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NEW INEQUALITIES IN TERMS OF OPERATOR m-CONVEX FUNCTIONS IN HILBERT SPACE

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Abstract. In this study, we obtained some new inequalities via operator m-convex functions. Afterwards, we generalized and revised some theorems and lemmas in operator m-convex functions.

1. Introduction

We know the below inequality in literature as Hermite-Hadamard,

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

where $f:\mathbb{R}\to\mathbb{R}$ is any convex function, $a,b\in\mathbb{R}$. It satisfies approximates of the mean value of $f:[a,b]\to\mathbb{R}$ continuous convex function. Let $A,B\in B(H)$ be selfadjoint operators, where H is a Hilbert space, $B(H)$ is all bounded operators from H to H. Then for every $x\in H$

$$A \leq B \qquad \text{means that} \qquad \langle Ax,x\rangle \leq \langle Bx,x\rangle$$

or

$$B \leq A \qquad \text{means that} \qquad \langle Bx,x\rangle \leq \langle Ax,x\rangle$$

Let $(H,\langle\cdot,\cdot\rangle)$ be a complex Hilbert space and A be a selfadjoint operator on it. The Gelfand transformation sets up a Φ *-isometrical isomorphism $C(Sp(A))$ among $C^*(A)$ the $C^*(A)$-algebra $C(Sp(A))$ is $C^*(A)$-algebra of all continuous complex-valued functions on spectrum A. Let $f,g\in C(Sp(A))$ and $\alpha,\beta\in\mathbb{C}$

i. $\Phi(\alpha f+\beta g)=\alpha\Phi(f)+\beta\Phi(g)$
ii. $\Phi(fg)=\Phi(f)\Phi(g)$ and $\Phi(f^*)=\Phi(f)^*$
iii. $\| \Phi(f) \| = \| f \| := sup_{t\in Sp(A)} \mid f(t) \mid$
iv. $\Phi(f_0)=1$ and $\Phi(f_1)=A$ where $f_0(t)=1, f_1(t)=t$ for $t\in Sp(A)$

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which is the element of $\Phi(f)$ of $C^*(A)$, is continuous functional calculus. In this conditions, if for all $t \in Sp(A)$, $f(t) \geq 0$, then $f(A) \geq 0$, namely $f(A)$ is positive on H. Let $f, g : Sp(A) \to \mathbb{R}$ be two functions. If for every $t \in Sp(A)$ $f(t) \leq g(t)$, then $f(A) \leq g(A)$ in the operator order $B(H)$.

Definition 1. [1] *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be continuous function and $\lambda \in [0,1]$. If for all $A, B \in B(H)$ selfadjoint operator, where spectra in I,*

$$f((1-\lambda)A + \lambda B) \leq (1-\lambda)f(A) + \lambda f(B)$$

inequality satisfies, then we say the f is an operator convex function.

Definition 2. [7] *Let $f : [a,b] \to \mathbb{R}$ be any function and $m \in [0,1]$. If for all $x, y \in [a,b]$ and $t \in [0,1]$, the following inequality holds. Then we say the function m-convex*

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y).$$

Now, we give an example of m-convex function [6]*. The function*

$$f(x) = \begin{cases} \frac{1}{2}x & for \quad 0 \leq x < 1 \\ \frac{3}{2}x - \frac{1}{2} & for \quad 1 \leq x \leq 2 \end{cases}$$

is m-convex for every $m \in (0, 1/2]$.

Definition 3. [2] *Assume that, $f : I \subset \mathbb{R} \to \mathbb{R}$ be a continuous function and $\lambda \in [0,1]$. Then if for all $A, B \in B(H)$ selfadjoint operators, whose spectra in I and $m \in [0,1]$,*

$$f((1-\lambda)A + m\lambda B) \leq (1-\lambda)f(A) + m\lambda f(B)$$

inequality holds, we say the f function is an operator m-convex.

