

Research Article

Integral operators on Riesz-Morrey spaces

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ABSTRACT. The main result of this paper is the boundedness of integral operators on Riesz-Morrey spaces. As applications of the main result, we extend Hardy's inequalities to Riesz-Morrey spaces and establish the boundedness of the Hadamard fractional integrals on Riesz-Morrey spaces.

Keywords: Integral operator, Hardy's inequalities, Hadamard fractional integrals, Riesz-Morrey spaces, Morrey spaces.

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1. INTRODUCTION

In this paper, we study the boundedness of some integral operators on Riesz-Morrey spaces defined on $(0, \infty)$. We also extend Hardy's inequalities and the boundedness of the Hadamard fractional integrals to Riesz-Morrey spaces.

The study of Riesz-Morrey spaces is initiated by a surprising result from Riesz in [15]. This result states that for any cube the Lebesgue spaces on that cube can be characterized by so called the Riesz norm, see [18]. The studies of the Riesz-Morrey had been revitalized by another interesting result in [18] which shows that the family of Riesz-Morrey spaces is a bridge connecting Lebesgue and Morrey spaces. The Morrey spaces are introduced by Morrey in [13] to study the solutions of some quasilinear elliptic partial differential equations. For the history and developments of the Morrey spaces, see [1, 17].

The Riesz-Morrey spaces had been further investigated in [19, 20] which show that the non-triviality of the Riesz-Morrey spaces and the mapping properties of the Calderón-Zygmund operators on the Riesz-Morrey spaces.

In this paper, we aim to study the integral operators for Riesz-Morrey spaces defined on $(0, \infty)$. Notice that the results in [18, 20] are mainly Riesz-Morrey spaces defined on \mathbb{R}^n and cubes on \mathbb{R}^n while they are still valid for the Riesz-Morrey spaces defined on $(0, \infty)$.

We obtain a general result assuring the boundedness of some integral operators on Riesz-Morrey spaces. This general result is obtained by studying the boundedness of dilation operators on Riesz-Morrey spaces. This idea had been used to obtain the boundedness of some integral operators on function spaces, such as the amalgam spaces and grand Morrey spaces in [7] and [10], respectively.

As applications of this general result, we obtain the Hardy inequalities on Riesz-Morrey spaces. We also establish the boundedness of the Hadamard fractional integrals on Riesz-Morrey spaces.

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This paper is organized as follows. We give the definition of Riesz-Morrey spaces and the boundedness of dilation operators on Riesz-Morrey spaces in Section 2. The main result of the boundedness of integral operators on Riesz-Morrey spaces is established in Section 3. The Hardy's inequalities and the boundedness of the Hadamard fractional integrals on Riesz-Morrey spaces are also established in that section.

2. DEFINITIONS AND PRELIMINARIES

Let $\mathbb{I} = \{(a, b) : a, b \in (0, \infty), a < b\}$ be the set of connected open intervals on $(0, \infty)$. For any $I \in \mathbb{I}$, let $|I|$ denote the Lebesgue measure of I . For any $t \in (0, \infty)$ and $I = (a, b) \in \mathbb{I}$, write $I_t = (ta, tb)$. We have $|I_t| = t|I|$.

We now give the definition of the Riesz-Morrey spaces defined on $(0, \infty)$.

Definition 2.1. Let $\alpha \in \mathbb{R}$, $p, q \in [1, \infty]$, the Riesz-Morrey space $RM_{p,q,\alpha}$ consists of all Lebesgue measurable functions f satisfying

$$(2.1) \quad \|f\|_{RM_{p,q,\alpha}} = \sup \left(\sum_k |I_k|^{1-p\alpha-\frac{p}{q}} \|f\chi_{I_k}\|_{L^q}^p \right)^{\frac{1}{p}}, \quad p \neq \infty,$$

$$(2.2) \quad \|f\|_{RM_{\infty,q,\alpha}} = \sup_{I \in \mathbb{I}} |I|^{-\alpha-\frac{1}{q}} \|f\chi_I\|_{L^q}, \quad p = \infty,$$

where the supremum in (2.1) is taken over all families of pairwise disjoint intervals $\{I_k\}_{k \in \mathbb{N}}$.

