On selective sequential separability of function spaces with the compact-open topology

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Abstract

For a Tychonoff space \( X \), we denote by \( C_k(X) \) the space of all real-valued continuous functions on \( X \) with the compact-open topology. A subset \( A \subset X \) is said to be sequentially dense in \( X \) if every point of \( X \) is the limit of a convergent sequence in \( A \). A space \( C_k(X) \) is selectively sequentially separable (in Scheepers’ terminology: \( C_k(X) \) satisfies \( S_{\text{fin}}(S, S) \)) if whenever \((S_n : n \in \mathbb{N})\) is a sequence of sequentially dense subsets of \( C_k(X) \), one can pick finite \( F_n \subset S_n \) (\( n \in \mathbb{N} \)) such that \( \bigcup_{n \in \mathbb{N}} F_n \) is sequentially dense in \( C_k(X) \). In this paper, we give a characterization for \( C_k(X) \) to satisfy \( S_{\text{fin}}(S, S) \).

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1. Introduction

If \( X \) is a topological space and \( A \subseteq X \), then the sequential closure of \( A \), denoted by \([A]_{\text{seq}}\), is the set of all limits of sequences from \( A \). A set \( D \subseteq X \) is said to be sequentially dense if \( X = [D]_{\text{seq}} \). A space \( X \) is called sequentially separable if it has a countable sequentially dense set [26, 27].

Let \( X \) be a topological space, and \( x \in X \). Consider the following collections:

- \( \Omega_x = \{A \subseteq X : x \in \overline{A} \setminus A\} \);
- \( \Gamma_x = \{A \subseteq X : x = \lim A\} \).

Note that if \( A \in \Gamma_x \), then there exists \( \{a_n\} \subset A \) converging to \( x \). So, simply \( \Gamma_x \) may be the set of non-trivial convergent sequences to \( x \).

Many topological properties are defined or characterized in terms of the following classical selection principles. Let \( A \) and \( B \) be sets consisting of families of subsets of an infinite set \( X \). Then:

- \( S_1(A, B) \) is the selection hypothesis: for each sequence \( (A_n : n \in \mathbb{N}) \) of elements of \( A \) there is a sequence \( \{b_n\}_{n \in \mathbb{N}} \) such that for each \( n \), \( b_n \in A_n \), and \( \{b_n : n \in \mathbb{N}\} \) is an element of \( B \).
- \( S_{\text{fin}}(A, B) \) is the selection hypothesis: for each sequence \( (A_n : n \in \mathbb{N}) \) of elements of \( A \) there is a sequence \( \{B_n\}_{n \in \mathbb{N}} \) of finite sets such that for each \( n \), \( B_n \subseteq A_n \), and \( \bigcup_{n \in \mathbb{N}} B_n \in B \).

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In this paper, by a cover we mean a cover $U$ with $X \notin U$.
A cover $U$ of a space $X$ is called:
- a $k$-cover if each compact subset $C$ of $X$ is contained in an element of $U$;
- a $\gamma_k$-cover if $U$ is infinite and for each compact subset $C$ of $X$ the set $\{U \in U : C \notin U\}$ is finite.

Note that a $\gamma_k$-cover is a $k$-cover, and a $k$-cover is infinite. A compact space has no $k$-covers.

For a Tychonoff space $X$, we denote by $C_k(X)$ the space of all real-valued continuous functions on $X$ with the compact-open topology. Subbase open sets of $C_k(X)$ are of the form $[A,U] = \{f \in C(X) : f(A) \subset U\}$, where $A$ is a compact subset of $X$ and $U$ is a non-empty open subset of $\mathbb{R}$. Sometimes we will write the basic neighborhood of a point $f \in C_k(X)$ as $\langle f, A, \epsilon \rangle$ where $\langle f, A, \epsilon \rangle := \{g \in C(X) : |f(x) - g(x)| < \epsilon \ \forall x \in A\}$, $A$ is a compact subset of $X$ and $\epsilon > 0$.

For a topological space $X$ we denote:
- $\Gamma_k$ — the family of open $\gamma_k$-covers of $X$;
- $\mathcal{X}$ — the family of open $k$-covers of $X$;
- $\mathcal{X}_{c_0}^\infty$ — the family of countable co-zero $k$-covers of $X$;
- $\mathcal{D}$ — the family of dense subsets of $C_k(X)$;
- $\mathcal{S}$ — the family of sequentially dense subsets of $C_k(X)$;
- $\mathcal{K}(X)$ — the family of all non-empty compact subsets of $X$.

