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SOLUTIONS OF THE RADIAL SCHRÖDINGER EQUATION IN HYPERGEOMETRIC AND DISCRETE FRACTIONAL FORMS

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Abstract. The purpose of this present paper is to obtain the hypergeometric and discrete fractional solutions of the radial Schrödinger equation by using the nabla discrete fractional calculus operator.

1. Introduction

Recently, it is possible to see many scientific works related to fractional calculus, discrete fractional calculus (*fractional sum and difference calculus*) and Schrödinger equation that is the main equation of our study. Sumudu convolution and shift properties were investigated and, Sumudu transform method was applied to Bessel functions and equations [6]. The Sumudu operator was used to the classes of fractional differential equations [15]. The solution of generalized fractional kinetic equation involving the generalized Lauricella functions was obtained [12]. The analytical solution for the fractional radial diffusion equation was derived by means of Hankel and Sumudu transforms [8]. A sufficient condition to guarantee the solution of the constant coefficient fractional differential equation via the Sumudu transform was presented [11]. Existence and uniqueness of solutions to a boundary value problem for a discrete fractional mixed type sum-difference equation with the nonlinear term dependent on a fractional difference of lower order were introduced via Schauder's fixed point theorem and contraction mapping principle [16]. Chen and Tang studied on the discrete fractional boundary value problem in detail, and exhibited the uniqueness and multiplicity of the solutions for the discrete fractional boundary value problem [10]. Continuous dependence of solutions on the initial conditions for nabla fractional difference equations was given, and the linear variation of parameters formula for nabla fractional difference equations involving Riemann-Liouville type fractional differences was also presented [18]. Existence of multiple solutions for a fractional difference boundary value problem with p-Laplacian operator was

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obtained [13]. A new class of 3-point boundary value problems of nonlinear fractional difference equations was defined, and existence and uniqueness of solutions were proved by means of the Banach fixed-point theorem, and existence of the positive solutions was also proved by means of the Krasnoselskii's fixed-point theorem [21]. The stability of the equilibrium solution of the νth order linear system of fractional-order difference equations was studied [1]. Two new monotonicity concepts for a nonnegative or nonpositive valued function defined on a discrete domain were investigated, and examples to illustrate connections between these new monotonicity concepts and the traditional ones were given [4]. Some important results for nabla and delta fractional difference were obtained [5]. Equations of motion in mass-spring-damper system were solved by applying nabla discrete fractional operator[19]. Tselios and Simos introduced new symplectic-schemes for the numerical solution of the radial Shrödinger equation, and developed symplectic integrators for Hamiltonian systems [22]. Exact bound-state solutions in the generalized harmonic-oscillator elementary were expressed via a broad class of the regular potentials [25]. The exact solutions of the bound states of the Schrödinger equation for the modified Kratzer potential plus a new ring-shaped potential were exhibited analytically by means of the Nikiforov-Uvarov method [9]. Semiclassical limit of the nonlinear Schrödinger equation was studied in a radial potential [7]. Holmer and Roudenko interested in a sharp condition for scattering of the radial $3D$ cubic nonlinear Schrödinger equation [14]. Explicit solutions of second-order linear ordinary differential equations were obtained by using fractional calculus techniques [23]. The N-method was used for the radial component of the fractional Schrödinger equation [20]. And, we aim to solve the radial Schrödinger equation by means of Leibniz rule with nabla discrete fractional calculus operator in this study.

2. Preliminaries

We present some properties of the fractional calculus and discrete fractional calculus via this section.

Lemma 1. *Let $F(u)$ and $G(u)$ be single-valued and analytic functions. If F_ν and G_ν exist, Leibniz rule in fractional calculus is given as*

$$(FG)_\nu = \sum_{n=0}^{\infty} \binom{\nu}{n} F_{\nu-n} G_n \quad \left(\binom{\nu}{n} = \frac{\Gamma(\nu+1)}{\Gamma(\nu+1-n)\Gamma(n+1)} \right), \tag{2.1}$$

where $\nu \in \mathbf{R}$, $z \in \mathbf{C}$ and $\left|\binom{\nu}{n}\right| < \infty$ [23].

