Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 48 (6) (2019), 1761 – 1766
DOI : 10.15672/HJMS.2018.635

Research Article

On selective sequential separability of function spaces with the compact-open topology

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Abstract

For a Tychonoff space X, we denote by $C_k(X)$ the space of all real-valued continuous functions on X with the compact-open topology. A subset $A \subset X$ is said to be sequentially dense in X if every point of X is the limit of a convergent sequence in A. A space $C_k(X)$ is selectively sequentially separable (in Scheepers' terminology: $C_k(X)$ satisfies $S_{fin}(\mathcal{S}, \mathcal{S})$) if whenever $(S_n : n \in \mathbb{N})$ is a sequence of sequentially dense subsets of $C_k(X)$, one can pick finite $F_n \subset S_n$ $(n \in \mathbb{N})$ such that $\bigcup\{F_n : n \in \mathbb{N}\}$ is sequentially dense in $C_k(X)$. In this paper, we give a characterization for $C_k(X)$ to satisfy $S_{fin}(\mathcal{S}, \mathcal{S})$.

Mathematics Subject Classification (2010). 54C25, 54C35, 54C40, 54D20

Keywords. compact-open topology, function space, selectively sequentially separable, $S_1(\mathcal{S}, \mathcal{S})$, sequentially dense set, property α_2, property α_4

1. Introduction

If X is a topological space and $A \subseteq X$, then the sequential closure of A, denoted by $[A]_{seq}$, is the set of all limits of sequences from A. A set $D \subseteq X$ is said to be sequentially dense if $X = [D]_{seq}$. A space X is called sequentially separable if it has a countable sequentially dense set [26, 27].

Let X be a topological space, and $x \in X$. Consider the following collections:

- $\Omega_x = \{A \subseteq X : x \in \overline{A} \setminus A\}$;
- $\Gamma_x = \{A \subseteq X : x = \lim A\}$.

Note that if $A \in \Gamma_x$, then there exists $\{a_n\} \subset A$ converging to x. So, simply Γ_x may be the set of non-trivial convergent sequences to x.

Many topological properties are defined or characterized in terms of the following classical selection principles. Let $\mathcal{A}$ and $\mathcal{B}$ be sets consisting of families of subsets of an infinite set X. Then:

$S_1(\mathcal{A}, \mathcal{B})$ is the selection hypothesis: for each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $\{b_n\}_{n\in\mathbb{N}}$ such that for each n, $b_n \in A_n$, and $\{b_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

$S_{fin}(\mathcal{A}, \mathcal{B})$ is the selection hypothesis: for each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $\{B_n\}_{n\in\mathbb{N}}$ of finite sets such that for each n, $B_n \subseteq A_n$, and $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$.

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Received: 13.05.2018; Accepted: 09.07.2018

In this paper, by a cover we mean a cover $\mathcal{U}$ with $X \notin \mathcal{U}$.

A cover $\mathcal{U}$ of a space X is called:

• a k-cover if each compact subset C of X is contained in an element of $\mathcal{U}$;

• a γ_k-cover if $\mathcal{U}$ is infinite and for each compact subset C of X the set $\{U \in \mathcal{U} : C \nsubseteq U\}$ is finite.

Note that a γ_k-cover is a k-cover, and a k-cover is infinite. A compact space has no k-covers.

For a Tychonoff space X, we denote by $C_k(X)$ the space of all real-valued continuous functions on X with the compact-open topology. Subbase open sets of $C_k(X)$ are of the form $[A, U] = \{f \in C(X) : f(A) \subset U\}$, where A is a compact subset of X and U is a non-empty open subset of $\mathbb{R}$. Sometimes we will write the basic neighborhood of a point $f \in C_k(X)$ as $\langle f, A, \epsilon\rangle$ where $\langle f, A, \epsilon\rangle := \{g \in C(X) : |f(x) - g(x)| < \epsilon \ \forall x \in A\}$, A is a compact subset of X and $\epsilon > 0$.