A space $X$ is said to be a $\gamma_k$-set if each $k$-cover $U$ of $X$ contains a countable set $\{U_n : n \in \mathbb{N}\}$ which is a $\gamma_k$-cover of $X$ [9].

2. Main definitions and notation

- A space $X$ is $R$-separable, if $X$ satisfies $S_1(\mathcal{D}, \mathcal{D})$ ([[2], Definition 47])
- A space $X$ is selectively separable ($M$-separable), if $X$ satisfies $S_{fin}(\mathcal{D}, \mathcal{D})$.
- A space $X$ is selectively sequentially separable ($M$-sequentially separable), if $X$ satisfies $S_{fin}(\mathcal{S}, \mathcal{S})$ ([[4], Definition 1.2]).

For a topological space $X$ we have the next relations of selectors for sequences of dense sets of $X$.

$S_1(\mathcal{S}, \mathcal{S}) \Rightarrow S_{fin}(\mathcal{S}, \mathcal{S}) \Rightarrow S_{fin}(\mathcal{D}, \mathcal{D}) \Leftrightarrow S_1(\mathcal{D}, \mathcal{D})$

We write $\Pi(A_x, \mathcal{B}_x)$ without specifying $x$, we mean $\langle \forall x \rangle \Pi(A_x, \mathcal{B}_x)$.

- A space $X$ has property $\alpha_2$ ($\alpha_2$ in the sense of Arhangel’skii), if $X$ satisfies $S_1(\Gamma_x, \Gamma_x)$ [1].
- A space $X$ has property $\alpha_4$ ($\alpha_4$ in the sense of Arhangel’skii), if $X$ satisfies $S_{fin}(\Gamma_x, \Gamma_x)$ [1].

So we have three types of topological properties described through the selection principles:
- local properties of the form $S_n(\Phi_x, \Psi_x)$;
- global properties of the form $S_n(\Phi, \Psi)$;
- semi-local properties of the form $S_n(\Phi, \Psi_x)$.

In a series of papers it was demonstrated that $\gamma$-covers, Borel covers, $k$-covers play a key role in function spaces ([5], [10]-[8], [13]-[15], [18]-[25] and many others). We continue to investigate applications of $k$-covers in function spaces with the compact-open topology.

A great attention has recently received the notions of selective separability and selective sequential separability ($S_{fin}(\mathcal{S}, \mathcal{S})$) [[2, 3, 6, 7]. In this paper, we give characterizations for $C_k(X)$ to satisfy $S_{fin}(\mathcal{S}, \mathcal{S}), S_{fin}(\mathcal{S}, \Gamma_x)$, and $S_{fin}(\Gamma_x, \Gamma_x)$. 

3. Main results

Definition 3.1. A $\gamma_k$-cover $\mathcal{U}$ of co-zero sets of $X$ is $\gamma_k$-shrinkable if there exists a $\gamma_k$-cover $\{F(U) : U \in \mathcal{U}\}$ of zero-sets of $X$ with $F(U) \subset U$ for every $U \in \mathcal{U}$.

Note that every $\gamma_k$-shrinkable cover contains a countable $\gamma_k$-shrinkable cover.

For a topological space $X$ we denote:

- $\Gamma^h_k$ — the family of $\gamma_k$-shrinkable covers of $X$.

-Similar to the proof that $S_1(X, \Gamma_k) = S_{fin}(X, \Gamma_k)$ ([9, Theorem 5]), we prove the following.

Lemma 3.2. For a space $X$ the following are equivalent:

1. $X$ satisfies $S_{fin}(\Gamma_k^h, \Gamma_k)$;
2. $X$ satisfies $S_1(\Gamma_k^h, \Gamma_k)$.

Proof. (1) $\Rightarrow$ (2). Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of (countable) $\gamma_k$-shrinkable covers of $X$; suppose that for each $n \in \mathbb{N}$, $\mathcal{U}_n = \{U_{n,m} : m \in \mathbb{N}\}$. Let $V_{n,m} = U_{1,m} \cap \ldots \cap U_{n,m}$ and let $V_n = \{V_{n,m} : m \in \mathbb{N}\}$. Then $(V_n : n \in \mathbb{N})$ is a sequence of $\gamma_k$-shrinkable covers of $X$.