Theorem 1. [4] *Let $f, g : I \subset \mathbb{R} \to \mathbb{R}$ be operator s_1 and s_2 convex functions, respectively. Then for every positive A, B operator in Hilbert space whose spectra in I, the following inequality holds for any $x \in H$, with $\|x\|$=1,*

$$\begin{aligned}\int_0^1 \langle f(tA + (1-t)B)x, x\rangle \langle g(tA + (1-t)B)x, x\rangle dt \quad \leq \quad & \frac{1}{s_1 + s_2 + 1} M(A,B)(x) + \\ & \beta(s_1 + 1, s_2 + 1) N(A,B)(x)\end{aligned}$$

where

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt, x > y \qquad (\textit{Beta function})$$

and

$$M = M(A,B)(x) = \langle f(A)x, x\rangle \langle g(A)x, x\rangle + \langle f(B)x, x\rangle \langle g(B)x, x\rangle$$

$$N = N(A,B)(x) = \langle f(A)x, x\rangle \langle g(B)x, x\rangle + \langle f(B)x, x\rangle \langle g(A)x, x\rangle.$$

Theorem 2. [4] *Let $f, g : I \subset \mathbb{R} \to \mathbb{R}$ be operator s_1 and s_2 convex functions, respectively. Then for every positive A, B operator in Hilbert space whose spectra in I, the following inequality holds for any $x \in H$, $\|x\|=1$,*

$$\begin{aligned} & 2^{s_1+s_2-1}\left\langle f\left(\frac{a+b}{2}\right)x,x\right\rangle\left\langle g\left(\frac{a+b}{2}\right)x,x\right\rangle \\ \leq & \int_0^1 \langle f(tA+(1-t)B)x,x\rangle\langle g(tA+(1-t)B)x,x\rangle \\ & +\beta(s_1+1,s_2+1)M(A,B)(x)+\frac{1}{s_1+s_2+1}N(A,B)(x). \end{aligned} \tag{1.2}$$

where

$$\beta(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt, x>y \qquad (\textit{Beta function})$$

and

$$M=M(A,B)(x)=\langle f(A)x,x\rangle\langle g(A)x,x\rangle+\langle f(B)x,x\rangle\langle g(B)x,x\rangle$$

$$N=N(A,B)(x)=\langle f(A)x,x\rangle\langle g(B)x,x\rangle+\langle f(B)x,x\rangle\langle g(A)x,x\rangle$$

Theorem 3. [4] *Let $A, B \in K \subseteq B(H)^+$, $B(H)^+$ is set of all positive operators. Then $AB+BA$ is positive if only if*

$$f(A+B) \leq f(A)+f(B)$$

Where f is nonnegative operator function on $[0,\infty)$.

Theorem 4. [1] *Let f be an operator convex function on the I. Under the circumstances, for all selfadjoint A, B operators, whose spectra in I, below inequality holds,*

$$\begin{aligned} f(\frac{A+B}{2}) & \leq \frac{1}{2}[f(\frac{3A+B}{4})+(\frac{A+3B}{4})] \\ & \leq \int_0^1 f((1-t)A+tB)dt \\ & \leq \frac{1}{2}[f(\frac{A+B}{2})+(\frac{f(A)+f(B)}{2})] \\ & \leq \frac{f(A)+f(B)}{2} \end{aligned}$$

2. RESULTS

Lemma 1. *If f is operator m-convex on $[0,\infty)$ and non-decreasing function for operator in K, $\frac{1}{m}\langle Ax,x\rangle \subset I$ and $m \in (0,1]$ then $f(A)$ is positive for every $A \in K$.*

Proof. Since f is operator m-convex and $A \in K$, we can write the following inequality,

$$\begin{aligned} f(A) &= f\Big(\frac{tA + m(1-t)A + m((1-t)\frac{A}{m} + tA)}{m+1}\Big) \\ &\leq f(tA + m(1-t)A + m((1-t)\frac{A}{m} + tA)) \\ &\leq tf(A) + m(1-t)f(A) + (1-t)f(A) + mtf(A) \end{aligned}$$

So, we have

$$\begin{aligned} f(A) &\leq f(A)(m+1) \\ 0 &\leq f(A) \end{aligned}$$

Consequently, $f(A)$ is positive. □

Lemma 2. *Let $I \subset [0, \infty)$, and $f : I \to \mathbb{R}$ be continuous function. Then for all $A, B \in K \subset B(H)^+$, f is an operator m-convex on I if only if for every $x \in H$ with $\|x\| = 1$,*

$$\varphi_{x,A,B}(t) = \langle f((1-t)A + mtB)x, x\rangle$$

defined $\varphi_{x,A,B} : [0,1] \to \mathbb{R}$ function is m-convex.

