Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 48 (6) (2019), 1695 – 1705
DOI : 10.15672/HJMS.2018.632

Research Article

On third Hankel determinants for subclasses of analytic functions and close-to-convex harmonic mappings

Yong Sun*[1], Zhi-Gang Wang[2], Antti Rasila[3]

[1] *School of Science, Hunan Institute of Engineering, Xiangtan, 411104, Hunan, People's Republic of China*
[2] *School of Mathematics and Computing Science, Hunan First Normal University, Changsha, 410205, Hunan, People's Republic of China*
[3] *Department of Mathematics and Systems Analysis, Aalto University, P. O. Box 11100, FI-00076 Aalto, Finland*

Abstract

In this paper, we obtain the upper bounds to the third Hankel determinants for convex functions of order α and bounded turning functions of order α. Furthermore, several relevant results on a new subclass of close-to-convex harmonic mappings are obtained. Connections of the results presented here to those that can be found in the literature are also discussed.

Mathematics Subject Classification (2010). 30C45, 30C50, 58E20, 30C80

Keywords. Univalent function, starlike function, convex function, bounded turning function, close-to-convex function, harmonic mapping, Hankel determinant

1. Introduction

Let $\mathcal{A}$ be the class of functions *analytic* in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of univalent functions.

A function $f \in \mathcal{A}$ is said to be starlike of order α $(0 \leq \alpha < 1)$, if it satisfies the following condition:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{D}).$$

We denote by $\mathcal{S}^*(\alpha)$ the class of starlike functions of order α.

Denote by $\mathcal{K}(\alpha)$ the class of functions $f \in \mathcal{A}$ such that

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (-1/2 \leq \alpha < 1;\ z \in \mathbb{D}).$$

*Corresponding Author.
Email addresses: yongsun2008@foxmail.com (Y. Sun), wangmath@163.com (Z. Wang), antti.rasila@iki.fi (A. Rasila)
Received: 13.02.2018; Accepted: 21.06.2018

In particular, functions in $\mathcal{K}(-1/2)$ are known to be close-to-convex but are not necessarily starlike in $\mathbb{D}$. For $0 \le \alpha < 1$, functions in $\mathcal{K}(\alpha)$ are known to be convex of order α in $\mathbb{D}$.

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}(\alpha)$, consisting of functions whose derivative have a positive real part of α $(0 \le \alpha < 1)$, if it satisfies the following condition:

$$\Re(f'(z)) > \alpha \quad (z \in \mathbb{D}).$$

Choosing $\alpha = 0$, we denote the $\mathcal{S} := \mathcal{S}^*(0)$, $\mathcal{K} := \mathcal{K}(0)$ and $\mathcal{R} := \mathcal{R}(0)$, the classes of starlike, convex and bounded turning functions, respectively.

Let $\mathcal{H}$ denote the class of all *complex-valued harmonic mappings* f in $\mathbb{D}$ normalized by the condition $f(0) = f_z(0) - 1 = 0$. It is well-known that such functions can be written as $f = h + \overline{g}$, where h and g are analytic functions in $\mathbb{D}$. We call h the analytic part and g the co-analytic part of f, respectively. Let $\mathcal{S}_H$ be the subclass of $\mathcal{H}$ consisting of univalent and sense-preserving mappings. Such mappings can be written in the form

$$f(z) = h(z) + \overline{g(z)} = z + \sum_{k=2}^{\infty} a_k z^k + \sum_{k=1}^{\infty} \overline{b_k z^k} \quad (|b_1| < 1;\ z \in \mathbb{D}). \tag{1.2}$$

Harmonic mapping f is called locally univalent and sense-preserving in $\mathbb{D}$ if and only if $|h'(z)| > |g'(z)|$ holds for $z \in \mathbb{D}$. Observe that $\mathcal{S}_H$ reduces to $\mathcal{S}$, the class of normalized univalent analytic functions, if the co-analytic part g vanishes. The family of all functions $f \in \mathcal{S}_H$ with the additional property that $f_{\overline{z}}(0) = 0$ is denoted by $\mathcal{S}_H^0$. For further information about planar harmonic mappings, see e.g. [10, 13, 33].

Recall that a function $f \in \mathcal{H}$ is close-to-convex in $\mathbb{D}$ if it is univalent and the range $f(\mathbb{D})$ is a close-to-convex domain, i.e., the complement of $f(\mathbb{D})$ can be written as the union of nonintersecting half-lines. A normalized analytic function f in $\mathbb{D}$ is close-to-convex in $\mathbb{D}$ if there exists a convex analytic function in $\mathbb{D}$, not necessarily normalized, ϕ such that $\Re(f'(z)/\phi'(z)) > 0$. In particular, if $\phi(z) = z$, then for any $f \in \mathcal{A}$, $\Re(f'(z)) > 0$ implies f is close-to-convex in $\mathbb{D}$, see [37]. We refer to [6, 20, 29, 34, 35] for discussion and basic results on close-to-convex harmonic mappings.