Since $X$ satisfies $S_{fin}(\Gamma_k^h, \Gamma_k)$ choose for each $n \in \mathbb{N}$ a finite subset $W_n$ of $V_n$ such that $\bigcup_{n \in \mathbb{N}} W_n$ is a $\gamma_k$-cover of $X$. (Note that some $W_n$’s can be empty.)

As $\bigcup_{n \in \mathbb{N}} W_n$ is infinite and all $W_n$’s are finite, there exists a sequence $m_1 < m_2 < \ldots < m_p < \ldots$ in $\mathbb{N}$ such that for each $i \in \mathbb{N}$ we have $W_{m_i} \setminus \bigcup_{j<i} W_{m_j} \neq \emptyset$. Choose an element $W_{m_i} \in \bigcup_{j<i} W_{m_j}$, $i \in \mathbb{N}$, and fix its presentation $W_{m_i} = U_{1,k_{m_i}} \cap U_{2,k_{m_i}} \cap \ldots \cap U_{n,m_{k_{m_i}}}$ as above.

Since each infinite subset of a $\gamma_k$-cover is also a $\gamma_k$-cover, we have that the set $\{W_{m_i} : i \in \mathbb{N}\}$ is a $\gamma_k$-cover of $X$. For each $n \leq m_1$ let $U_n \in \mathcal{U}_n$, be the $n$-th coordinate of $W_{m_1}$ in the chosen representation of $W_{m_1}$, and for each $n \in (m_i, m_{i+1}]$, $i \geq 1$, let $U_n \in \mathcal{U}_n$, be the $n$-th coordinate of $W_{m_{i+1}}$ in the above representation of $W_{m_{i+1}}$. Observe that each $U_n \supset W_{m_{i+1}}$. Therefore, we obtain a sequence $(U_n : n \in \mathbb{N})$ of elements, one from each $\mathcal{U}_n$, which form a $\gamma_k$-cover of $X$ and show that $X$ satisfies $S_1(\Gamma_k^h, \Gamma_k)$. \hfill $\square$

The symbol $0$ denotes the constantly zero function in $C_k(X)$. Because $C_k(X)$ is homogeneous we can work with $0$ to study local and semi-local properties of $C_k(X)$.

Theorem 3.3. For a Tychonoff space $X$ the following statements are equivalent:

1. $C_k(X)$ satisfies $S_1(\Gamma_0, \Gamma_0)$ [property $\alpha_2$];
2. $X$ satisfies $S_1(\Gamma_k^h, \Gamma_k)$.

Proof. (1) $\Rightarrow$ (2). Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of (countable) $\gamma_k$-shrinkable covers of $X$; suppose that for each $n \in \mathbb{N}$, $\mathcal{U}_n = \{U_{n,m} : m \in \mathbb{N}\}$ and $\{F(U_{n,m}) : U_{n,m} \in \mathcal{U}_n\}$ is a $\gamma_k$-cover of zero-sets of $X$ with $F(U_{n,m}) \subset U_{n,m}$ for every $U_{n,m} \in \mathcal{U}_n$. For each $n, m \in \mathbb{N}$ we fix $m \in \mathbb{N}$ such that $f_{n,m} | F(U_{n,m}) \equiv 0$, $f_{n,m} | (X \setminus U_{n,m}) \equiv 1$. Consider $S_n = \{f_{n,m} : m \in \mathbb{N}\}$. Since $\{F(U_{n,m}) : U_{n,m} \in \mathcal{U}_n\}$ is a $\gamma_k$-cover of $X$, then $S_n \in \Gamma_0$ for each $n \in \mathbb{N}$. By (1), there is $\{f_{n,m(n)} : n \in \mathbb{N}\}$ such that $f_{n,m(n)} \in S_n$ and $\{f_{n,m(n)} : n \in \mathbb{N}\} \subset \Gamma_k$. We show that $\{U_{n,m(n)} : n \in \mathbb{N}\} \subset \Gamma_k$. Suppose $A \in \mathbb{K}(X)$ and $W = [A, (-\frac{1}{2}, \frac{1}{2})]$ is a base neighborhood of $0$ then there exists $n' \in \mathbb{N}$ such that $f_{n,m(n)} \in W$ for every $n > n'$. It follows that $A \subset U_{n,m(n)}$ for every $n > n'$.