Proof. We assume that f is operator m- convex, then we prove the $\varphi_{x,A,B}(\cdot)$ function is m-convex. For $A, B \in K$ whose spectra are in $I, t_1, t_2 \in [0, 1]$ and $\lambda, \gamma \geq 0$ with $\lambda + m\gamma = 1$, we can write the below equalities

$$\begin{aligned} \varphi_{x,A,B}(\lambda t_1 + m\gamma t_2) &= \Big\langle f\Big(\big[1 - (\lambda t_1 + m\gamma t_2)\big]A + m\big[\lambda t_1 + m\gamma t_2\big]B\Big)x, x\Big\rangle \\ &= \Big\langle f\Big(A - \lambda t_1 A - m\gamma t_2 A + m\lambda t_1 B + m^2\gamma t_2 B\Big)x, x\Big\rangle \\ &= \Big\langle f\Big(1.A + \lambda\big[-t_1 A + mt_1 B\big] + m\gamma\big[-t_2 A + mt_2 B\big]\Big)x, x\Big\rangle. \end{aligned}$$

If we take $\lambda + m\gamma$ instead of 1, then we obtain the following equalities

$$\begin{aligned} &\varphi_{x,A,B}(\lambda t_1 + m\gamma t_2) \\ &= \Big\langle f\Big((\lambda + m\gamma)A + \lambda\big[-t_1 A + mt_1 B\big] + m\gamma\big[-t_2 A + mt_2 B\big]\Big)x, x\Big\rangle \\ &= \Big\langle f\Big(\lambda A + m\gamma A + \lambda\big[-t_1 A + mt_1 B\big] + m\gamma\big[-t_2 A + mt_2 B\big]\Big)x, x\Big\rangle \\ &= \Big\langle f\Big(\lambda\big[A - t_1 A + mt_1 B\big] + m\gamma\big[A - t_2 A + mt_2 B\big]\Big)x, x\Big\rangle \\ &= \Big\langle f\Big(\lambda\big[(1-t_1)A + mt_1 B\big] + m\gamma\big[(1-t_2)A + mt_2 B\big]\Big)x, x\Big\rangle \end{aligned}$$

Due to f is operator m-convex

$$\varphi_{x,A,B}(\lambda t_1 + m\gamma t_2) \leq \lambda\varphi_{x,A,B}(t_1) + m\gamma\varphi_{x,A,B}(t_2)$$

inequality holds, so $\varphi_{x,A,B}(\cdot)$ function is m-convex. Now we assume vice versa, namely if $\varphi_{x,A,B}$ is m-convex then we will show that f is operator m-convex. Let for all $t_1, t_2 \in [0,1]$, $m \in [0,1]$ and $A, B \in K$

$$C := (1-t_1)A + mt_1B$$

$$D := (1-t_2)A + mt_2B$$

In this case, we calculate

$$\begin{aligned}
& \varphi_{x,C,D}(\lambda) = \langle f((1-\lambda)C + m\lambda D)x, x\rangle \\
= \;& \langle f((1-\lambda)[(1-t_1)A + mt_1B] + m\lambda[(1-t_2)A + mt_2B])x.x\rangle \\
= \;& \langle f(A - t_1A + mt_1B - \lambda A + \lambda t_1A - m\lambda t_1B + m\lambda A - m\lambda t_2A + m^2\lambda t_2B)x, x\rangle \\
= \;& \langle f(A(1-t_1) - \lambda A(1-t_1) + m\lambda A(1-t_2) + mt_1B + m^2\lambda t_2B - m\lambda t_1B)x, x\rangle \\
= \;& \langle f(-\lambda((1-t_1)A + mt_1B) + A(1-t_1) + mt_1B + m\lambda(A(1-t_2) + mt_2B))x, x\rangle \\
= \;& \langle f((1-\lambda)((1-t_1)A + mt_1B) + m\lambda((1-t_2)A + mt_2B))x, x\rangle \\
\leq \;& (1-\lambda)\langle f(C)x, x\rangle + m\lambda\langle f(D)x, x\rangle
\end{aligned}$$