For a harmonic mapping $f = h + \overline{g}$ in $\mathbb{D}$, a basic result in [28] (see also [27]) shows that if at least one of the analytic functions h and g is convex, then f is univalent whenever it is locally univalent in $\mathbb{D}$. It is natural to study the univalence of $f = h + \overline{g}$ in $\mathbb{D}$ if it is locally univalent and sense-preserving, and analytic function h is univalent and close-to-convex. Motivated by this idea, we next consider the following subclass of $\mathcal{H}$.

Definition 1.1. For $\alpha \in \mathbb{R}$ with $-1/2 \le \alpha < 1$, let $\mathcal{M}(\alpha)$ denote the class of harmonic mapping $f = h + \overline{g}$ in $\mathbb{D}$ of the form (1.2), with $h'(0) \ne 0$, which satisfy

$$\Re\left(1 + \frac{zh''(z)}{h'(z)}\right) > \alpha \quad \text{and} \quad g'(z) = zh'(z) \quad (z \in \mathbb{D}).$$

By making use of the similar arguments to those in the proof of [7, Theorem 1], one can easily obtain the close-to-convexity of the class $\mathcal{M}(\alpha)$. For special values of α, many authors have studied the class of close-to-convex harmonic mappings, see e.g. [5, 9, 28, 29, 38].

Pommerenke (see [31, 32]) defined the Hankel determinant $H_{q,n}(f)$ as

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix} \quad (q, n \in \mathbb{N}).$$

Problems involving Hankel determinants $H_{q,n}(f)$ in geometric function theory originate from the work of, e.g., Hadamard, Polya and Edrei (see [11, 14]), who used them in study of singularities of meromorphic functions. For example, they can be used in showing that a function of bounded characteristic in $\mathbb{D}$, i.e., a function which is a ratio of two bounded analytic functions with its Laurent series around the origin having integral coefficients, is rational [8]. Pommerenke [31] proved that the Hankel determinants of univalent functions satisfy the inequality $|H_{q,n}(f)| < Kn^{-(\frac{1}{2}+\beta)q+\frac{3}{2}}$, where $\beta > 1/4000$ and K depends only on q. Furthermore, Hayman [17] has proved a stronger result for areally mean univalent functions, i.e., the estimate $H_{2,n}(f) < An^{1/2}$, where A is an absolute constant.

We note that $H_{2,1}(f)$ is the well-known *Fekete-Szegő functional*, see [15, 21, 22]. The sharp upper bounds on $H_{2,2}(f)$ were obtained by the authors of articles [3, 18, 19, 23] for various classes of functions.

By the definition, $H_{3,1}(f)$ is given by

$$H_{3,1}(f) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}.$$

Note that for $f \in \mathcal{A}$, $a_1 = 1$ so that

$$H_{3,1}(f) = -a_2^2 a_5 + 2a_2 a_3 a_4 - a_3^3 + a_3 a_5 - a_4^2.$$

Obviously, the case of the upper bounds on $H_{3,1}(f)$ is much more difficult than the cases of $H_{2,1}(f)$ and $H_{2,2}(f)$. In 2010, Babalola [2] has studied the max $|H_{3,1}(f)|$ for the classes of convex and bounded turning functions.

Theorem 1.2. *Let $h \in \mathcal{K}$ and $g \in \mathcal{R}$, respectively. Then*

$$|H_{3,1}(h)| \leq \frac{32 + 33\sqrt{3}}{72\sqrt{3}} \approx 0.714 \qquad \text{and} \qquad |H_{3,1}(g)| \leq \frac{2736\sqrt{3} + 675\sqrt{5}}{4860\sqrt{3}} \approx 0.742.$$

In 2017, Zaprawa [40] proved that

Theorem 1.3. *Let $h \in \mathcal{K}$ and $g \in \mathcal{R}$, respectively. Then*

$$|H_{3,1}(h)| \leq \frac{49}{540} \approx 0.0907, \qquad \text{and} \qquad |H_{3,1}(g)| \leq \frac{41}{60} \approx 0.683.$$

Recently, Orhan and Zaprawa [30] proved that

Theorem 1.4. *Let $h \in \mathcal{K}(\alpha)$. Then*

$$|H_{3,1}(h)| \leq \begin{cases} \frac{1}{540}(1-\alpha)^2(49 - 102\alpha + 40\alpha^2 - 8\alpha^3), & -1/2 \leq \alpha \leq 0, \\ \frac{1}{540}(1-\alpha)^2(49 - 16\alpha), & 0 \leq \alpha < 1. \end{cases}$$

Raza and Malik [36] have obtained the upper bound on $|H_{3,1}(f)|$ for a class of analytic functions that is related to the lemniscate of Bernoulli. Also, Bansal *et al.* [4] obtained the following results.