(2) $\Rightarrow$ (1). Let $S_n \in \Gamma_0$ for every $n \in \mathbb{N}$; suppose that for each $n \in \mathbb{N}$, $S_n = \{f_{n,j} : j \in \mathbb{N}\}$. Consider $V_n = \{f_{n,j}^{-1}\left((-\frac{1}{n}, \frac{1}{n})\right) : f_{n,j} \in S_n\}$ for each $n \in \mathbb{N}$.

Let $J = \{n \in \mathbb{N} : f_{n,j}^{-1}\left((-\frac{1}{n}, \frac{1}{n})\right) = X$ for some $j \in \mathbb{N}\}$. If $J$ is finite, then we can ignore such finitely many $n$. If $J$ is infinite, then for some $j_n \in J$, $f_{n,j_n} \to 0$ uniformly. Thus, without loss of generality, we may assume $f_{n,j}^{-1}\left((-\frac{1}{n}, \frac{1}{n})\right) \neq X$ for each $n, j \in J$.

Note that $W_n = \{f_{n,j}^{-1}\left((-\frac{1}{n+m}, \frac{1}{n+m})\right) : f_{n,j} \in S_n\}$ is a $\gamma_k$-cover of zero-sets of $X$. Hence, $\mathcal{V}_n \in \Gamma_k^h$ for each $n \in \mathbb{N}$. By (2), there is $\{f_{n,j(n)} : n \in \mathbb{N}\}$ such that $\{f_{n,j(n)}^{-1}\left((-\frac{1}{n}, \frac{1}{n})\right) : \}$
Every selectively sequentially separable space is $C_X$. For a Tychonoff space $f$ there is a number $h \in S$ such that $f_{n,j(n)} = S$ for each $n > n'$. There is $n'' > n'$ such that $\frac{1}{n''} < \epsilon$, hence, $f_{n,j(n)} \in [A, (-\epsilon, \epsilon)]$ for each $n > n''$.

**Proposition 3.4** ([3, Proposition 4.2]). Every selectively sequentially separable space is sequentially separable.

We shall prove the following theorem under the condition that the space $C_k(X)$ is sequentially separable.

**Theorem 3.5.** For a Tychonoff space $X$ such that $C_k(X)$ is sequentially separable the following statements are equivalent:

1. $C_k(X)$ satisfies $S_1(8,8)$;
2. $C_k(X)$ satisfies $S_1(8,\Gamma_0)$;
3. $C_k(X)$ satisfies $S_1(\Gamma_0,\Gamma_0)$ [property $\alpha_2$];
4. $X$ satisfies $S_1(\Gamma^{sh}_k,\Gamma_k)$;
5. $C_k(X)$ satisfies $S_{fin}(8,8)$ [selectively sequentially separable];
6. $C_k(X)$ satisfies $S_{fin}(8,\Gamma_0)$;
7. $C_k(X)$ satisfies $S_{fin}(\Gamma_0,\Gamma_0)$ [property $\alpha_4$];
8. $X$ satisfies $S_{fin}(\Gamma^{sh}_k,\Gamma_k)$.

**Proof.** (1) $\Rightarrow$ (4). Let $\{U_i\} \subset \Gamma^{sh}_k$, $U_i = \{U_i^m : m \in \mathbb{N}\}$ for each $i \in I$ and let $S = \{h_m : m \in \mathbb{N}\}$ be a countable sequentially dense subset of $C_k(X)$.

For each $i, m \in \mathbb{N}$ we fix $f_i^m \in C(X)$ such that $f_i^m \mid F(U_i^m) = h_m$ and $f_i^m \mid (X \setminus U_i^m) = 1$. Let $S_i = \{f_i^m : m \in \mathbb{N}\}$. Since $S$ is a countable sequentially dense subset of $C_k(X)$, we have that $S_i$ is a countable sequentially dense subset of $C_k(X)$ for each $i \in I$. Let $h \in C(X)$, there is a set $\{h_m : s \in \mathbb{N}\} \subset S$ such that $\{h_m\}_{s \in \mathbb{N}}$ converges to $h$. Let $K$ be a compact subset of $X$, $\epsilon > 0$ and let $W = (h, K, \epsilon)$ be a base neighborhood of $h$, then there is a number $m_0$ such that $K \subset F(U_i^m)$ for $m > m_0$ and $h_{m_0} \in W$ for $m_s > m_0$. Since $f_i^m \mid K = h_m \mid K$ for each $m_s > m_0$, $f_i^m \in W$ for each $m_s > m_0$. It follows that there is a sequence $\{f_i^{m_s}\}_{s \in \mathbb{N}}$ converging to $h$.