So proof is completed. □

Theorem 5. *Let $I \subset [0,\infty)$, $f : I \to \mathbb{R}$ be nondecreasing and operator m-convex function. For $A, B \in K \subseteq B(H)^+$, whose spectra in I, $\frac{1}{m}\langle Ax, x\rangle, \frac{1}{m}\langle Bx, x\rangle \in \mathbb{R}$, $m \in (0,1]$*

$$\begin{aligned}
\langle f\Big(\frac{A+mB}{2}\Big)x, x\rangle \;&\leq\; \frac{1}{2}\int_0^1 \Big\langle f\Big(\big[tA + m(1-t)B\big] + m\big[f((1-t)\frac{A}{m}) + tB\big]\Big)x, x\Big\rangle \\
&\leq\; \frac{1}{2}\Big[\frac{\langle f(A)x, x\rangle + m\langle f(B)x, x\rangle}{2} + m\Big(\frac{\langle f(\frac{A}{m})x, x\rangle + \langle f(\frac{B}{m})x, x\rangle}{2}\Big)\Big]
\end{aligned}$$

inequalities hold.

Proof. For $t \in [0,1]$ and $x \in H$ with $\|x\| = 1$.

$$\langle [tA + m(1-t)B]x, x\rangle = t\langle Ax, x\rangle + m(1-t)\langle Bx, x\rangle$$

Afterwards since $\langle Ax, x\rangle \in Sp(A)$ and $\langle Bx, x\rangle \in Sp(B)$, continuity of f,

$$\int_0^1 f(tA + (1-t)B)dt$$

operator valued integral exist. Due to f is an operator m-convex, for $A, B \in K$ and $t \in [0,1]$, we can write the below inequality,

$$f(tA + m(1-t)B) \leq tf(A) + m(1-t)f(B)$$

Then,

$$\begin{aligned}\langle f(\frac{A+mB}{2})x,x\rangle &= \langle f(\frac{tA+m(1-t)B+m[(1-t)\frac{A}{m}+tB]}{2})x,x\rangle\\ &\leq \frac{1}{2}\Big[\langle f(tA+m(1-t)B)+mf[(1-t)\frac{A}{m}+tB]x,x\rangle\Big]\\ &+ \quad m(1-t)\langle f(\frac{A}{m})x,x\rangle+mt\langle f(\frac{B}{m})x,x\rangle dt\\ &= \frac{1}{2}\Big[\frac{\langle f(A)x,x\rangle}{2}+m\frac{\langle f(B)x,x\rangle}{2}+m(\frac{\langle f(\frac{A}{m})x,x\rangle}{2}+\frac{\langle f(\frac{B}{m})x,x\rangle}{2})\Big]\end{aligned}$$

So we proved the desire consequent. □

Remark 1. *If we take $m=1$ in Theorem(5), then we obtain Hermite-Hadamard Type inequality.*

Theorem 6. *Let $f,g:I\subset[0,\infty)\to\mathbb{R}$ are nondecreasing operator m_1 and m_2-convex function, respectively, $m_1,m_2\in(0,1]$. Then for any $A,B\in K\subset B(H)^+$ operators, whose spectra in I, and $x\in H$ with$||x||=1$,*

$$\int_0^1[\langle f(tA+m_1(1-t)B)x,x\rangle\langle g(tA+m_2(1-t)B)x,x\rangle]dt\leq\Big(\frac{K+m_1m_2S}{3}\Big)+\Big(\frac{m_2L+m_1R}{6}\Big)$$

inequality holds, where

$$\begin{aligned}K &= K(A)(x)=\langle f(A)x,x\rangle\langle g(A)x,x\rangle\\ L &= L(A,B)(x)=\langle f(A)x,x\rangle\langle g(B)x,x\rangle\\ R &= R(A,B)(x)=\langle f(B)x,x\rangle\langle g(A)x,x\rangle\\ S &= S(B)(x)=\langle f(B)x,x\rangle\langle g(B)x,x\rangle\end{aligned}$$