Theorem 1.5. *Let $h \in \mathcal{K}(-1/2)$ and $g \in \mathcal{R}$, respectively. Then*

$$|H_{3,1}(h)| \leq \frac{180 + 69\sqrt{15}}{32\sqrt{15}} \approx 3.609, \qquad |H_{3,1}(g)| \leq \frac{439}{540} \approx 0.813.$$

For the class $\mathcal{R}(\alpha)$, Vamshee Krishna *et al.* [39] proved that

Theorem 1.6. *Let $g \in \mathcal{R}(\alpha)$ with $\alpha \in [0, 1/4]$. Then*

$$|H_{3,1}(g)| \leq \frac{(1-\alpha)^2}{3}\left[\frac{8(1-\alpha)}{9} + \frac{1}{4}\left(\frac{5-4\alpha}{3}\right)^{\frac{3}{2}} + \frac{4}{5}\right].$$

In the present investigation, our goal is to discuss the upper bounds to the third Hankel determinants for the subclasses of univalent functions: $\mathcal{K}(\alpha)$ and $\mathcal{R}(\alpha)$. Furthermore, we develop similar results on the Hankel determinants $|H_{3,1}(h)|$ and $|H_{3,1}(g)|$ in the context of the close-to-convex harmonic mappings $f = h + \overline{g} \in \mathcal{M}(\alpha)$.

2. Preliminary results

Denote by $\mathcal{P}$ the class of *Carathéodory functions* p normalized by

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \quad \text{and} \quad \Re(p(z)) > 0 \quad (z \in \mathbb{D}). \tag{2.1}$$

Following results are the well known for functions belonging to the class $\mathcal{P}$.

Lemma 2.1 ([12]). *If $p \in \mathcal{P}$ is of the form (2.1), then*

$$|p_n| \leq 2 \qquad (n \in \mathbb{N}). \tag{2.2}$$

The inequality (2.2) is sharp and the equality holds for the function

$$\phi(z) = \frac{1+z}{1-z} = 1 + 2\sum_{n=1}^{\infty} z^n.$$

Lemma 2.2 ([26]). *If $p \in \mathcal{P}$ is of the form (2.1), then holds the sharp estimate*

$$|p_n - p_k p_{n-k}| \leq 2 \qquad (n,\ k \in \mathbb{N},\ n > k). \tag{2.3}$$

Lemma 2.3 ([16]). *If $p \in \mathcal{P}$ is of the form (2.1), then holds the sharp estimate*

$$|p_n - \mu p_k p_{n-k}| \leq 2 \qquad (n,\ k \in \mathbb{N},\ n > k;\ 0 \leq \mu \leq 1). \tag{2.4}$$

Lemma 2.4 ([24,25]). *If $p \in \mathcal{P}$ is of the form (2.1), then there exist x, z such that $|x| \leq 1$ and $|z| \leq 1$,*

$$2p_2 = p_1^2 + (4 - p_1^2)x, \tag{2.5}$$

and

$$4p_3 = p_1^3 + 2p_1(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z. \tag{2.6}$$

3. Bounds of Hankel determinants for analytic functions

In this section, we assume that

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{K}(\alpha) \qquad \text{and} \qquad g(z) = z + \sum_{k=2}^{\infty} c_k z^k \in \mathcal{R}(\alpha).$$

Theorem 3.1. *Let $g \in \mathcal{R}(\alpha)$ with $0 \leq \alpha < 1$. Then*

$$\left|H_{3,1}(g)\right| \leq \frac{1}{60}(1-\alpha)^2(36 - 20\alpha + 5|1 - 4\alpha|). \tag{3.1}$$

Proof. Let $g \in \mathcal{R}(\alpha)$ and

$$p(z) = \frac{1}{1-\alpha}\left(g'(z) - \alpha\right) = 1 + \sum_{k=1}^{\infty} p_k z^k \in \mathcal{P} \quad (0 \leq \alpha < 1;\ z \in \mathbb{D}).$$

then

$$(k+1)c_{k+1} = (1-\alpha)p_k \qquad (k \in \mathbb{N}). \tag{3.2}$$

Putting it into the definition of $H_{3,1}(g)$, we have

$$\begin{aligned}
H_{3,1}(g) &= \frac{1}{2160}(1-\alpha)^2\Big\{(1-\alpha)\big[-108p_1^2p_4 + 180p_1p_2p_3 - 80p_2^3\big] + 144p_2p_4 - 135p_3^2\Big\} \\
&= \frac{1}{2160}(1-\alpha)^2\Big\{108(1-\alpha)p_4(p_2 - p_1^2) + 80(1-\alpha)p_2(p_4 - p_2^2) \\
&\quad - 135p_3(p_3 - p_1p_2) - 45(1-4\alpha)p_2(p_4 - p_1p_3) + (1+8\alpha)p_2p_4\Big\}.
\end{aligned}$$

By using Lemma 2.1 and Lemma 2.3 and triangle inequality, we obtain the estimate (3.1) of $H_{3,1}(g)$. This completes the proof.

□

Remark 3.2. By setting $\alpha = 0$ and $\alpha = 1/4$ in Theorem 3.1, respectively, the bounds of $H_{3,1}(g)$ in (3.1) improved the results of the Theorem 1.5 and Theorem 1.6.

In 1960, Lawrence Zalcman posed a conjecture that the coefficients of $\mathcal{S}$ satisfy the sharp inequality

$$|a_n^2 - a_{2n-1}| \leq (n-1)^2 \qquad (n \in \mathbb{N}),$$

with equality only for the Koebe function $k(z) = z/(1-z)^2$ and its rotations. We call $J_n(f) = a_n^2 - a_{2n-1}$ the Zalcman functional for $f \in \mathcal{S}$.