The above definition is adapted from the Riesz-Morrey spaces defined on \mathbb{R}^n and cubes in [18].

The results obtained in [18] are for Riesz-Morrey spaces defined on \mathbb{R}^n and cubes. With some obvious modifications, they are also valid for Riesz-Morrey spaces defined on $(0, \infty)$. For instance, we see that $RM_{p,q,0} = L^p$ and $RM_{p,q,\frac{1}{p}-\frac{1}{q}} = L^q$ [18, Theorem 1]. Notice that the Riesz-Morrey spaces are non-trivial, that is, the family of the Riesz-Morrey spaces is not entirely consisting of Lebesgue spaces, see [20].

We now study the boundedness of dilation operators on the Riesz-Morrey space $RM_{p,q,\alpha}$.

Let f be a Lebesgue measurable function and $t \in (0, \infty)$. Write $D_t f(y) = f(ty)$, $y \in (0, \infty)$. The following results give the mapping properties of the dilation operator D_t on $RM_{p,q,\alpha}$.

Lemma 2.1. Let $\alpha \in \mathbb{R}$, $p, q \in [1, \infty]$.

(1) If $p \neq \infty$, then for any $t \in (0, \infty)$,

$$\|D_t f\|_{RM_{p,q,\alpha}} = t^{-\frac{1}{p}+\alpha} \|f\|_{RM_{p,q,\alpha}}.$$

(2) If $p = \infty$, then for any $t \in (0, \infty)$, we have

$$\|D_t f\|_{RM_{\infty,q,\alpha}} = t^\alpha \|f\|_{RM_{\infty,q,\alpha}}.$$

Proof. We first consider the case $p \neq \infty$. For any families of pairwise disjoint intervals $\{I_k\}_{k \in \mathbb{N}}$, by using the substitution $z = ty$, we obtain

$$\int_{I_k} |D_t f(y)|^q dy = \int_{(I_k)_t} |f(z)|^q t^{-1} dz.$$

We have $\|(D_t f)\chi_{I_k}\|_{L^q} = t^{-\frac{1}{q}}\|f\chi_{(I_k)_t}\|_{L^q}$ when $q \neq \infty$ and $\|(D_t f)\chi_k\|_{L^\infty} = \|f\chi_{(I_k)_t}\|_{L^\infty}$. Consequently,

$$\begin{aligned} |I_k|^{1-p\alpha-\frac{p}{q}}\|(D_t f)\chi_{I_k}\|_{L^q}^p &= |I_k|^{1-p\alpha-\frac{p}{q}}t^{-\frac{p}{q}}\|f\chi_{(I_k)_t}\|_{L^q}^p \\ &= |(I_k)_t|^{1-p\alpha-\frac{p}{q}}t^{-1+p\alpha+\frac{p}{q}}t^{-\frac{p}{q}}\|f\chi_{(I_k)_t}\|_{L^q}^p \\ &= t^{-1+p\alpha}|(I_k)_t|^{1-p\alpha-\frac{p}{q}}\|f\chi_{(I_k)_t}\|_{L^q}^p. \end{aligned}$$

By taking summation over $k \in \mathbb{N}$ on both sides of the above equalities, we obtain

$$\sum_k |I_k|^{1-p\alpha-\frac{p}{q}}\|(D_t f)\chi_{I_k}\|_{L^q}^p = t^{-1+p\alpha} \sum_k |(I_k)_t|^{1-p\alpha-\frac{p}{q}}\|f\chi_{(I_k)_t}\|_{L^q}^p.$$

As $\{I_k\}_{k \in \mathbb{N}}$ are pairwise disjoint, $\{(I_k)_t\}$ are also pairwise disjoint, Definition 2.1 yields

$$\left(\sum_k |I_k|^{1-p\alpha-\frac{p}{q}}\|(D_t f)\chi_{I_k}\|_{L^q} \right)^{\frac{1}{p}} \leq t^{-\frac{1}{p}+\alpha}\|f\|_{RM_{p,q,\alpha}}.$$