For a topological space X we denote:

- Γ_k — the family of open γ_k-covers of X;
- $\mathcal{K}$ — the family of open k-covers of X;
- $\mathcal{K}^{\omega}_{cz}$ — the family of countable co-zero k-covers of X;
- $\mathcal{D}$ — the family of dense subsets of $C_k(X)$;
- $\mathcal{S}$ — the family of sequentially dense subsets of $C_k(X)$;
- $\mathbb{K}(X)$ — the family of all non-empty compact subsets of X.

A space X is said to be a γ_k-set if each k-cover $\mathcal{U}$ of X contains a countable set $\{U_n : n \in \mathbb{N}\}$ which is a γ_k-cover of X [9].

2. Main definitions and notation

- A space X is R-separable, if X satisfies $S_1(\mathcal{D}, \mathcal{D})$ ([2, Definition 47]).
- A space X is selectively separable (M-separable), if X satisfies $S_{fin}(\mathcal{D}, \mathcal{D})$.
- A space X is selectively sequentially separable (M-sequentially separable), if X satisfies $S_{fin}(\mathcal{S}, \mathcal{S})$ ([4, Definition 1.2]).

For a topological space X we have the next relations of selectors for sequences of dense sets of X.

$$\begin{array}{ccccccc} S_1(\mathcal{S},\mathcal{S}) & \Rightarrow & S_{fin}(\mathcal{S},\mathcal{S}) & \Rightarrow & S_{fin}(\mathcal{S},\mathcal{D}) & \Leftarrow & S_1(\mathcal{S},\mathcal{D}) \\ \Uparrow & & \Uparrow & & \Uparrow & & \Uparrow \\ S_1(\mathcal{D},\mathcal{S}) & \Rightarrow & S_{fin}(\mathcal{D},\mathcal{S}) & \Rightarrow & S_{fin}(\mathcal{D},\mathcal{D}) & \Leftarrow & S_1(\mathcal{D},\mathcal{D}) \end{array}$$

We write $\Pi(\mathcal{A}_x, \mathcal{B}_x)$ without specifying x, we mean $(\forall x)\Pi(\mathcal{A}_x, \mathcal{B}_x)$.

- A space X has *property* α_2 (α_2 in the sense of Arhangel'skii), if X satisfies $S_1(\Gamma_x, \Gamma_x)$ [1].
- A space X has *property* α_4 (α_4 in the sense of Arhangel'skii), if X satisfies $S_{fin}(\Gamma_x, \Gamma_x)$ [1].

So we have three types of topological properties described through the selection principles:

- local properties of the form $S_*(\Phi_x, \Psi_x)$;
- global properties of the form $S_*(\Phi, \Psi)$;
- semi-local properties of the form $S_*(\Phi, \Psi_x)$.

In a series of papers it was demonstrated that γ-covers, Borel covers, k-covers play a key role in function spaces ([5],[10]-[8], [13]-[15], [18]-[25] and many others). We continue to investigate applications of k-covers in function spaces with the compact-open topology.

A great attention has recently received the notions of selective separability and selective sequential separability ($S_{fin}(\mathcal{S}, \mathcal{S})$) [2, 3, 6, 7]. In this paper, we give characterizations for $C_k(X)$ to satisfy $S_{fin}(\mathcal{S}, \mathcal{S})$, $S_{fin}(\mathcal{S}, \Gamma_x)$, and $S_{fin}(\Gamma_x, \Gamma_x)$.

3. Main results

Definition 3.1. A γ_k-cover $\mathcal{U}$ of co-zero sets of X is γ_k**-shrinkable** if there exists a γ_k-cover $\{F(U) : U \in \mathcal{U}\}$ of zero-sets of X with $F(U) \subset U$ for every $U \in \mathcal{U}$.

Note that every γ_k-shrinkable cover contains a countable γ_k-shrinkable cover.

