A Note on Error Functions and Some Related Special Functions

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Abstract - In this note, priority is given to presenting essential information on the fundamental error functions, which play particularly different roles in the natural and engineering sciences, as well as some special functions closely related to them. Subsequently, various special relationships among these functions are highlighted, and finally, some selected applications of these relationships — emphasized through several references — are also proposed for interested readers.

Keywords The error functions, the Faddeeva function, the Dawson function (or integral), the Taylor-Maclaurin series, the Gausian function and integral, Special functions and series.

1. Introduction and literature survey

As is well known from the literature, special functions — although often regarded as elementary functions with specific names and notations defined by common consensus — constitute a broad and comprehensive class of mathematical functions. They play fundamental and far-reaching roles, particularly in mathematics and physics, as well as in nearly all areas of science, engineering, and technology, in both theoretical and applied contexts. Moreover, depending on the field of application and the level of mathematical rigor involved, the relevant functions may be defined in terms of real or complex variables (parameters).

In sapecial, special functions frequently arise as solutions to various types of equations, systems of equations, and inequalities in different types of spaces. They also appear in diverse areas of mathematical analysis, including differential relationships describing changes between dependent and independent variables, series expansions, antiderivatives, and integral theory, along with its applications. Their theoretical and applied significance in mathematics (and associated sciences) , physics, and mathematical physics has continued to deepen over time, while their influence has also expanded into several branches of engineering sciences. In particular, in light of the extensive theory of complex functions, the analytic investigation of special functions has become increasingly detailed and sophisticated.

Historically, the fundamental theory of special functions reached a major peak between 1800 and 1900, and its impact has naturally persisted in many modern applications. Today, new mathematical phenomena, models, theories, and approximation methods — together with their potential influence on other disciplines — continue to drive scientific and engineering research. In particular, some of the essential works given in [1, 2, 3, 4, 5, 6, 7, 8, 9] represent the primary and most fundamental references on a number of special functions, providing comprehensive and significant coverage within the existing literature.

At the same time, these developments impose important responsibilities on mathematicians and computer scientists in both theoretical investigations and computational implementations. For example, the theory of complex functions plays a crucial role in mathematics and physics. As is known, complex functions describe transformations between two two-dimensional planes, namely the z-plane and the w-plane, and enable the relationships between these planes to be examined from various perspectives. See [10, 11, 12, 13].

Although such transformations often involve highly intricate structures, they also possess rich analytical and geometric properties, depending on the objectives and requirements of the scientific fields to which they are applied. In these complex investigations, researchers who derive (or organize) the relevant transformation relationships within the associated spaces can develop appropriate models, algorithms, and computational tools that support and advance related disciplines. For further details, readers (*or* researchers) may consult the relevant studies presenting various types of applications in [14, 15, 16, 17, 18].

In addition, with respect to various applications, the concepts of series, series of functions, and related transformations in science and engineering are important and closely related to special functions. See [19, 20].

Now, let us turn to some information about error functions, a special class of functions that has made undeniable contributions to a wide range of applications in various scientific fields, including mathematics, probability, and physics, as well as to some of their inferences and applications.

2. The main error functions and some of related special functions

Although classical error functions appear in the mathematical literature primarily in two basic forms, many other functions can be constructed from them. These special functions include both real-valued (parameterized) functions and complex-valued functions, the latter occupying a particularly important place in the literature.