By taking the supremum over all families of pairwise disjoint intervals $\{I_k\}_{k \in \mathbb{N}}$ on both sides of the above inequality, we obtain

$$(2.3) \quad \|D_t f\|_{RM_{p,q,\alpha}} \leq t^{-\frac{1}{p}+\alpha}\|f\|_{RM_{p,q,\alpha}}.$$

Notice that $D_{\frac{1}{t}}(D_t f) = f$. Therefore, (2.3) gives

$$\|f\|_{RM_{p,q,\alpha}} = \|D_{\frac{1}{t}}(D_t f)\|_{RM_{p,q,\alpha}} \leq \left(\frac{1}{t}\right)^{-1+p\alpha} \|D_t f\|_{RM_{p,q,\alpha}}.$$

That is,

$$(2.4) \quad t^{-\frac{1}{p}+\alpha}\|f\|_{RM_{p,q,\alpha}} \leq \|D_t f\|_{RM_{p,q,\alpha}}.$$

Thus, (2.3) and (2.4) yield

$$\|D_t f\|_{RM_{p,q,\alpha}} = t^{-\frac{1}{p}+\alpha}\|f\|_{RM_{p,q,\alpha}}.$$

We consider $p = \infty$. For any $I \in \mathbb{I}$, we have

$$\begin{aligned} |I|^{-\alpha-\frac{1}{q}}\|(D_t f)\chi_I\|_{L^q} &= |I|^{-\alpha-\frac{1}{q}}t^{-\frac{1}{q}}\|f\chi_{I_t}\|_{L^q} \\ &= t^{\alpha-\frac{1}{q}+\frac{1}{q}}|I_t|^{-\alpha-\frac{1}{q}}\|f\chi_{I_t}\|_{L^q} \\ &\leq t^\alpha\|f\|_{RM_{\infty,q,\alpha}}. \end{aligned}$$

By taking the supremum over $I \in \mathbb{I}$, we obtain

$$\|D_t f\|_{RM_{\infty,q,\alpha}} \leq t^\alpha\|f\|_{RM_{\infty,q,\alpha}}.$$

As $D_{\frac{1}{t}}(D_t f) = f$, we have

$$\|D_t f\|_{RM_{\infty,q,\alpha}} = t^\alpha\|f\|_{RM_{\infty,q,\alpha}}.$$

□

In the next section, we apply the above result to obtain the boundedness of some integral operators on Riesz-Morrey spaces.

3. MAIN RESULTS

The boundedness of integral operators on the Riesz-Morrey space $RM_{p,q,\alpha}$ is established in this section. By using this result, we extend Hardy's inequities to the Riesz-Morrey space $RM_{p,q,\alpha}$. We also obtain the boundedness of the Hadamard fractional integrals on Riesz-Morrey space $RM_{p,q,\alpha}$.

The following is the main result of this paper. It gives criteria for the boundedness of integral operators on Riesz-Morrey spaces.

Theorem 3.1. *Let $\alpha \in \mathbb{R}$, $p, q \in [1, \infty]$ and $K : (0, \infty) \times (0, \infty) \rightarrow \mathbb{R}$ be a Lebesgue measurable function. If for any $s > 0$, K satisfies*

$$(3.5) \quad K(sx, sy) = s^{-1}K(x, y), \quad \forall x, y \in (0, \infty)$$

and

$$(3.6) \quad \int_0^\infty |K(t, 1)|t^{-\frac{1}{p}+\alpha} dt < \infty,$$

then the integral operator

$$Tf(y) = \int_0^\infty K(x, y)f(x)dx, \quad y \in (0, \infty)$$

is bounded on $RM_{p,q,\alpha}$.