For a topological space X we denote:

- Γ_k^{sh} — the family of γ_k-shrinkable covers of X.

-Similar to the proof that $S_1(\mathcal{K}, \Gamma_k) = S_{fin}(\mathcal{K}, \Gamma_k)$ ([9, Theorem 5]), we prove the following.

Lemma 3.2. *For a space X the following are equivalent:*

(1) *X satisfies $S_{fin}(\Gamma_k^{sh}, \Gamma_k)$;*
(2) *X satisfies $S_1(\Gamma_k^{sh}, \Gamma_k)$.*

Proof. (1) $\Rightarrow$ (2). Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of (countable) γ_k-shrinkable covers of X; suppose that for each $n \in \mathbb{N}$, $\mathcal{U}_n = \{U_{n,m} : m \in \mathbb{N}\}$. Let $V_{n,m} = U_{1,m} \cap ... \cap U_{n,m}$ and let $\mathcal{V}_n = \{V_{n,m} : m \in \mathbb{N}\}$. Then $(\mathcal{V}_n : n \in \mathbb{N})$ is a sequence of γ_k-shrinkable covers of X. Since X satisfies $S_{fin}(\Gamma_k^{sh}, \Gamma_k)$ choose for each $n \in \mathbb{N}$ a finite subset $\mathcal{W}_n$ of $\mathcal{V}_n$ such that $\bigcup_{n\in\mathbb{N}} \mathcal{W}_n$ is a γ_k-cover of X. (Note that some $\mathcal{W}_n$'s can be empty.)

As $\bigcup_{n\in\mathbb{N}} \mathcal{W}_n$ is infinite and all $\mathcal{W}_n$'s are finite, there exists a sequence $m_1 < m_2 < ... < m_p < ...$ in $\mathbb{N}$ such that for each $i \in \mathbb{N}$ we have $\mathcal{W}_{m_i} \setminus \bigcup_{j<i} \mathcal{W}_{m_j} \neq \emptyset$. Choose an element $W_{m_i} \in \mathcal{W}_{m_i} \setminus \bigcup_{j<i} \mathcal{W}_{m_j}$, $i \in \mathbb{N}$, and fix its representation $W_{m_i} = U_{1,k_{m_i}} \cap U_{2,k_{m_i}} \cap ... \cap U_{m_i,k_{m_i}}$ as above.

Since each infinite subset of a γ_k-cover is also a γ_k-cover, we have that the set $\{W_{m_i} : i \in \mathbb{N}\}$ is a γ_k-cover of X. For each $n \leq m_1$ let $U_n \in \mathcal{U}_n$ be the n-th coordinate of W_{m_1} in the chosen representation of W_{m_1}, and for each $n \in (m_i, m_{i+1}]$, $i \geq 1$, let $U_n \in \mathcal{U}_n$ be the n-th coordinate of $W_{m_{i+1}}$ in the above representation of $W_{m_{i+1}}$. Observe that each $U_n \supset W_{m_{i+1}}$. Therefore, we obtain a sequence $(U_n : n \in \mathbb{N})$ of elements, one from each $\mathcal{U}_n$, which form a γ_k-cover of X and show that X satisfies $S_1(\Gamma_k^{sh}, \Gamma_k)$. □

The symbol $\mathbf{0}$ denotes the constantly zero function in $C_k(X)$. Because $C_k(X)$ is homogeneous we can work with $\mathbf{0}$ to study local and semi-local properties of $C_k(X)$.

