theta_t]/(x^s-1)$ is well defined.

The ring $R_s^l=(D_t[x,\theta_t]/(x^s-1))^l$ is a left $R_s=D_t[x,\theta_t]/(x^s-1)$ module by the following multiplication on the left $f(x)(g_1(x),...,g_l(x))=(f(x)g_1(x),...f(x)g_l(x))$. If the map Λ_t is defined by

$$\Lambda_t: D_t^n\longrightarrow R_s^l$$

$(e_{0,0},...,e_{0,l-1},e_{1,0},...,e_{1,l-1},...,e_{s-1,0},...,e_{s-1,l-1})\mapsto(c_0(x),...,c_{l-1}(x))$ such that $c_j(x)=\sum_{i=0}^{s-1}e_{i,j}x^i\in R_s$ where $j=0,1,...,l-1$ then the map Λ_t gives a one to one correspondence D_t^n and the ring R_s^l.

Theorem 33. *A subset $C^{(t)}$ of D_t^n is a skew quasi-cyclic code of length $n=sl$ and index l if and only if $\Lambda_t(C^{(t)})$ is a left R_s-submodule of R_s^l.*

Definition 34. *Let θ_t be an automorphism of D_t, λ_t be a unit in D_t, $C^{(t)}$ be a linear code D_t. A linear code $C^{(t)}$ is said to be a skew constacyclic code if $C^{(t)}$ is closed under the $\theta_t-\lambda_t$-constacyclic shift $\tau_{\theta_t,\lambda_t}: D_t^n\longrightarrow D_t^n$ defined by*

$$\tau_{\theta_t,\lambda_t}(c_0,...,c_{n-1})=(\theta_t(\lambda_t c_{n-1}),\theta_t(c_0),...,\theta_t(c_{n-2}))$$

7. The Gray images of skew cyclic, quasi-cyclic and constacyclic codes over D_t

Proposition 35. *Let σ_{θ_t} be the skew cyclic shift on D_t^n, Let Φ_t be the Gray map from D_t^n to $Z_4^{(t+1)n}$ and φ be as in the preliminaries. Then*

$$\Phi_t \sigma_{\theta_t} = \upsilon \varphi \Phi_t$$

where υ is map such that $\upsilon(x_1, x_2, ..., x_{t+1}) = (x_1, x_{t+1}, x_t, ..., x_2)$ for $x_i \in Z_4^n, i = 1, ..., t+1$.

Proof. It is proved that as in the proof the Proposition 3. □

Theorem 36. *The Gray image of a skew cyclic code over D_t of length n is permutation equivalent to a quasi-cyclic code of index $t+1$ with length $(t+1)n$.*

Proof. It is proved that as in the proof the Theorem 4. □

Proposition 37. *Let $\tau_{\theta_t,s,l}$ be the skew quasi-cyclic shift, Γ be as in the preliminaries, Φ_t be the Gray map from D_t^n to $Z_4^{(t+1)n}$. Then*

$$\Phi_t \tau_{\theta_t,s,l} = \upsilon \Gamma \Phi_t$$

where υ is map such that $\upsilon(x_1, x_2, ..., x_{t+1}) = (x_1, x_{t+1}, x_t, ..., x_2)$ for $x_i \in Z_4^n, i = 1, ..., t+1$.

Theorem 38. *The Gray image of a skew quasi-cyclic code over D_t of length n is permutation equivalent to a l-quasi-cyclic code of index $t+1$ with length $(t+1)n$.*

Proposition 39. *Let $\tau_{\theta_t,\lambda}$ be the θ_t-λ_t-cyclic shift, let Φ_t be the Gray map from D_t^n to $Z_4^{(t+1)n}$ and σ_{λ_t} be constacyclic shift. Then*

$$\Phi_t \tau_{\theta_t,\lambda t} = \upsilon \Phi_t \sigma_{\lambda_t}$$

where υ is a map such as above.

Theorem 40. *The Gray image of a skew constacyclic code over D_t of length n is permutation equivalent to the Gray image of a constacyclic code over D_t of length n.*

8. Skew cyclic DNA codes over D_t

In this section, we introduce a family of DNA skew cyclic codes over D_t. We study its property of being reverse complement.

For all $x \in D_t$, we have

$$\theta_t(x) + \theta_t(\overline{x}) = 1$$

Theorem 41. *Let $C^{(t)} = \langle f_t(x) \rangle$ be a skew cyclic code over D_t of length n, where $f_t(x)$ is a monic polynomial in $C^{(t)}$ of minimal degree. If $C^{(t)}$ is reversible complement, the polynomial $f_t(x)$ is self reciprocal and $(1, 1, ..., 1) \in C^{(t)}$.*

Proof. Let $C^{(t)} = \langle f_t(x) \rangle$ be a skew cyclic code over D_t, where $f_t(x)$ is a monic polynomial in $C^{(t)}$. Since $(0,0,...,0) \in C^{(t)}$ and $C^{(t)}$ is reversible complement, we have $(\overline{0},\overline{0},...,\overline{0}) = (1,1,...,1) \in C^{(t)}$.