First, we denote by $\mathbf{R}$ the set of real numbers and by $\mathbf{C}$ the set of complex numbers. In regard to the esential information in [3, 5, 21, 22, 23, 24, 25, 26, 27], for a complex variable (*or* parameter)

$$z = x + iy \ \ (x, y \in \mathbf{R}\,;\ i^2 = -1),$$

the *Faddeeva* function $\boldsymbol{w}(z)$, the *Voigt* function $\boldsymbol{V}(x,y)$, the *imaginary Voigt* function $\boldsymbol{L}(x,y)$, the *imaginary error* function $\boldsymbol{erfi}(z)$ and the *Dawson* function (*or* integral) $\boldsymbol{D}(z)$ (*or* $\boldsymbol{F}(z)$) are all closely related. For these functions, one can establish a comprehensive set of interrelations, as summarized below:

$$\boldsymbol{w}(z) = exp(-z^2)\,\boldsymbol{erfc}(-iz) \tag{1}$$

$$= exp(-z^2)\,[1 - \boldsymbol{erf}(-iz)] \tag{2}$$

$$= exp(-z^2)\,[1 + \boldsymbol{erf}(iz)] \tag{3}$$

$$= exp(-z^2)\,[1 + i\,\boldsymbol{erfi}(z)] \tag{4}$$

$$= exp(-z^2) + \frac{2i}{\sqrt{\pi}}\boldsymbol{F}(z) \tag{5}$$

$$= \boldsymbol{V}(x,y) + i\,\boldsymbol{L}(x,y) \quad (y > 0). \tag{6}$$

Here, the special functions denoted by

$$\boldsymbol{erf}(z)\ \ ,\ \ \boldsymbol{erfc}(z)\ \text{ and }\ \boldsymbol{D}(z)$$

are defined by the following integrals:

$$\boldsymbol{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z exp(-s^2)\,ds, \tag{7}$$

$$\boldsymbol{erfc}(z) = 1 - \frac{2}{\sqrt{\pi}}\int_z^{\infty} exp(-s^2)\,ds, \tag{8}$$

and

$$\boldsymbol{D}(z) = exp(-z^2)\int_0^z exp(s^2)\,ds, \tag{9}$$

respectively. In the same time, the special functions:

$$\boldsymbol{erf}(z)\ \text{ and }\ \boldsymbol{erfc}(z)$$

are also called as the *error* function and the *complementary error* function in the literature. Furthermore, one can easily see the relationships:

$$\boldsymbol{w}(z) = exp(-z^2)\,\boldsymbol{erfc}(-iz) \tag{10}$$

$$\equiv exp(-z^2)\left(1 + \frac{2i}{\sqrt{\pi}}\int_0^z exp(s^2)\,ds\right). \tag{11}$$

Based upon the special information between (7) and (11), the Dawson function (*or* integral) may also describe as a special function consisting of complex (*or* real) variable (*or* parameter).

Naturally, it is closely related to the functions indicated in (1)-(6), and it is easily seen that

$$\boldsymbol{D}(z) = exp(-z^2)\int_0^z exp(s^2)\,ds \tag{12}$$

$$= \frac{1}{2}\sqrt{\pi}\,exp(-z^2)\,\boldsymbol{erfi}(z). \tag{13}$$

We further emphasize that all the mathematical expressions (or relations) between (1) and (13) are connected through special functions that exhibit various direct or indirect relationships with many other special mathematical expressions (or relations). For example, the Dawson function arises in a wide range of physical phenomena, including the computation of the Voigt line shape, heat conduction problems, and the theory of electrical oscillations in certain special vacua. Accordingly, each of the following statements, which is of considerable importance, is also valid:

$$\boldsymbol{D}(-z) = -\boldsymbol{D}(z)\,, \tag{14}$$

$$\boldsymbol{D}'(z) + 2z\boldsymbol{D}(z) = 1\,, \tag{15}$$

$$\boldsymbol{D}(z) = \sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}2^{\ell}}{(2\ell+1)!!}\,z^{2\ell+1} \tag{16}$$

$$= z - \frac{2}{3}z^3 + \frac{4}{15}z^5 - \frac{8}{105}z^7 + \cdots\ ,$$

and

$$\int \boldsymbol{D}(z)\,dz = \frac{1}{2}\,z^2\ {}_2F_2\left(1,1;\frac{3}{2},2;-z^2\right). \tag{17}$$

As is well known, the mathematical notation $n!!$ denotes the double factorial of a positive integer n, defined as the product of all positive integers less than or equal to n that have the same parity (odd or even) as n. The notation used in equation (17) also represents a (generalized) hypergeometric function, and the associated functions satisfy hypergeometric equations that possess many other special functions as particular or limiting cases.