Theorem 3.3. *For a Tychonoff space X the following statements are equivalent:*

(1) *$C_k(X)$ satisfies $S_1(\Gamma_\mathbf{0}, \Gamma_\mathbf{0})$ [property α_2];*
(2) *X satisfies $S_1(\Gamma_k^{sh}, \Gamma_k)$.*

Proof. (1) $\Rightarrow$ (2). Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of (countable) γ_k-shrinkable covers of X; suppose that for each $n \in \mathbb{N}$, $\mathcal{U}_n = \{U_{n,m} : m \in \mathbb{N}\}$ and $\{F(U_{n,m}) : U_{n,m} \in \mathcal{U}_n\}$ is a γ_k-cover of zero-sets of X with $F(U_{n,m}) \subset U_{n,m}$ for every $U_{n,m} \in \mathcal{U}_n$. For each $n, m \in \mathbb{N}$ we fix $f_{n,m} \in C(X)$ such that $f_{n,m} \upharpoonright F(U_{n,m}) \equiv 0$, $f_{n,m} \upharpoonright (X \setminus U_{n,m}) \equiv 1$. Consider $S_n = \{f_{n,m} : m \in \mathbb{N}\}$. Since $\{F(U_{n,m}) : U_{n,m} \in \mathcal{U}_n\}$ is a γ_k-cover of X, then $S_n \in \Gamma_\mathbf{0}$ for each $n \in \mathbb{N}$. By (1), there is $\{f_{n,m(n)} : n \in \mathbb{N}\}$ such that $f_{n,m(n)} \in S_n$ and $\{f_{n,m(n)} : n \in \mathbb{N}\} \in \Gamma_\mathbf{0}$. We show that $\{U_{n,m(n)} : n \in \mathbb{N}\} \in \Gamma_k$. Suppose $A \in \mathbb{K}(X)$ and $W = [A, (-\frac{1}{2}, \frac{1}{2})]$ is a base neighborhood of $\mathbf{0}$ then there exists $n' \in \mathbb{N}$ such that $f_{n,m(n)} \in W$ for every $n > n'$. It follows that $A \subset U_{n,m(n)}$ for every $n > n'$.

(2) $\Rightarrow$ (1). Let $S_n \in \Gamma_\mathbf{0}$ for every $n \in \mathbb{N}$; suppose that for each $n \in \mathbb{N}$, $S_n = \{f_{n,j} : j \in \mathbb{N}\}$. Consider $\mathcal{V}_n = \{f_{n,j}^{-1}((-\frac{1}{n}, \frac{1}{n})) : f_{n,j} \in S_n\}$ for each $n \in \mathbb{N}$.

Let $J = \{n \in \mathbb{N} : f_{n,j}^{-1}((-\frac{1}{n}, \frac{1}{n})) = X$ for some $j \in \mathbb{N}\}$. If J is finite, then we can ignore such finitely many n. If J is infinite, then for some j_n $(n \in J)$, $f_{n,j_n} \to \mathbf{0}$ uniformly. Thus, without loss of generality, we may assume $f_{n,j}^{-1}((-\frac{1}{n}, \frac{1}{n})) \neq X$ for each $n, j \in \mathbb{N}$.

Note that $\mathcal{W}_n = \{f_{n,j}^{-1}([-\frac{1}{n+1}, \frac{1}{n+1}]) : f_{n,j} \in S_n\}$ is a γ_k-cover of zero-sets of X. Hence, $\mathcal{V}_n \in \Gamma_k^{sh}$ for each $n \in \mathbb{N}$. By (2), there is $\{f_{n,j(n)} : n \in \mathbb{N}\}$ such that $\{f_{n,j(n)}^{-1}((-\frac{1}{n}, \frac{1}{n})) :$

$n \in \mathbb{N}\} \in \Gamma_k$. We show that $\{f_{n,j(n)} : n \in \mathbb{N}\} \in \Gamma_{\mathbf{0}}$. Let $[A, (-\epsilon, \epsilon)]$ be a base neighborhood of $\mathbf{0}$ where $A \in \mathbb{K}(X)$ and $\epsilon > 0$. There is $n' \in \mathbb{N}$ such that $A \subset f^{-1}_{n,j(n)}((-\frac{1}{n}, \frac{1}{n}))$ for each $n > n'$. There is $n'' > n'$ such that $\frac{1}{n''} < \epsilon$, hence, $f_{n,j(n)} \in [A, (-\frac{1}{n''}, \frac{1}{n''})] \subset [A, (-\epsilon, \epsilon)]$ for each $n > n''$. □

Proposition 3.4 ([3, Proposition 4.2])**.** *Every selectively sequentially separable space is sequentially separable.*

We shall prove the following theorem under the condition that the space $C_k(X)$ is sequentially separable.