We observe that $H_{3,1}(f)$ $(f \in \mathcal{A})$ can be written in the form

$$H_{3,1}(f) = a_3(a_2a_4 - a_3^2) + a_4(a_2a_3 - a_4) - a_5J_2(f),$$

and equivalently,

$$H_{3,1}(f) = a_3J_3(f) + a_4(2a_2a_3 - a_4) - a_5a_2^2.$$

An analogous calculation can be applied to the Zalcman functional $J_n(f)$ for the classes of starlike, convex and bounded turning functions of order α.

Theorem 3.3. *The following estimates hold for analytic functions:*

(1) *If* $f \in \mathcal{S}^*(\alpha)$ $(0 \leq \alpha < 1)$, *then* $|J_3(f)| \leq \frac{1}{2}(1-\alpha)(8-7\alpha)$.
(2) *If* $h \in \mathcal{K}(\alpha)$ $(-1/2 \leq \alpha < 1)$, *then* $|J_3(h)| \leq \frac{1}{360}(1-\alpha)(127-109\alpha)$.
(3) *If* $g \in \mathcal{R}(\alpha)$ $(0 \leq \alpha < 1)$, *then* $|J_n(g)| \leq \frac{2}{2n-1}(1-\alpha) \qquad (n \geq 2)$.

Proof. Let $h \in \mathcal{K}(\alpha)$ and

$$p(z) = \frac{1}{1-\alpha}\left(1 + \frac{zh''(z)}{h'(z)} - \alpha\right) \quad (-1/2 \leq \alpha < 1;\ z \in \mathbb{D}),$$

then, we have

$$p(z) = 1 + p_1z + p_2z^2 + \dots \qquad \text{and} \qquad \Re(p(z)) > 0 \qquad (z \in \mathbb{D}).$$

By elementary calculations, we obtain

$$n(n-1)a_n = (1-\alpha)\sum_{k=1}^{n-1} ka_kp_{n-k} \qquad (n \geq 2). \tag{3.3}$$

It follows from (3.3) that

$$\begin{cases} a_2 = \frac{1}{2}(1-\alpha)p_1, \\ a_3 = \frac{1}{6}(1-\alpha)\big[(1-\alpha)p_1^2 + p_2\big], \\ a_4 = \frac{1}{24}(1-\alpha)\big[(1-\alpha)^2p_1^3 + 3(1-\alpha)p_1p_2 + 2p_3\big], \\ a_5 = \frac{1}{120}(1-\alpha)\big[(1-\alpha)^3p_1^4 + 6(1-\alpha)^2p_1^2p_2 + 8(1-\alpha)p_1p_3 + 3(1-\alpha)p_2^2 + 6p_4\big]. \end{cases} \tag{3.4}$$

From (3.4), we have

$$\begin{aligned} J_3(h) &= \frac{1}{360}(1-\alpha)\Big\{-7(1-\alpha)^3p_1^4 - 2(1-\alpha)^2p_1^2p_2 - (1-\alpha)p_2^2 + 24(1-\alpha)p_1p_3 + 18p_4\Big\} \\ &= \frac{1}{360}(1-\alpha)\Big\{-\frac{63}{4}(1-\alpha)\big[p_2 - \frac{2}{3}(1-\alpha)p_1^2\big]^2 + 24(1-\alpha)p_1\big[p_3 - \frac{2}{3}(1-\alpha)p_1p_2\big] \\ &\quad + \frac{21}{2}(1-\alpha)p_2\big[p_2 - \frac{2}{3}(1-\alpha)p_1^2\big] + \frac{17}{4}(1-\alpha)p_2^2 + 18p_4\Big\}. \end{aligned}$$

By using Lemma 2.1 and Lemma 2.3, we obtain the bound for the Zalcman functional $J_3(h)$.

For $f \in \mathcal{S}^*(\alpha)$, combining the Alexander relation $b_k(f) = ka_k(h)$ $(k \in \mathbb{N})$ and (3.4), yields

$$\begin{aligned} J_3(f) &= \frac{1}{24}(1-\alpha)\Big\{-5(1-\alpha)^3p_1^4 - 6(1-\alpha)^2p_1^2p_2 - 3(1-\alpha)p_2^2 + 8(1-\alpha)p_1p_3 + 6p_4\Big\} \\ &= \frac{1}{24}(1-\alpha)\Big\{-5(1-\alpha)\big[p_2-(1-\alpha)p_1^2\big]^2 + 8(1-\alpha)p_1\big[p_3-(1-\alpha)p_1p_2\big] \\ &\quad + 8(1-\alpha)p_2\big[p_2-(1-\alpha)p_1^2\big] + 6\big[p_4-(1-\alpha)p_2^2\big]\Big\}. \end{aligned}$$

Again, by using Lemma 2.1 and Lemma 2.3, we obtain the bound for the Zalcman functional $J_3(f)$.