Proof. Let $f \in RM_{p,q,\alpha}$. By using the substitution $x = yt$, $t > 0$, we find that

$$\int_0^\infty |K(x, y)f(x)|dx = \int_0^\infty |K(yt, y)f(yt)|ydt.$$

Thus, (3.5) guarantees that

$$\begin{aligned} \int_0^\infty |K(x, y)f(x)|dx &= \int_0^\infty |K(yt, y)f(yt)|ydt \\ &= \int_0^\infty y^{-1}|K(t, 1)D_t f(y)|ydt \\ &= \int_0^\infty |K(t, 1)D_t f(y)|dt. \end{aligned}$$

For any family of pairwise disjoint intervals $\{I_k\}_{k \in \mathbb{N}}$, we have

$$\chi_{I_k}(y) \int_0^\infty |K(x, y)f(x)|dx = \int_0^\infty |K(t, 1)D_t f(y)|\chi_{I_k}(y)dt.$$

Minkowski's inequality for integrals [4, (6.19) (a)] assures that

$$\begin{aligned} &\left(\int_0^\infty \left(\chi_{I_k}(y) \int_0^\infty |K(x, y)f(x)|dx \right)^q dy \right)^{\frac{1}{q}} \\ &= \left(\int_0^\infty \left(\int_0^\infty |K(t, 1)D_t f(y)|\chi_{I_k}(y)dt \right)^q dy \right)^{\frac{1}{q}} \\ &\leq \int_0^\infty \left(\int_0^\infty |K(t, 1)|^q |D_t f(y)|^q \chi_{I_k}(y)dy \right)^{\frac{1}{q}} dt \\ &= \int_0^\infty |K(t, 1)| \left(\int_0^\infty |D_t f(y)|^q \chi_{I_k}(y)dy \right)^{\frac{1}{q}} dt. \end{aligned}$$

Consequently, by using Minkowski's inequality in [4, (6.19) (a)] with $d\mu$ being the counting measure, we have

$$\begin{aligned} & \left(\sum_k |I_k|^{1-p\alpha-\frac{p}{q}} \left(\int_0^\infty \left(\chi_{I_k}(y) \int_0^\infty |K(x,y)f(x)|dx \right)^q dy \right)^{\frac{p}{q}} \right)^{\frac{1}{p}} \\ & \leq \left(\sum_k |I_k|^{1-p\alpha-\frac{p}{q}} \left(\int_0^\infty |K(t,1)| \left(\int_0^\infty |D_t f(y)|^q \chi_{I_k}(y) dy \right)^{\frac{1}{q}} dt \right)^p \right)^{\frac{1}{p}} \\ & \leq \int_0^\infty \left(|K(t,1)|^p \sum_k |I_k|^{1-p\alpha-\frac{p}{q}} \left(\int_0^\infty |D_t f(y)|^q \chi_{I_k}(y) dy \right)^{\frac{p}{q}} \right)^{\frac{1}{p}} dt. \end{aligned}$$

Definition 2.1 yields

$$\begin{aligned} & \left(\sum_k |I_k|^{1-p\alpha-\frac{p}{q}} \left(\int_0^\infty \left(\chi_{I_k}(y) \int_0^\infty |K(x,y)f(x)|dx \right)^q dy \right)^{\frac{p}{q}} \right)^{\frac{1}{p}} \\ & \leq \int_0^\infty |K(t,1)| \|D_t f\|_{RM_{p,q,\alpha}} dt. \end{aligned}$$

As $f \in RM_{p,q,\alpha}$, Lemma 2.1 guarantees that $D_t f \in RM_{p,q,\alpha}$ and

$$\begin{aligned} & \left(\sum_k |I_k|^{1-p\alpha-\frac{p}{q}} \left(\int_0^\infty \left(\chi_{I_k}(y) \int_0^\infty |K(x,y)f(x)|dx \right)^q dy \right)^{\frac{p}{q}} \right)^{\frac{1}{p}} \\ (3.7) \quad & \leq \int_0^\infty |K(t,1)| t^{-\frac{1}{p}+\alpha} \|f\|_{RM_{p,q,\alpha}} dt. \end{aligned}$$

Thus, (3.6) asserts that for any $y \in (0, \infty)$

$$\int_0^\infty |K(x,y)f(x)|dx$$

is well defined and $\int_0^\infty |K(x,\cdot)f(x)|dx$ belongs to $RM_{p,q,\alpha}$. Consequently, $Tf(\cdot)$ is well defined and belongs to $RM_{p,q,\alpha}$. Moreover, (3.7) assures that $T : RM_{p,q,\alpha} \rightarrow RM_{p,q,\alpha}$ is bounded. \square

The following sections give the applications of Theorem 3.1 on Hardy's inequalities and the boundedness of the Hadamard fractional integrals on Riesz-Morrey spaces.