Theorem 3.5. *For a Tychonoff space X such that $C_k(X)$ is sequentially separable the following statements are equivalent:*

(1) *$C_k(X)$ satisfies $S_1(\mathcal{S}, \mathcal{S})$;*
(2) *$C_k(X)$ satisfies $S_1(\mathcal{S}, \Gamma_{\mathbf{0}})$;*
(3) *$C_k(X)$ satisfies $S_1(\Gamma_{\mathbf{0}}, \Gamma_{\mathbf{0}})$ [property α_2];*
(4) *X satisfies $S_1(\Gamma_k^{sh}, \Gamma_k)$;*
(5) *$C_k(X)$ satisfies $S_{fin}(\mathcal{S}, \mathcal{S})$ [selectively sequentially separable];*
(6) *$C_k(X)$ satisfies $S_{fin}(\mathcal{S}, \Gamma_{\mathbf{0}})$;*
(7) *$C_k(X)$ satisfies $S_{fin}(\Gamma_{\mathbf{0}}, \Gamma_{\mathbf{0}})$ [property α_4];*
(8) *X satisfies $S_{fin}(\Gamma_k^{sh}, \Gamma_k)$.*

Proof. (1) $\Rightarrow$ (4). Let $\{\mathcal{U}_i\} \subset \Gamma_k^{sh}$, $\mathcal{U}_i = \{U_i^m : m \in \mathbb{N}\}$ for each $i \in \mathbb{N}$ and let $S = \{h_m : m \in \mathbb{N}\}$ be a countable sequentially dense subset of $C_k(X)$.

For each $i, m \in \mathbb{N}$ we fix $f_i^m \in C(X)$ such that $f_i^m \upharpoonright F(U_i^m) = h_m$ and $f_i^m \upharpoonright (X \setminus U_i^m) = 1$. Let $S_i = \{f_i^m : m \in \mathbb{N}\}$. Since S is a countable sequentially dense subset of $C_k(X)$, we have that S_i is a countable sequentially dense subset of $C_k(X)$ for each $i \in \mathbb{N}$. Let $h \in C(X)$, there is a set $\{h_{m_s} : s \in \mathbb{N}\} \subset S$ such that $\{h_{m_s}\}_{s\in\mathbb{N}}$ converges to h. Let K be a compact subset of X, $\epsilon > 0$ and let $W = \langle h, K, \epsilon\rangle$ be a base neighborhood of h, then there is a number m_0 such that $K \subset F(U_i^m)$ for $m > m_0$ and $h_{m_s} \in W$ for $m_s > m_0$. Since $f_i^{m_s} \upharpoonright K = h_{m_s} \upharpoonright K$ for each $m_s > m_0$, $f_i^{m_s} \in W$ for each $m_s > m_0$. It follows that a sequence $\{f_i^{m_s}\}_{s\in\mathbb{N}}$ converges to h.

Since $C_k(X)$ satisfies $S_1(\mathcal{S}, \mathcal{S})$, there is a sequence $\{f_i^{m(i)}\}_{i\in\mathbb{N}}$ such that for each i, $f_i^{m(i)} \in S_i$, and $\{f_i^{m(i)} : i \in \mathbb{N}\}$ is an element of $\mathcal{S}$.

We show that $\{U_i^{m(i)} : i \in \mathbb{N}\}$ is a γ_k-cover of X.