For $g \in \mathcal{R}(\alpha)$, according to the formula (3.2), we have

$$\begin{aligned} J_n(g) &= \frac{1}{n^2}(1-\alpha)^2p_{n-1}^2 - \frac{1}{2n-1}(1-\alpha)p_{2n-2} \\ &= -\frac{1}{2n-1}(1-\alpha)\Big[p_{2n-2} - \frac{2n-1}{n^2}(1-\alpha)p_{n-1}^2\Big]. \end{aligned}$$

In view of

$$0 < \frac{2n-1}{n^2}(1-\alpha) < 1 \qquad (0 \le \alpha < 1;\ n \ge 2),$$

and, by Lemma 2.3, we have the desired bound of the Zalcman functional $J_n(g)$. This completes the proof.

□

Remark 3.4. By setting $\alpha = -1/2$ for the class $\mathcal{K}(\alpha)$ in Theorem 3.3, we obtain the known results [1, Theorem 2.3]. Furthermore, using the similar argument in Theorem 3.3, we may obtain the bounds of the Zalcman functional $J_2(f)$ and $J_2(h)$: If $f \in \mathcal{S}^*(\alpha)$ $(0 \le \alpha < 1)$, then $J_2(f) \le 1-\alpha$. If $h \in \mathcal{K}(\alpha)$ $(-1/2 \le \alpha < 1)$, then $J_2(h) \le \frac{1}{3}(1-\alpha)$.

4. Bounds of Hankel determinants for $\mathcal{M}(\alpha)$

In this section, we obtain upper bounds for the Hankel determinants $|H_{3,1}(h)|$ and $|H_{3,1}(g)|$ of close-to-convex harmonic mappings $f = h + \overline{g} \in \mathcal{M}(\alpha)$.

Theorem 4.1. *Let $f = h + \overline{g} \in \mathcal{M}(\alpha)$ be of the form* (1.2). *Then*

$$|H_{3,1}(h)| \le \frac{1}{540}(1-\alpha)^2(37-4\alpha), \qquad (-1/2 \le \alpha < 1) \tag{4.1}$$

and

$$|H_{3,1}(g)| \le \begin{cases} \frac{1}{30}(1-\alpha), & -1/2 \le \alpha \le 0, \\ \frac{1}{30}(1-\alpha)(1+2\alpha), & 0 < \alpha < 1. \end{cases}$$

Proof. Let $f = h + \overline{g} \in \mathcal{M}(\alpha)$. By using the above values of a_2, a_3, a_4 and a_5 from (3.4), and by a routine computation, we obtain

$$\begin{aligned} H_{3,1}(h) = \frac{1}{8640}(1-\alpha)^2\Big\{&-(1-\alpha)^4p_1^6 + 6(1-\alpha)^3p_1^4p_2 + 12(1-\alpha)^2p_1^3p_3 - 21(1-\alpha)^2p_1^2p_2^2 \\ &- 36(1-\alpha)p_1^2p_4 + 36(1-\alpha)p_1p_2p_3 - 4(1-\alpha)p_2^3 + 72p_2p_4 - 60p_3^2\Big\}. \end{aligned} \tag{4.2}$$

From (4.2), we give the decomposition for functional $H_{3,1}(h)$ as follows

$$\begin{aligned}H_{3,1}(h) = \frac{1}{8640}(1-\alpha)^2\Big\{&8(1-\alpha)\big[p_2-\frac{1}{2}(1-\alpha)p_1^2\big]^3-60\big[p_3-\frac{1}{2}(1-\alpha)p_1p_2\big]^2\\&+48\big[p_2-\frac{1}{2}(1-\alpha)p_1^2\big]\big[p_4-\frac{1}{2}(1-\alpha)p_1p_3\big]\\&+24\big[p_2-\frac{1}{2}(1-\alpha)p_1^2\big]\big[p_4-\frac{1}{2}(1-\alpha)p_2^2\big]\Big\}.\end{aligned}$$

We note that

$$0 \le \frac{1}{2}(1-\alpha) \le 1 \quad \text{for} \quad -\frac{1}{2} \le \alpha < 1,$$

by triangle inequality and Lemmas 2.1-2.3, we can obtain the estimate of $H_{3,1}(h)$.

By the power series representations of h and g for $f = h + \overline{g} \in \mathcal{M}(\alpha)$, we see that

$$b_1 = 0, \qquad (k+1)b_{k+1} = ka_k \quad \text{for} \quad k \ge 1,$$

which yields

$$\begin{cases} b_2 &= \frac{1}{2}a_1 = \frac{1}{2}, \\ b_3 &= \frac{2}{3}a_2 = \frac{1}{3}(1-\alpha)p_1, \\ b_4 &= \frac{3}{4}a_3 = \frac{1}{8}(1-\alpha)\big[(1-\alpha)p_1^2+p_2\big], \\ b_5 &= \frac{4}{5}a_4 = \frac{1}{30}(1-\alpha)\big[(1-\alpha)^2p_1^3+3(1-\alpha)p_1p_2+2p_3\big]. \end{cases}$$

Then, by using (2.5) and (2.6) in Lemma 2.4, we obtain that for some x and z such that $|x| \le 1$ and $|z| \le 1$,

$$\begin{aligned}H_{3,1}(g) &= 2b_2b_3b_4 - b_3^3 - b_2^2b_5 = b_3b_4 - b_3^3 - \frac{1}{4}b_5\\&= \frac{1}{2160}(1-\alpha)\Big\{\big(-8\alpha^2-2\alpha+1\big)p_1^3+9(4-p_1^2)\big[p_1(x^2-2\alpha x)-2(1-|x|^2)z\big]\Big\}.\end{aligned}$$