Proof. For $t\in[0,1]$ and $A,B\in K$,

$$\langle(tA+m(1-t)B)x,x\rangle=t\langle Ax,x\rangle+m(1-t)\langle Bx,x\rangle$$

then $t\langle Ax,x\rangle,m(1-t)\langle Bx,x\rangle\in I$ and continuity f,g

$$\int_0^1 f(tA+m_1(1-t)B)dt,\ \int_0^1 g(tA+m_2(1-t)B)dt,\ \int_0^1(fg)(tA+m(1-t)B)dt$$

integrals exist. Since f,g operator m_1 and m_2 convex functions, respectively, we can write the following inequalities,

$$\langle f(tA+m_1(1-t)B)x,x\rangle\leq t\langle f(A)x,x\rangle+m_1(1-t)\langle f(B)x,x\rangle$$

$$\langle g(tA+m_2(1-t)B)x,x\rangle\leq t\langle g(A)x,x\rangle+m_2(1-t)\langle g(B)x,x\rangle.$$

If we multiply side by side, we have the following inequality

$$\begin{aligned}&\langle f(tA+m_1(1-t)B)x,x\rangle\langle g(tA+m_2(1-t)B)x,x\rangle\\ &\quad\leq t^2\langle f(A)x,x\rangle\langle g(A)x,x\rangle+tm_2(1-t)\langle f(A)x,x\rangle\langle g(B)x,x\rangle\end{aligned}$$

$$+tm_1(1-t)\langle f(B)x,x\rangle\langle g(A)x,x\rangle + m_1m_2(1-t)^2\langle f(B)x,x\rangle\langle g(B)x,x\rangle.$$

If we integrate the inequality over $[0,1]$, with respect to t,

$$\int_0^1 \langle f(tA+m_1(1-t)B)x,x\rangle\langle g(tA+m_2(1-t)B)x,x\rangle dt \leq (\frac{K}{3}) + (\frac{m_2L}{6}) + (\frac{m_1R}{6}) + (\frac{m_1m_2S}{3})$$

then we obtain the proof of theorem. Here,

$$\begin{aligned} K &= K(A)(x) = \langle f(A)x,x\rangle\langle g(A)x,x\rangle \\ L &= L(A,B)(x) = \langle f(A)x,x\rangle\langle g(B)x,x\rangle \\ R &= R(A,B)(x) = \langle f(B)x,x\rangle\langle g(A)x,x\rangle \\ S &= S(B)(x) = \langle f(B)x,x\rangle\langle g(B)x,x\rangle \end{aligned}$$

□

Remark 2. *If we take $m_1, m_2 = 1$ in Theorem(6), then for $s_1, s_2 = 1$ in (1.2) we obtain the same inequality.*

Theorem 7. *$f, g : I \subset [0,\infty) \to \mathbb{R}$ are nondecreasing, operator m_1 and m_2-convex functions, respectively. If for all $A, B \in B(H)^+$ operators, which $Sp(A), Sp(B) \subset I$. $\frac{1}{m}\langle Ax,x\rangle, \frac{1}{m}\langle Bx,x\rangle \subset I$ and $m_1, m_2 \in (0,1]$ then we obtain the following inequality.*

$$\begin{aligned} &\langle f(\tfrac{A+m_1B}{2})x,x\rangle\langle g(\tfrac{A+m_2B}{2})x,x\rangle \\ &\leq \frac{1}{2}\int_0^1 \langle f(tA+m_1(1-t)B)x,x\rangle\langle g(tA+m_2(1-t)B)x,x\rangle dt \\ &\leq \frac{K}{12} + \frac{m_2L}{6} + \frac{m_1R}{6} + \frac{m_1m_2S}{12} \end{aligned}$$

where $x \in H, ||x|| = 1$.