There is a sequence $\{f_{i_j}^{m(i_j)}\}$ converges to $\mathbf{0}$. Let K be a compact subset of X and let $U = \langle \mathbf{0}, K, (-1, 1)\rangle$ be a base neighborhood of $\mathbf{0}$. Then there exists $j_0 \in \mathbb{N}$ such that $f_{i_j}^{m(i_j)} \in U$ for each $j > j_0$. It follows that $K \subset U_{i_j}^{m(i_j)}$ for $j > j_0$. By Lemma 3.2, $S_{fin}(\Gamma_k^{sh}, \Gamma_k) = S_1(\Gamma_k^{sh}, \Gamma_k)$.

(4) $\Leftrightarrow$ (3). By Theorem 3.3.

(3) $\Rightarrow$ (2) is immediate.

(2) $\Rightarrow$ (1). For each $n \in \mathbb{N}$, let S_n be a sequentially dense subset of $C_k(X)$ and let $\{h_n : n \in \mathbb{N}\}$ be sequentially dense in $C_k(X)$. Take a sequence $\{f_n^m : m \in \mathbb{N}\} \subset S_n$ such that $f_n^m \mapsto h_n$ $(m \mapsto \infty)$. Then $f_n^m - h_n \mapsto \mathbf{0}$ $(m \mapsto \infty)$. Hence, there exists $f_n^{m_n}$ such that $f_n^{m_n} - h_n \mapsto \mathbf{0}$ $(n \mapsto \infty)$. We see that $\{f_n^{m_n} : n \in \mathbb{N}\}$ is sequentially dense. Let $h \in C_k(X)$ and take a sequence $\{h_{n_j} : j \in \mathbb{N}\} \subset \{h_n : n \in \mathbb{N}\}$ converging to h. Then, $f_{n_j}^{m_{n_j}} = (f_{n_j}^{m_{n_j}} - h_{n_j}) + h_{n_j} \mapsto h$ $(j \mapsto \infty)$.

(4) $\Leftrightarrow$ (8). By Lemma 3.2.

The proofs of the remaining implications are similar to those proved above. □

Recall that the i-weight $iw(X)$ of a space X is the smallest infinite cardinal number τ such that X can be mapped by a one-to-one continuous mapping onto a Tychonoff space of the weight not greater than τ.

It is well known that if X is hemicompact then $C_k(X)$ is metrizable. It follows that $C_k(X)$ is sequential separable for a hemicompact space X with $iw(X) = \aleph_0$. But, for general case, the author does not know the answer to the next question.

Question 1. Characterize a Tychonoff space X such that a space $C_k(X)$ is sequential separable ?

Proposition 3.6 ([3, Corollary 4.8 (Dow-Barman)])**.** *Every Fréchet-Urysohn separable T_2 space is selectively separable (hence, selectively sequentially separable).*

It is well known that a Tychonoff space X the space $C_k(X)$ is Fréchet-Urysohn if and only if X satisfies $S_1(\mathcal{K}, \Gamma_k)$ ([11]).

A Tychonoff space X the space $C_k(X)$ is separable if and only if $iw(X) = \aleph_0$ [16].

Question 2. Is there a Tychonoff space X with $iw(X) = \aleph_0$ such that $C_k(X)$ satisfies $S_1(\mathcal{S}, \mathcal{S})$, but $C_k(X)$ is not Fréchet-Urysohn (i.e. X satisfies $S_1(\Gamma_k^{sh}, \Gamma_k)$, but it has not property $S_1(\mathcal{K}, \Gamma_k)$) ?

Acknowledgment. I would like to thank the anonymous referee who read carefully the manuscript and helped me to simplify and improve the presentation of the results of the paper.

References

[1] A.V. Arhangel'skii, *The frequency spectrum of a topological space and the classification of spaces*, Soviet Math. Dokl. **13**, 1186–1189, 1972.

[2] A. Bella, M. Bonanzinga, and M. Matveev, *Variations of selective separability*, Topol. App. **156**, 1241–1252, 2009.