By Lemma 2.1, we may assume that $|p_1| = c \in [0,2]$. By applying the triangle inequality in above relation with $\mu = |x|$, we obtain

$$\big|H_{3,1}(g)\big| \le \frac{1}{2160}(1-\alpha)\Big\{|8\alpha^2+2\alpha-1|c^3+9(4-c^2)\big[(c-2)\mu^2+2\alpha c\mu+2\big]\Big\} =: Q(c,\mu).$$

Let

$$\varphi(\mu) = (c-2)\mu^2+2\alpha c\mu+2, \qquad (0 \le c \le 2,\ 0 \le \mu \le 1).$$

If $\alpha \in [-1/2, 0]$ and $c \in [0,2]$, then $\varphi(\mu)$ is a non-increasing function, so $\varphi(\mu) \le \varphi(0) = 2$. If $\alpha \in (0,1)$ and $c \in [0,2]$, $\mu \in [0,1]$, then it is clear that $2\alpha(2-c\mu)+(2-c)\mu^2 \ge 0$. Consequently,

$$(c-2)\mu^2+2\alpha c\mu+2 \le 4\alpha+2.$$

Thus, we have

$$\varphi(\mu) \le T(\alpha) := \begin{cases} 2, & -1/2 \le \alpha \le 0, \\ 4\alpha+2, & 0 < \alpha < 1. \end{cases}$$

Furthermore, we have

$$\big|H_{3,1}(g)\big| \le Q(c,\mu) \le \frac{1}{2160}(1-\alpha)\Big\{|8\alpha^2+2\alpha-1|c^3+9(4-c^2)T(\alpha)\Big\}.$$

Let

$$\chi(c) = |8\alpha^2+2\alpha-1|c^3+9(4-c^2)T(\alpha), \qquad (0 \le c \le 2).$$

If $\alpha \in [-1/2, 0]$, then

$$\chi(c) = |8\alpha^2+2\alpha-1|c^3-18c^2+72 \qquad (0 \le c \le 2).$$

We note that

$$|8\alpha^2 + 2\alpha - 1| = (1 + 2\alpha)(1 - 4\alpha) \in [0,\ 9/8], \qquad (-1/2 \le \alpha \le 0).$$

There are critical points of $\chi(c)$: 0 and $c_1 = 12/(1 - 2\alpha - 8\alpha^2)$ which is greater than or equal to 32/3. Consequently, $\chi(c)$ is decreasing for $c \in [0, 2]$, so $\chi(c) \le \chi(0) = 72$. Thus, we obtain the following bound

$$|H_{3,1}(g)| \le \frac{1}{30}(1 - \alpha), \qquad (-1/2 \le \alpha \le 0).$$

If $\alpha \in (0, 1)$, then

$$\chi(c) = |8\alpha^2 + 2\alpha - 1|c^3 - 18(1 + 2\alpha)c^2 + 72(1 + 2\alpha) \qquad (0 \le c \le 2).$$

We note that

$$|8\alpha^2 + 2\alpha - 1| = (1 + 2\alpha) \cdot |1 - 4\alpha| \in [0,\ 9], \qquad (0 < \alpha < 1).$$

There are critical points of $\chi(c)$: 0 and $c_2 = 12/|1 - 4\alpha|$ which is greater than 4. Consequently, for $\alpha \in (0, 1)$ and $c \in [0, 2]$, we get

$$\chi(c) \le \max\left\{\chi(0),\ \chi(2)\right\} = \max\left\{72(1 + 2\alpha),\ 8|8\alpha^2 + 2\alpha - 1|\right\} = 72(1 + 2\alpha).$$

Thus, we obtain the following bound

$$|H_{3,1}(g)| \le \frac{1}{30}(1 - \alpha)(1 + 2\alpha), \qquad (0 < \alpha < 1).$$

This completes the proof. □

Remark 4.2. By setting $\alpha = 0$ and $\alpha = -1/2$ in Theorem 4.1, respectively, we have

$$|H_{3,1}(h)|_{\alpha=0} \le \frac{37}{540} \approx 0.0685, \qquad |H_{3,1}(h)|_{\alpha=-1/2} \le \frac{13}{80} = 0.1625,$$

and they are much better than Theorem 1.2, Theorem 1.3 and Theorem 1.5.

Furthermore, we note that

$$37 - 4\alpha \le 49 - 102\alpha + 40\alpha^2 - 8\alpha^3 \qquad \text{for} \qquad -1/2 \le \alpha \le 0,$$

and

$$37 - 4\alpha \le 49 - 16\alpha \qquad \text{for} \qquad 0 \le \alpha < 1,$$

the bounds of $H_{3,1}(h)$ in (4.1) improved the result of the Theorem 1.4.