Proof. Since f, g are nondecreasing, operator m_1 and m_2-convex functions, respectively, for $t \in [0,1]$, $x \in H$ and $\|x\| = 1$ we can easily calculate the following inequality:

$$\langle f(\frac{tA+m_1(1-t)B+m_1((1-t)\frac{A}{m_1}+tB)}{2})x,x\rangle \times\langle g(\frac{tA+m_2(1-t)B+m_2((1-t)\frac{A}{m_2}+tB)}{2})x,x\rangle$$

$$\leq \Big[\frac{1}{2}f\Big(tA+m_1(1-t)B+m_1((1-t)\frac{A}{m_1}+tB)\Big)\Big]\Big[\frac{1}{2}g\Big(tA+m_2(1-t)B +m_2((1-t)\frac{A}{m_2}+tB)\Big)\Big]$$

$$\begin{aligned} = \ & \frac{1}{4}\Big[\langle f(tA+m_1(1-t)B)x,x\rangle\langle g(tA+m_2(1-t)B)x,x\rangle\Big] & (2.1)\\ & +\frac{1}{4}\Big[\langle f(tA+m_1(1-t)B)x,x\rangle\langle g(m_2((1-t)\frac{A}{m_2}+tB))x,x\rangle\Big] & (2.2)\\ & +\frac{1}{4}\Big[\langle f(m_1((1-t)\frac{A}{m_1}+tB))x,x\rangle\langle g(m_2((1-t)\frac{A}{m_2}+tB))x,x\rangle\Big] & (2.3)\\ & +\frac{1}{4}\Big[\langle f(m_1((1-t)\frac{A}{m_1}+tB))x,x\rangle\langle g(tA+m_2(1-t)B)x,x\rangle\Big]. & (2.4)\end{aligned}$$

If we keep constant $(2.1), (2.2)$ and for $(2,3), (2,4)$ we use the operator m_1, m_2-convex function of f, g respectively, then we have below inequality

$$\langle f(\frac{tA+m_1(1-t)B+m_1((1-t)\frac{A}{m_1}+tB)}{2})x,x\rangle \times\langle g(\frac{tA+m_2(1-t)B+m_2((1-t)\frac{A}{m_2}+tB)}{2})x,x\rangle$$

$$\begin{aligned} \leq \ & \frac{1}{4}\Big[\langle f(tA+m_1(1-t)B)x,x\rangle\langle g(tA+m_2(1-t)B)x,x\rangle\Big]\\ & +\frac{1}{4}\Big[\langle f(m_1((1-t)\frac{A}{m_1}+tB))x,x\rangle\langle g(m_2((1-t)\frac{A}{m_2}+tB))x,x\rangle\Big]\\ & +\frac{1}{4}\Big[\Big(t\langle f(A)x,x\rangle+m_1(1-t)\langle f(B)x,x\rangle\Big)\Big((1-t)\langle g(A)x,x\rangle+m_2t\langle g(B)x,x\rangle\Big)\Big]\\ & +\frac{1}{4}\Big[\Big((1-t)\langle f(A)x,x\rangle+m_1t\langle f(B)x,x\rangle\Big)\Big(t\langle g(A)x,x\rangle+m_2(1-t)\langle g(B)x,x\rangle\Big)\Big].\end{aligned}$$

Finally, if we integrate above inequality over $[0,1]$ with respect to t and

$$\int_0^1 f\Big(tA+m_1(1-t)B\Big)dt=\int_0^1 f\Big((1-t)A+m_1tB\Big)dt$$

we use the equality, then we complete the proof of the theorem. □

Remark 3. *If we choose $m_1, m_2 = 1$ in Theorem(7), then for $s_1, s_2 = 1$ in (1.3) we obtain the same inequalities.*

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