Lemma 2. *For a constant k* [23],

$$(e^{ku})_\nu = k^\nu e^{ku} \quad (k \neq 0, \nu \in \mathbf{R}, u \in \mathbf{C}), \tag{2.2}$$

$$(e^{-ku})_\nu = e^{-i\pi\nu} k^\nu e^{-ku} \quad (k \neq 0, \nu \in \mathbf{R}, u \in \mathbf{C}), \tag{2.3}$$

$$(u^k)_\nu = e^{-i\pi\nu} \frac{\Gamma(\nu-k)}{\Gamma(-k)} u^{k-\nu} \quad \left(\nu \in \mathbf{R}, u \in \mathbf{C}, \left| \frac{\Gamma(\nu-k)}{\Gamma(-k)} \right| < \infty \right). \tag{2.4}$$

Definition 1. *The rising factorial power* $u^{\overline{n}}$ *is defined by*

$$u^{\overline{n}} = u(u+1)(u+2)...(u+n-1) \quad (n \in \mathbf{N}, u^{\overline{0}} = 1). \tag{2.5}$$

If $u \in \mathbf{R}\backslash\{..., -2, -1, 0\}$ *and* $\nu \in \mathbf{R}$, *and so, "u to the* ν *rising"is*

$$u^{\overline{\nu}} = \frac{\Gamma(u+\nu)}{\Gamma(u)} \quad (0^{\overline{\nu}} = 0), \tag{2.6}$$

and,

$$\nabla(u^{\overline{\nu}}) = \nu u^{\overline{\nu-1}}, \tag{2.7}$$

where $\nabla F(u) = F(u) - F(u-1)$ [3].

Definition 2. *Consider* $m \in \mathbf{R}$ *and* $\nu \in \mathbf{R}^+$ *such that* $0 < n-1 \leq \nu < n$ $(n \in \mathbf{Z})$. *The fractional sum of function* F *with* ν*-th order is*

$$\nabla_m^{-\nu} F(u) = \sum_{k=m}^{u} \frac{[u-f(k)]^{\overline{\nu-1}}}{\Gamma(\nu)} F(k), \tag{2.8}$$

where $u \in \mathbf{N}_m = \{m, m+1, m+2, ...\}$, *and* $f(u) = u - 1$.
The fractional difference of function F *with* ν*-th order is*

$$\nabla_m^{\nu} F(u) = \nabla^n \nabla_m^{-(n-\nu)} F(u) = \nabla^n \sum_{k=m}^{u} \frac{[u-f(k)]^{\overline{n-\nu-1}}}{\Gamma(n-\nu)} F(k), \tag{2.9}$$

where $F : \mathbf{N}_m \to \mathbf{R}$ [2].

Definition 3. *The discrete shift operator* t *is given by*

$$t^n F(u) = F(u+n), \tag{2.10}$$

where $n \in \mathbf{N}$ [19].

Theorem 1. *Let* $\nu, \rho > 0$ *and* A, B *are scalars. Thus,*

$$\nabla^{-\nu}\nabla^{-\rho} F(u) = \nabla^{-(\nu+\rho)} F(u) = \nabla^{-\rho}\nabla^{-\nu} F(u), \tag{2.11}$$

$$\nabla^{\nu}[AF(u) + BG(u)] = A\nabla^{\nu} F(u) + B\nabla^{\nu} G(u), \tag{2.12}$$

$$\nabla\nabla^{-\nu} F(u) = \nabla^{-(\nu-1)} F(u), \tag{2.13}$$

$$\nabla^{-\nu}\nabla F(u) = \nabla^{1-\nu} F(u) - \binom{u+\nu-2}{u-1} F(0), \tag{2.14}$$

where $F, G : \mathbf{N}_0 \to \mathbf{R}$ [17].

Lemma 3. *The power rule is as follows:*

$$\nabla_m^{-\nu}(u-m+1)^{\overline{\rho}} = \frac{\Gamma(\rho+1)}{\Gamma(\nu+\rho+1)}(u-m+1)^{\overline{\nu+\rho}} \quad (\forall u \in \mathbf{N}_m, \nu > 0), \tag{2.15}$$

where $\nu, \rho \in \mathbf{R}$ [2].

Lemma 4. *The Leibniz rule with the nabla discrete fractional calculus operator is defined as*

$$\nabla_0^{\nu}(FG)(u) = \sum_{n=0}^{u} \binom{\nu}{n} [\nabla_0^{\nu-n} F(u-n)][\nabla^n G(u)] \quad (\nu > 0, u \in \mathbf{Z}^+), \tag{2.16}$$

where $F, G : \mathbf{N}_0 \to \mathbf{R}$ [24].

Lemma 5. *Consider* $F(u)$ *is analytic and single-valued. In the fractional calculus, the following equalities are available:*

$$[F_\nu(u)]_\rho = F_{\nu+\rho}(u) = [F_\rho(u)]_\nu, \tag{2.17}$$

$$(\nu, \rho \in \mathbf{R}, u \in \mathbf{N}, F_\nu(u) \neq 0, F_\rho(u) \neq 0),$$

where $F_\nu = d^\nu F/du^\nu$ [20].