Remark 4.3. For $H_{3,1}(g)$ in Theorem 4.1, if we apply the method in Theorem 3.1, then

$$\begin{aligned} H_{3,1}(g) &= 2b_2b_3b_4 - b_3^3 - b_2^2b_5 = b_3b_4 - b_3^3 - \frac{1}{4}b_5 \\ &= \frac{1}{540}(1 - \alpha)\Big\{ - 2(1 - \alpha)^2p_1^3 - 9\big[p_3 - (1 - \alpha)p_1p_2\big]\Big\} \\ &= \frac{1}{540}(1 - \alpha)\Big\{3(1 - \alpha)p_1\big[p_2 - \frac{2}{3}(1 - \alpha)p_1^2\big] - 9\big[p_3 - \frac{2}{3}(1 - \alpha)p_1p_2\big]\Big\}. \end{aligned}$$

By using Lemmas 2.1 and 2.3, we have

$$|H_{3,1}(g)| \le \frac{1}{90}(1 - \alpha)(5 - 2\alpha).$$

Obviously,

$$\frac{1}{90}(1 - \alpha)(5 - 2\alpha) > \frac{1}{30}(1 - \alpha) \quad \text{for} \quad -1/2 \le \alpha \le 0,$$

$$\frac{1}{90}(1 - \alpha)(5 - 2\alpha) \ge \frac{1}{30}(1 - \alpha)(2\alpha + 1) \quad \text{for} \quad 0 < \alpha \le 1/4,$$

and

$$\frac{1}{90}(1 - \alpha)(5 - 2\alpha) < \frac{1}{30}(1 - \alpha)(2\alpha + 1) \quad \text{for} \quad 1/4 < \alpha < 1.$$

Hence, we can get the better upper bounds for $H_{3,1}(g)$ in Corollary 4.4.

Corollary 4.4. *Let* $f = h + \overline{g} \in \mathcal{M}(\alpha)$ *be of the form* (1.2). *Then*

$$|H_{3,1}(g)| \leq \begin{cases} \frac{1}{30}(1-\alpha), & -1/2 \leq \alpha \leq 0, \\ \frac{1}{30}(1-\alpha)(2\alpha+1), & 0 < \alpha \leq 1/4, \\ \frac{1}{90}(1-\alpha)(5-2\alpha), & 1/4 < \alpha < 1. \end{cases}$$

Corollary 4.5. *Let* $f = h + \overline{g} \in \mathcal{M}(-1/2)$ *be of the form* (1.2). *Then*

$$|H_{3,1}(h)| \leq \frac{13}{80} = 0.1625, \qquad |H_{3,1}(g)| \leq \frac{1}{20} = 0.05.$$

Remark 4.6. From the upper bounds of $H_{3,1}(h)$ and $H_{3,1}(g)$ in Corollary 4.5, we note that the former is much larger than the latter, this implies that the analytic part h accounts for absolute advantage than the co-analytic part g for the harmonic mappings $f = h + \overline{g} \in \mathcal{M}(\alpha)$.

Acknowledgment. The authors would like to thank the referees for their valuable comments and suggestions, which essentially improved the quality of this paper. The present investigation was supported by the *Foundation of Educational Committee of Hunan Province* under Grant no. 15C1089.

References

[1] Y. Abu Muhanna, L. Li, and S. Ponnusamy, *Extremal problems on the class of convex functions of order -1/2*, Arch. Math. (Basel) **103** (6), 461–471, 2014.

[2] K.O. Babalola, *On* $H_3(1)$ *Hankel determinant for some classes of univalent functions*, in: Inequal. Theory and Appl. **6**, 1–7, editors: Y. J. Cho, J.K. Kim and S.S. Dragomir, 2010.

[3] D. Bansal, *Upper bound of second Hankel determinant for a new class of analytic functions*, Appl. Math. Lett. **26** (1), 103–107, 2013.

[4] D. Bansal, S. Haharana, and J.K. Prajapat, *Third order Hankel determinant for certain univalent functions*, J. Korean Math. Soc. **52**, 1139–1148, 2015.

[5] S.V. Bharanedhar and S. Ponnusamy, *Coefficient conditions for harmonic univelent mappings and hypergeometric mappings*, Rocky Mt. J. Math. **44**, 753–777, 2014.

[6] D. Bshouty, S.S. Joshi, and S.B. Joshi, *On close-to-convex harmonic mappings*, Complex Var. Elliptic Equ. **58**, 1195–1199, 2013.

[7] D. Bshouty and A. Lyzzaik, *Close-to-convexity criteria for planar harmonic mappings*, Complex Appl. Oper. Theory **5**, 767–774, 2011.

[8] D.G. Cantor, *Power series with integral coefficients*, Bull. Amer. Math. Soc. **69**, 362–366, 1963.

[9] J. Chen, A. Rasila, and X. Wang, *Coefficient estimates and radii problems for certain classes of polyharmonic mappings*, Complex Var. Elliptic Equ. **60**, 354–371, 2015.

[10] J. Clunie and T. Sheil-Small, *Harmonic univalent functions*, Ann. Acad. Sci. Fenn. Ser. A. I Math. **9**, 3–25, 1984.

[11] P. Dienes, *The Taylor Series*, Dover, New York, 1957.

[12] P.L. Duren, *Univalent functions*, Springer Verlag, New York Inc., 1983.