3.1. Hardy inequality. The Hardy's inequality is one of the most important inequalities in analysis. For the history, development and applications in analysis of Hardy's inequalities, see [3, 6, 12, 14]. The Hardy's inequality is presented in terms of the boundedness of the Hardy operator. We recall the definition of the well known Hardy operator.

For any locally integrable function f on $(0, \infty)$, the Hardy operator is defined as

$$Hf(y) = \frac{1}{y} \int_0^y f(x)dx, \quad y \in (0, \infty).$$

We now extend Hardy's inequalities to Riesz-Morrey spaces.

Theorem 3.2. *Let $\alpha \in \mathbb{R}$ and $p, q \in [1, \infty]$. If $1 + \alpha > \frac{1}{p}$, then there is a constant $C > 0$ such that for any $f \in RM_{p,q,\alpha}$*

$$\|Hf\|_{RM_{p,q,\alpha}} \leq C \|f\|_{RM_{p,q,\alpha}}.$$

Proof. We see that

$$Hf(x) = \int_0^\infty K_0(x, y)f(x)dx,$$

where

$$K_0(x, y) = \frac{1}{y}\chi_{\{(x,y):0 < x < y\}}.$$

Obviously, for any $s \in (0, \infty)$

$$K_0(sx, sy) = \frac{1}{sy}\chi_{\{(x,y):0 < sx < sy\}} = s^{-1}K_0(x, y).$$

Thus, it satisfies (3.5). Since $1 + \alpha > \frac{1}{p}$, we find that

$$\int_0^\infty |K_0(t, 1)|t^{-\frac{1}{p}+\alpha} dt = \int_0^1 t^{-\frac{1}{p}+\alpha} dt = \frac{t^{-\frac{1}{p}+\alpha+1}}{-\frac{1}{p}+\alpha+1} \Big|_0^1 = \frac{1}{-\frac{1}{p}+\alpha+1}.$$

Therefore, (3.6) is also fulfilled. Theorem 3.1 yields the boundedness of $H : RM_{p,q,\alpha} \rightarrow RM_{p,q,\alpha}$. \square

3.2. Hadamard fractional integrals. The Hadamard fractional integrals were introduced by Hadamard in [5]. It provides several applications in applied analysis and inspires the studies of some new fractional integrals, see [2, 8, 9, 11, 16].

We recall the definitions of the Hadamard fractional integrals [2]. Let $\beta \in (0, \infty)$ and $\mu \in \mathbb{R}$. For any locally integrable function f , define

$$\mathcal{J}_{0+,\mu}^\beta f(y) = \frac{1}{\Gamma(\beta)} \int_0^y \left(\frac{x}{y}\right)^\mu \left(\log \frac{y}{x}\right)^{\beta-1} f(x) \frac{dx}{x}, \quad y > 0.$$

When $\mu = 0$, we see that

$$\mathcal{J}_{0+,0}^\beta f(y) = \frac{1}{\Gamma(\beta)} \int_0^y \left(\log \frac{y}{x}\right)^{\beta-1} f(x) \frac{dx}{x}, \quad y > 0$$

and $\mathcal{J}_{0+,0}^\beta$ is the Hadamard fractional integral introduced and studied by Hadamard in [5]. Thus, $\mathcal{J}_{0+,\mu}^\beta$ is a generalization of the Hadamard fractional integral. The reader is referred to [2, 16] for the studies of the Hadamard fractional integrals.

We now use Theorem 3.1 to establish the boundedness of the Hadamard fractional integrals on Riesz-Morrey spaces.