Let $f_t(x) = 1 + a_1^t x + ... + a_{r-1}^t x^{r-1} + x^r$. Since $C^{(t)}$ is reversible complement, we have $f_t^{rc}(x) \in C^{(t)}$. That is

$$f_t^{rc}(x) = 1 + x + ... + x^{n-r-2} + 0x^{n-r-1} + \overline{a^t}_{r-1} x^{n-r} + ... + \overline{a^t}_1 x^{n-2} + 0x^{n-1}$$

Since $C^{(t)}$ is a linear code, we have $f_t^{rc}(x) - \frac{x^n-1}{x-1} \in C^{(t)}$. This implies that

$$-x^{n-r-1} + (\overline{a^t}_{r-1} - 1)x^{n-r} + ... + (\overline{a^t}_1 - 1)x^{n-2} - x^{n-1} \in C^{(t)}$$

By multiplying on the right by x^{r+1-n}, we have

$$-1 + (\overline{a^t}_{r-1} - 1)\theta_t(1)x + ... + (\overline{a^t}_1 - 1)\theta_t^{r-1}(1)x^{r-1} - \theta_t^r(1)x^r \in C^{(t)}$$

By using $a + \overline{a} = 1$, for $a \in D_t$, we have

$$-1 - a_{r-1}^t x - a_{r-2}^t x^2 - ... - a_1^t x^{r-1} - x^r = 3f_t^*(x) \in C^{(t)}$$

Since $C^{(t)} = \langle f_t(x) \rangle$, there exist $q_t(x) \in D_t[x, \theta_t]$ such that $3f_t^*(x) = q_t(x)f_t(x)$. Since $\deg f_t(x) = \deg f_t^*(x)$, we have $q_t(x) = 1$. Since $3f_t^*(x) = f_t(x)$, we have $f_t^*(x) = 3f_t(x)$. So, $f_t(x)$ is self reciprocal. □

Theorem 42. *Let $C^{(t)} = \langle f_t(x) \rangle$ be a skew cyclic code over D_t of length n, where $f_t(x)$ is a monic polynomial in $C^{(t)}$ of minimal degree. If $(1,1,...,1) \in C^{(t)}$ and $f_t(x)$ is self reciprocal, then $C^{(t)}$ is reversible complement.*

Proof. Let $f_t(x) = 1 + a_1^t x + ... + a_{r-1}^t x^{r-1} + x^r$ be a monic polynomial of the minimal degree.

Let $c_t(x) \in C^{(t)}$. So, $c_t(x) = q_t(x)f_t(x)$, where $q_t(x) \in D_t[x, \theta_t]$. By using Lemma 19, we have $c_t^*(x) = (q_t(x)f_t(x))^* = q_t^*(x)f_t^*(x)$. Since $f_t(x)$ is self reciprocal, so $c_t^*(x) = q_t^*(x)e_t f_t(x)$, where $e_t \in Z_4 \backslash \{0\}$. Therefore $c_t^*(x) \in C^{(t)} = \langle f_t(x) \rangle$. Let $c_t(x) = c_0^t + c_1^t x + ... + c_r^t x^r \in C^{(t)}$. Since $C^{(t)}$ is a cyclic code, we get

$$c_t(x)x^{n-r-1} = c_0^t x^{n-r-1} + c_1^t x^{n-r} + ... + c_r^t x^{n-1} \in C^{(t)}$$

The vector correspond to this polynomial is

$$(0,0,...,0,c_0^t,c_1^t,...,c_r^t) \in C^{(t)}$$

Since $(1,1,...,1) \in C^{(t)}$ and $C^{(t)}$ linear, we have

$$(1,1,...,1) - (0,0,...,0,c_0^t,c_1^t,...,c_r^t) = (1,...,1,1-c_0^t,...,1-c_r^t) \in C^{(t)}$$

By using $a + \overline{a} = 1$, for $a \in D_t$, we get

$$(1,1,...,1,\overline{c_0^t},...,\overline{c_r^t}) \in C^{(t)}$$

which is equal to $(c_t(x)^*)^{rc}$. This shows that $((c_t(x)^*)^{rc})^* = c_t(x)^{rc} \in C^{(t)}$. □

9. MDS codes over D_t

In this section, we investigate some properties of MDS codes over D_t.