Indeed, the special form presented in (17), together with other mathematical constructs such as Mittag-Leffler functions, generalized hypergeometric functions, and H-functions, offers different perspectives on special functions in both theory and applications across various fields of mathematics, physics, and engineering. It should be emphasized that each of these mathematical forms involves series representations of different types.

Consequently, such series are frequently studied in approximation theory, where computer-aided methods play a crucial role, particularly in advanced numerical computations. The principal references [28-32] provide researchers — especially those working on special functions — with a broad range of alternative approaches and methodologies.

3. Some essential results in relation with the error functions

In the second part, we have discussed the main error functions and some special functions related to them. In the literature, particularly when considering the integrals given in equations (1)-(13), it is clear that an exponential-type function of the form given by

$$\mathcal{G}(s) = exp(-s^2), \tag{18}$$

which is known as the Gaussian function, plays fundamental rolles in the construction of these special functions [5].

At the same time, with the aid of the function $\mathcal{G}(s)$ defined in (18), the following improper integral

$$\int_{-\infty}^{\infty} \mathcal{G}(s)\, ds \tag{19}$$

is known as the mathematical literature as the Gaussian integral. It is well known that this integral is convergent and its value is equal to $\sqrt{\pi}$.

Furthermore, since the function $\mathcal{G}(s)$ is an even function, this result can be expressed as

$$\int_{-\infty}^{\infty} \mathcal{G}(s)\, ds = 2\int_{0}^{\infty} \mathcal{G}(s)\, ds = \sqrt{\pi}. \tag{20}$$

By combining the result in (20) with the definition of the Gaussian function defined in (18), we then obtain

$$\frac{2}{\sqrt{\pi}}\int_{0}^{\infty} \mathcal{G}(s)\, ds = \frac{2}{\sqrt{\pi}}\left(\int_{0}^{z} \mathcal{G}(s)\, ds + \int_{z}^{\infty} \mathcal{G}(s)\, ds\right),$$

which yields that

$$\frac{2}{\sqrt{\pi}}\left(\int_{0}^{z} \mathcal{G}(s)\, ds + \int_{z}^{\infty} \mathcal{G}(s)\, ds\right) = 1,$$

where z is any complex (or real) number. This identity immediately implies that

$$erfc(z) = 1 - erf(z), \tag{21}$$

which is only one of the main relationships between the error function $erf(\cdot)$ and the complementary error function $erf\,c(\cdot)$ as defined in (7) and (8). See more detailed information in [3, 5].

As is well known, the relevant function series, together with MATLAB (or similar computer programs), play a very important role in approximate calculations. To pursue this goal and to clarify the relationships between the series expansions of the functions appearing in (1)–(13), we also present some remarks.

By making use of the familiar Taylor-Maclaurin expansion of a n-times differentiable function $\varphi := \varphi(s)$, given by

$$\varphi(s) = \sum_{\ell=0}^{\infty} \left(\frac{d^{\ell}\varphi}{ds^{\ell}}\Big|_{s:=0}\right)\frac{s^{\ell}}{\ell!},$$

the series expansion:

$$exp(s) = \sum_{\ell=0}^{\infty} \frac{s^{\ell}}{\ell!}$$

can be easily obtained. From the series above, it evidently follows that

$$exp(-s^2) = \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{\ell!} s^{2\ell}. \tag{22}$$

By considering the series expansion in (22) for the main error function defined in (7), the series representation of the error function, namely $\boldsymbol{erf}\,(z)$ can be expressed as

$$\boldsymbol{erf}(z) = \frac{2}{\sqrt{\pi}}\sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{(2\ell+1)\ell!} s^{2\ell+1}, \tag{23}$$

which in turn gives the series expansion of the complementary error function

$$\boldsymbol{erfc}(z) = 1 - \frac{2}{\sqrt{\pi}}\sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{(2\ell+1)\ell!} s^{2\ell+1}, \tag{24}$$

by making use of the main relations presented by (21). Here we specially emphasize that, in constructing the special series given in (16), there exists an important implicit and previously unstated combination of the series presented in (22)–(24). This observation also plays a significant role in determining the series representations of the other special functions introduced in (1)–(17).