Theorem 3.3. *Let $\alpha \in \mathbb{R}$, $p, q \in [1, \infty]$, $\beta \in (0, \infty)$ and $\mu \in \mathbb{R}$. If $\beta > 0$ and $\mu - \frac{1}{p} + \alpha > 0$, then there is a constant $C > 0$ such that for any $f \in RM_{p,q,\alpha}$*

$$\|\mathcal{J}_{0+,\mu}^\beta f\|_{RM_{p,q,\alpha}} \leq C \|f\|_{RM_{p,q,\alpha}}.$$

Proof. Write

$$K_1(x, y) = \chi_{\{(x,y):0 < x < y\}} \frac{1}{\Gamma(\beta)} \left(\frac{x}{y}\right)^\mu \left(\log \frac{y}{x}\right)^{\beta-1} \frac{1}{x}.$$

Thus,

$$\mathcal{J}_{0+,\mu}^\beta f(y) = \int_0^\infty K_1(x, y)f(x)dx.$$

For any $s > 0$, we have

$$\begin{aligned} K_1(sx, sy) &= \chi_{\{(x,y):0 < sx < sy\}} \frac{1}{\Gamma(\beta)} \left(\frac{sx}{sy}\right)^\mu \left(\log \frac{sy}{sx}\right)^{\beta-1} \frac{1}{sx} \\ &= \frac{1}{s} \chi_{\{(x,y):0 < x < y\}} \frac{1}{\Gamma(\beta)} \left(\frac{x}{y}\right)^\mu \left(\log \frac{y}{x}\right)^{\beta-1} \frac{1}{x} = s^{-1} K_1(x, y). \end{aligned}$$

Thus, K_1 satisfies (3.5).

Moreover, by using $t = e^{-u}$, we find that

$$\begin{aligned} &\int_0^\infty |K_1(t, 1)| t^{-\frac{1}{p} + \alpha} dt \\ &= \frac{1}{\Gamma(\beta)} \int_0^1 t^\mu \left(\log \frac{1}{t}\right)^{\beta-1} t^{-\frac{1}{p} + \alpha} \frac{dt}{t} \\ &= \frac{1}{\Gamma(\beta)} \int_0^\infty e^{-(\mu - \frac{1}{p} + \alpha)u} u^{\beta-1} du \\ (3.8) \quad &= \frac{1}{\Gamma(\beta)} \int_0^1 e^{-(\mu - \frac{1}{p} + \alpha)u} u^{\beta-1} du + \frac{1}{\Gamma(\beta)} \int_1^\infty e^{-(\mu - \frac{1}{p} + \alpha)u} u^{\beta-1} du. \end{aligned}$$

As $\beta > 0$, we have

$$(3.9) \quad \frac{1}{\Gamma(\beta)} \int_0^1 e^{-(\mu - \frac{1}{p} + \alpha)u} u^{\beta-1} du \leq C \int_0^1 u^{\beta-1} = C \frac{u^\beta}{\beta} \Big|_0^1 < C$$

for some $C > 0$.

Since $\mu - \frac{1}{p} + \alpha > 0$, there exist $\epsilon \in (0, \mu - \frac{1}{p} + \alpha)$ and constant $C > 0$ such that $u^{\beta-1} \leq C e^\epsilon$. We get

$$(3.10) \quad \frac{1}{\Gamma(\beta)} \int_1^\infty e^{-(\mu - \frac{1}{p} + \alpha)u} u^{\beta-1} du \leq C \int_1^\infty e^{-(\mu - \frac{1}{p} + \alpha - \epsilon)u} du < C$$

for some $C > 0$ because $\mu - \frac{1}{p} + \alpha - \epsilon > 0$. Therefore, (3.8), (3.9) and (3.10) assure that K_1 fulfills (3.6). Theorem 3.1 yields the boundedness of $\mathcal{J}_{0+, \mu}^\beta : RM_{p, q, \alpha} \rightarrow RM_{p, q, \alpha}$. \square

In particular, if $\beta > 0$ and $-\frac{1}{p} + \alpha > 0$, the Hadamard fractional integral $\mathcal{J}_{0+, 0}^\beta$ is bounded on the Riesz-Morrey space $RM_{p, q, \alpha}$.

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