It is well known that $C^{(i)}$ is a linear code of length n over D_i, where $i=1,2,...,t$ and d_{H_i} is the minimum distance, then

$$\left|C^{(i)}\right| \leq |D_i|^{n-d_{H_i}+1}$$

where $i=1,2,...,t$. So $d_{H_i} \leq n-\log_{|D_i|}\left|C^{(i)}\right|+1$, where $i=1,2,...,t$. This inequality is called Singleton bound. If $C^{(i)}$, where $i=1,2,...,t$ meet the Singleton bound, then $C^{(i)}$, where $i=1,2,...,t$ are called MDS codes.

Lemma 43. *Let C be a linear code of length n over Z_4, the C is a MDS code if and only if C is either Z_4^n with parameters $(n,4^n,1)$ or $\langle 1\rangle$ with parameters $(n,4,n)$ or $\langle 1\rangle^{\perp}$ with parameters $(n,4^{n-1},2)$, where 1 denotes the all 1 vectors, [17].*

We know that if $C^{(i)}$ is a linear code of length n over D_i, where $i=1,2,...,t$, then

$$C^{(i)}=(1-v_1-\cdots-v_i)\,C_1^{(i)}\oplus v_1C_2^{(i)}\oplus\cdots\oplus v_iC_{i+1}^{(i)}$$

where $C_j^{(i)}$ is a linear code of length n over Z_4, where $j=1,...,i+1$.

Let d_{H_i} be the Hamming distance of $C^{(i)}$. Then $d_{H_i}=\min\left\{d_{H_{i,j}}\right\}$ for $1\leq i\leq t$, $1\leq j\leq i+1$, where $d_{H_{i,j}}$ is Hamming distance of $C_j^{(i)}$. So the Singleton bound can be written as

$$d_{H_i} \leq n-\frac{1}{i+1}\sum_{j=1}^{i+1}\log_4\left|C_j^{(i)}\right|+1$$

Lemma 44. *Let $C^{(i)}$ be a MDS codes over D_i, where $i=1,2,...,t$.*

i. If $d_{H_i}=1$, then all of $C_j^{(i)}$, $j=1,...,i+1$, are MDS codes with parameters $(n,4^n,1)$.

ii. If $d_{H_i}=2$, then all of $C_j^{(i)}$, $j=1,...,i+1$, are MDS codes with parameters $(n,4^{n-1},2)$.

Proof. (i) If $d_{H_i}=1$, then $\sum_{j=1}^{i+1}\log_4\left|C_j^{(i)}\right|=(i+1)n$. Since $C^{(i)}$ is a MDS code over D_i, where $i=1,2,...,t$, but $|C_j^{(i)}|\leq 4^n$, then the identity is true iff $|C_j^{(i)}|=4^n$, where $1\leq i\leq t$, $1\leq j\leq i+1$. Therefore $C^{(i)}$ is a $(n,4^{(i+1)n},1)$ MDS code iff all of $C_j^{(i)}$ are $(n,4^n,1)$ MDS codes, where $1\leq i\leq t$, $1\leq j\leq i+1$.

(ii) If $d_{H_i}=2$, then $\sum_{j=1}^{i+1}\log_4\left|C_j^{(i)}\right|=(i+1)(n-1)$. Since $d_{H_i}=\min\left\{d_{H_{i,j}}\right\}$, then $d_{H_{i,j}}\geq 2$, for $1\leq i\leq t$, $j=1,...,i+1$. By using Singleton bound of code over Z_4, we get $|C_j^{(i)}|\leq 4^{n-d_{H_{i,j}}+1}$. For all i, since $d_{H_{i,j}}\geq 2$, we have $4^{n-d_{H_{i,j}}+1}\leq 4^{n-1}$. Then we have all of $C_j^{(i)}$ are $(n,4^{n-1},2)$,where $1\leq i\leq t$, $1\leq j\leq i+1$. $\square$

Theorem 45. *If $C^{(i)}$ is a MDS code over D_i, where $i = 1, 2, ..., t$, then there is at least one $C_j^{(i)}, 1 \leq i \leq t$, $j = 1, ..., i+1$, be MDS code.*

Proof. It is proved that as in the proof the Theorem 4.3 in [7]. □

Theorem 46. *If $C^{(i)}$ is a MDS code over D_i, where $i = 1, 2, ..., t$ and there exist i numbers MDS codes of $C_j^{(i)}, 1 \leq i \leq t, 1 \leq j \leq i+1$, then the other $C_j^{(i)}$ must be MDS code and all $C_j^{(i)}$ with same parameters.*

Proof. It is proved that as in the proof the Theorem 4.4 in [7]. □

Corollary 47. *$C^{(i)}$ is a MDS code over D_i iff all of $C_j^{(i)}$ for $1 \leq i \leq t$, $j = 1, ..., i+1$ are MDS codes over Z_4 with same parameters.*

10. Conclusion

In this paper, we generalize some results which are given in the papers [6] and [7], to the linear codes over D_i, where $i = 1, 2, ..., t$.

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