Since $C_k(X)$ satisfies $S_1(8,8)$, there is a sequence $\{f_i^{m(i)}\}_{i \in \mathbb{N}}$ such that for each $i, f_i^{m(i)} \in S_i$, and $\{f_i^{m(i)} : i \in \mathbb{N}\}$ is an element of $S$.

We show that $\{U_i^{m(i)} : i \in \mathbb{N}\}$ is a $\gamma_k$-cover of $X$.

There is a sequence $\{f_{ij}^{m(i)}\}$ converges to $0$. Let $K$ be a compact subset of $X$ and let $U = (0, K, (-1,1))$ be a base neighborhood of $0$. Then there exists $j_0 \in \mathbb{N}$ such that $f_{ij}^{m(i)} \in U$ for each $j > j_0$. It follows that $K \subset U_i^{m(i)}$ for $j > j_0$. By Lemma 3.2, $S_{fin}(\Gamma^{sh}_k,\Gamma_k) = S_1(\Gamma^{sh}_k,\Gamma_k)$.

(4) $\Leftrightarrow$ (3). By Theorem 3.3.

(3) $\Rightarrow$ (2). Immediate.

(2) $\Rightarrow$ (1). For each $n \in \mathbb{N}$, let $S_n$ be a sequentially dense subset of $C_k(X)$ and let $\{h_n : n \in \mathbb{N}\}$ be sequentially dense in $C_k(X)$. Take a sequence $\{f_i^m : m \in \mathbb{N}\} \subset S_n$ such that $f_i^m \rightarrow h_n$ ($m \rightarrow \infty$). Then $f_i^m - h_n \rightarrow 0$ ($m \rightarrow \infty$). Hence, there exists $f_{ij}^{m(i)}$ such that $f_{ij}^{m(i)} \rightarrow 0$ ($n \rightarrow \infty$). We see that $\{f_{ij}^{m(i)} : n \in \mathbb{N}\}$ is sequentially dense. Let $h \in C_k(X)$ and take a sequence $\{h_n : j \in \mathbb{N}\} \subset \{h_n : n \in \mathbb{N}\}$ converging to $h$. Then, $f_{ij}^{m(i)} = (f_{ij}^{m(i)} - h_n) + h_n \rightarrow h$ ($j \rightarrow \infty$).

$\Box$
Recall that the $i$-weight $iw(X)$ of a space $X$ is the smallest infinite cardinal number $\tau$ such that $X$ can be mapped by a one-to-one continuous mapping onto a Tychonoff space of the weight not greater than $\tau$.

It is well known that if $X$ is hemicompact then $C_k(X)$ is metrizable. It follows that $C_k(X)$ is sequential separable for a hemicompact space $X$ with $iw(X) = \aleph_0$. But, for general case, the author does not know the answer to the next question.

**Question 1.** Characterize a Tychonoff space $X$ such that a space $C_k(X)$ is sequential separable?

**Proposition 3.6 ([3, Corollary 4.8 (Dow-Barman)]).** Every Fréchet-Urysohn separable $T_2$ space is selectively separable (hence, selectively sequentially separable).

It is well known that a Tychonoff space $X$ the space $C_k(X)$ is Fréchet-Urysohn if and only if $X$ satisfies $S_1(X, \Gamma_k)$ ([11]).

A Tychonoff space $X$ the space $C_k(X)$ is separable if and only if $iw(X) = \aleph_0$ [16].

**Question 2.** Is there a Tychonoff space $X$ with $iw(X) = \aleph_0$ such that $C_k(X)$ satisfies $S_1(S, S)$, but $C_k(X)$ is not Fréchet-Urysohn (i.e. $X$ satisfies $S_1(\Gamma^s_k, \Gamma_k)$, but it has not property $S_1(K, \Gamma_k)$)?

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**References**