As a last special remark of this section, the series expansions obtained between (22) and (24) naturally provide a foundation for determining (or constructing) the corresponding series expansions of the special functions presented in (1)–(17). We leave the necessary investigations to interested researchers and instead focus on highlighting selected applications and properties of the special functions discussed in this section.

The mathematical forms presented above constitute a core component of many Gaussian-type models, together with their associated improper integrals and related special functions. Owing to their strong analytical properties, these forms are widely used across a broad range of disciplines.

In particular, in statistics and probability theory, they appear as probability density functions of normal distributions and underpin the central limit theorem. In mathematical physics, Gaussian functions arise naturally as Green's functions for the diffusion and heat equations and are closely related to the Weierstrass transformation. Beyond these areas, Gaussian functions play a significant role in quantum mechanics and quantum field theory, where they describe ground-state wave functions and vacuum states, as well as in computational chemistry, where Gaussian basis functions are employed in the construction of molecular orbitals.

Their importance also extends to engineering and the applied sciences, particularly in optics and signal processing through the use of Gaussian beams and filters. Furthermore, Gaussian functions are widely applied in image processing, computer vision, artificial neural networks, fluorescence microscopy, and geostatistics as effective tools for smoothing, modeling, and analyzing complex data and spatial patterns. For a more detailed discussion of their applications and the relationships among various classes of special functions, the reader is referred to the studies in [32, 33, 34, 35, 36].

4. Final words

As pointed out in the previous sections, error functions and related special functions play important roles in science and engineering. Their significance manifests itself through various methods and applications across the fields in which they are used. Since the fundamental concepts and applications associated with special functions are both extensive and diverse, we focus here on a single function: the Dawson function (or integral), and briefly discuss several related studies of interest to researchers. In particular, considerable research has emphasized the importance of identifying accurate, invertible, and integrable simple approximations of the Dawson function.

For instance, the reference [37] proposes several approximation orders with good accuracy; however, these approximations are neither sufficiently simple nor easily invertible or integrable. In [38], an accurate approximation of Dawson's integral is obtained by solving its differential equation using orthogonal rational Chebyshev functions of the second kind, but the resulting rational approximation is again difficult to invert and integrate. Similarly, the reference [39] includes rational approximations whose accuracy is verified only over finite intervals, rather than over the entire domain of the Dawson function.

In [40], a special family of rational functions is established for computing Dawson's integral. Although these approximations achieve low relative error, their expressions are too complicated to allow practical inversion or integration. In [41], the essential relationship between rapidly convergent exponential series for computing Dawson's integral is also considered. Since the approximation is expressed as an infinite sum of the form given by

$$\frac{1}{\sqrt{\pi}}\sum_{j=-\infty}^{\infty}\frac{1}{j}e^{-(x-jh)^2},$$

it is clearly neither invertible nor integrable.

In [42], an extension of the universal dispersion model is considered, where excitonic contributions are expressed as linear combinations of Gaussian and truncated Lorentzian terms. In this context, the real part of the dielectric function is written in terms of Dawson's function. The private study in [43] also proposes a rational approximation of Dawson's integral that can be used to compute the complex error function.

Finally, the private paper givn in [44] accentuates the importance of inverting Dawson's integral. In particular, in relativistic hydrodynamics, time t is expressed in terms of a scale factor r, which determines the shape of an entropy profile through Dawson's function. To fully solve the problem, it is necessary to invert this relationship to obtain r as a function of the parameter t, and thus determine the velocity gradient.

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