[13] P.L. Duren, *Harmonic mappings in the plane*, Cambridge University Press, Cambridge, 2004.

[14] A. Edrei, *Sur les déterminants récurrents et les singularités d'une fonction donnée par son développement de Taylor*, Compos. Math. **7**, 20–88, 1940.

[15] M. Fekete and G. Szegő, *Eine bemerkung über ungerade schlichte functions*, J. London Math. Soc. **8**, 85–89, 1933.

[16] T. Hayami and S. Owa, *Generalized Hankel determinant for certain classes*, Int. J. Math. Anal. **4** (52), 2573–2585, 2010.

[17] W.K. Hayman, *On the second Hankel determinant of mean univalent functions*, Proc. Lond. Math. Soc. **18** (3), 77–94, 1968.

[18] A. Janteng, S. Halim, and M. Darus, *Coefficient inequality for a function whose derivative has a positive real part*, J. Inequal. Pure Appl. Math. **7** (2), Art. 50, 5 pp, 2006.

[19] A. Janteng, S.A. Halim, and M. Darus, *Hankel determinant for starlike and convex functions*, Int. J. Math. Anal. **1** (13), 619–625, 2007.

[20] D. Kalaj, S. Ponnusamy, and M. Vuorinen, *Radius of close-to-convexity and fully starlikeness of harmonic mappings*, Complex Var. Elliptic Equ. **59**, 539–552, 2014.

[21] W. Koepf, *On the Fekete-Szegö problem for close-to-convex functions*, Proc. Amer. Math. Soc. **101**, 89–95, 1987.

[22] W. Koepf, *On the Fekete-Szegö problem for close-to-convex functions II*, Arch. Math. **49**, 420–433, 1987.

[23] S.K. Lee, V. Ravichandran, and S. Subramaniam, *Bounds for the second Hankel determinant of certain univalent functions*, J. Inequal. Appl. **2013**, Art. 281, 17 pp, 2013.

[24] R.J. Libera and E.J. Złotkiewicz, *Early coefficients of the inverse of a regular convex function*, Proc. Amer. Math. Soc. **85**, 225–230, 1982.

[25] R.J. Libera and E.J. Złotkiewicz, *Coefficient bounds for the inverse of a function with derivatives in P*, Proc. Am. Math. Soc. **87**, 251–257, 1983.

[26] A.E. Livingston, *The coefficients of multivalent close-to-convex functions*, Proc. Am. Math. Soc. **21**, 545–552, 1969.

[27] P.T. Mocanu, *Sufficient conditions of univalence for complex functions in the class* $\mathcal{C}^1$, Rev. Anal. Numér. Théor. Approx. **10**, 75–79, 1981.

[28] P.T. Mocanu, *Injectivity conditions in the complex plane*, Complex Appl. Oper. Theory **5**, 759–766, 2011.

[29] S. Nagpal and V. Ravichandran, *A subclass of close-to-convex harmonic mappings*, Complex Var. Elliptic Equ. **59**, 204–216, 2014.

[30] H. Orhan and P. Zaprawa, *Third Hankel determinants for starlike and convex functions*, Bull. Korean Math. Soc. **55**, 165–173, 2018.

[31] C. Pommerenke, *On the coefficients and Hankel determinant of univalent functions*, J. Lond. Math. Soc. **41**, 111–122, 1966.

[32] C. Pommerenke, *On the Hankel determinants of univalent functions*, Mathematika **14**, 108–112, 1967.

[33] S. Ponnusamy and A. Rasila, *Planar harmonic and quasiregular mappings*, in: Topics in modern function theory, Ramanujan Math. Soc. Lect. Notes Ser. **19**, 267–333, Ramanujan Math. Soc., Mysore, 2013.

[34] S. Ponnusamy and A. Sairam Kaliraj, *On harmonic close-to-convex functions*, Comput. Methods Funct. Theory **12**, 669–685, 2012.

[35] S. Ponnusamy and A. Sairam Kaliraj, *Constants and characterization for certain classes of univalent harmonic mappings*, Mediterr. J. Math. **12**, 647–665, 2015.

[36] M. Raza and S.N. Malik, *Upper bound of the third Hankel determinant for a class of analytic functions related with Lemniscate of Bernoulli*, J. Inequal. Appl. **2013**, Art. 412, 8 pp., 2013.

[37] T.J. Suffridge, *Some special classes of conformal mappings*, in: Handbook of complex analysis: geometric function theory (Edited by Kühnau), **2**, 309–338, Elsevier, Amsterdam, 2005.

[38] Y. Sun, Y. Jiang, and A. Rasila, *On a subclass of close-to-convex harmonic mappings*, Complex Var. Elliptic Equ. **61**, 1627–1643, 2016.

[39] D. Vamshee Krishna, B. Venkateswarlu, and T. RamReddy, *Third Hankel determinant for bounded turning functions of order alpha*, J. Nigerian Math. Soc. **34**, 121–127, 2015.

[40] P. Zaprawa, *Third Hankel determinants for subclasses of univalent functions*, Mediterr. J. Math. **14** (1), Art. 19, 10 pp., 2017.