[3] A. Bella, M. Bonanzinga, and M. Matveev, *Sequential+separable vs sequentially separable and another variation on selective separability*, Cent. Eur. J. Math. **11** (3), 530–538, 2013.

[4] A. Bella and C. Costantini, *Sequential Separability vs Selective Sequential Separability*, Filomat **29** (1), 121–124, 2015.

[5] A. Caserta, G. Di Maio, Lj.D.R. Kočinac, and E. Meccariello, *Applications of k-covers II*, Topol. App. **153**, 3277–3293, 2006.

[6] P. Gartside, J.T.H. Lo, and A. Marsh, *Sequential density*, Topol. App. **130**, 75–86, 2003.

[7] G. Gruenhage and M. Sakai, *Selective separability and its variations*, Topol. App. **158** (12), 1352–1359, 2011.

[8] Lj.D.R. Kočinac, *Closure properties of function spaces*, App. Gen. Top. **4** (2), 255–261, 2003.

[9] Lj.D.R. Kočinac, *γ-sets, γ_k-sets and hyperspaces*, Mathematica Balkanica **19**, 109–118, 2005.

[10] Lj.D.R. Kočinac, *Selection principles and continuous images*, Cubo Math. J. **8** (2), 23–31, 2006.

[11] S. Lin, C. Liu, and H. Teng, *Fan tightness and strong Fréchet property of $C_k(X)$*, Adv. Math. (Chinese) **23** (3), 234–237, 1994.

[12] G.Di Maio, Lj.D.R. Kočinac, and E. Meccariello, *Applications of k-covers*, Acta Math. Sin. (English Series) **22** (4), 1151–1160, 2006.

[13] G.Di Maio, Lj.D.R. Kočinac, and T. Nogura *Convergence properties of hyperspaces*, J. Korean Math. Soc. **44** (4), 845–854, 2007.

[14] A.J. Marsh, *Topology of function spaces*, Doctoral Dissertation, University of Pittsburgh, 2004.

[15] R.A. McCoy, *Function spaces which are k-spaces*, Topol. P. **5**, 139–146, 1980.

[16] N. Noble, *The density character of functions spaces*, Proc. Amer. Math. Soc. **42**, 228–233, 1974.

[17] A.V. Osipov, *Different kinds of tightness of a funtional space*, Tr. Inst. Mat. Mekh. (Russian), **22** (3), 192–199, 2016.

[18] A.V. Osipov, *Application of selection principles in the study of the properties of function spaces*, Acta Math. Hungar. **154** (2), 362–377, 2018.

[19] A.V. Osipov, *Classification of selectors for sequences of dense sets of $C_p(X)$*, Topology Appl. **242**, 20–32, 2018.

[20] A.V. Osipov, *The functional characterizations of the Rothberger and Menger properties*, Topology Appl. **243**, 146–152, 2018.

[21] A.V. Osipov, *Classification of selectors for sequences of dense sets of Baire functions*, submitted.

[22] A.V. Osipov and S. Özçağ, *Variations of selective separability and tightness in function spaces with set-open topologies*, Topology Appl. **217**, 38–50, 2017.

[23] A.V. Osipov and E.G. Pytkeev, *On sequential separability of functional spaces*, Topology Appl. **221**, 270–274, 2017.

[24] B.A. Pansera and V. Pavlović, *Open covers and function spaces*, Matematički Vesnik **58**, 57–70, 2006.

[25] M. Sakai, *k-Frechet-Urysohn Property of $C_k(X)$*, Topol. App. **154** (7), 1516–1520, 2007.

[26] G. Tironi and R. Isler, *On some problems of local approximability in compact spaces*, In: General Topology and its Relations to Modern Analysis and Algebra **III**, 443–446, Prague, August 30-September 3, 1971, Academia, Prague, 1972.

[27] A. Wilansky, *How separable is a space?*, Amer. Math. Monthly **79** (7), 764–765, 1972.