3. Main Results

In the x-dimensional space, radial Schrödinger equation is given by

$$g_2(u) + \frac{x-1}{u} g_1(u) + \left[\frac{2\mu}{\hbar^2}\left(E + e^2 \frac{\beta_c}{u^{c-2}}\right) - \frac{y(y-x-2)}{u^2}\right] g(u) = 0, \tag{3.1}$$

where constant β_c is $\beta_c = \frac{\Gamma(c/2)}{2\pi^{c/2}(c-2)\varepsilon_0}$ $(c > 2)$, $1 \leq x \leq 3$ and $0 \leq u \leq \infty$. For Eq. (3.1), we set

$$v = 2\beta u, \quad g = u^y e^{-\beta u} F, \quad \alpha = \frac{\mu e^2 \beta_c}{\hbar^2}$$

where $\beta^2 = -2\mu E/\hbar^2$. So, Eq. (3.1) becomes a singular differential equation as follows:

$$vF_2 + (\lambda - v)F_1 + \left(\omega v^{3-c} - \frac{\lambda}{2}\right)F = 0, \tag{3.2}$$

where $\lambda = 2y + x - 1$, $\omega = \frac{\alpha}{2^{3-c}\beta^{4-c}}$ [23].

Theorem 2. *We get* $c = 4$ *in Eq. (3.2), and so, we write*

$$vF_2 + (\lambda - v)F_1 + \left(\frac{\omega}{v} - \frac{\lambda}{2}\right)F = 0. \tag{3.3}$$

Eq. (3.3) has the following discrete fractional solutions:

$$F^I(v) = Av^{-(a+\frac{\lambda}{2})}(v^a e^v)_{-(1+t^{-1}a)}, \tag{3.4}$$

and,

$$F^{II}(v) = Bv^{-(b+\frac{\lambda}{2})}(v^b e^v)_{-(1+t^{-1}b)}, \tag{3.5}$$

where $v \in \mathbf{C}$, $F \in \{F : 0 \neq| F_\nu |< \infty, \nu \in \mathbf{R}\}$ *and* A, B, λ, a, b *are constants and,* t *is discrete shift operator.*

Proof. Consider $F = v^\eta G$ $(v \neq 0, G = G(v))$, and so,

$$vG_2 + (2\eta + \lambda - v)G_1 + \Big[(\eta^2 + \eta(\lambda - 1) + \omega)v^{-1} - (\eta + \frac{\lambda}{2})\Big]G = 0. \tag{3.6}$$

Suppose that $\eta^2 + \eta(\lambda - 1) + \omega = 0$ in Eq. (3.6), and so, $\eta = \frac{1-\lambda\pm\sigma}{2}$ where $\sigma = \sqrt{(\lambda-1)^2 - 4\omega}$.

(i.) If $\eta = \frac{1-\lambda+\sigma}{2}$, we have

$$vG_2 + (1 + \sigma - v)G_1 - \Big(\frac{1+\sigma}{2}\Big)G = 0. \tag{3.7}$$

When we use Eq. (2.16) (*nabla discrete fractional calculus operator*) for all of terms in Eq. (3.7),

$$vG_{2+\nu} + (\nu t + 1 + \sigma - v)G_{1+\nu} - \Big(\nu t + \frac{1+\sigma}{2}\Big)G_\nu = 0. \tag{3.8}$$

Let $\nu t + \frac{1+\sigma}{2} = 0$ in Eq. (3.8). Thus, $\nu = t^{-1}a$ $\Big(a = -\Big(\frac{1+\sigma}{2}\Big)\Big)$, and we obtain

$$H_1 - (av^{-1} + 1)H = 0 \quad \Big(H = H(v) = G_{(1+t^{-1}a)}, G = H_{-(1+t^{-1}a)}\Big). \tag{3.9}$$

The solution of Eq. (3.9) is as follows:

$$H(v) = Av^a e^v,$$

and, by substituting above assumptions, we write

$$F(v) = Av^{-(a+\frac{\lambda}{2})}(v^a e^v)_{-(1+t^{-1}a)}. \tag{3.10}$$

(ii.) By applying similar steps, the second discrete fractional solution is

$$F(v) = Bv^{-(b+\frac{\lambda}{2})}(v^b e^v)_{-(1+t^{-1}b)}. \tag{3.11}$$

where $b = -\Big(\frac{1-\sigma}{2}\Big)$. □

After, the hypergeometric forms of Eq. (3.10) and Eq. (3.11) are exhibited via the following theorems:

Theorem 3. *Let* $\mathbf{F}$ *be the Gauss hypergeometric function and* $|(v^a)_n| < \infty$ $(n \in \mathbf{N}, v \neq 0)$. *Thus, function* $F(v)$ *in Eq. (3.10) is written as*

$$F(v) = Av^{-\frac{\lambda}{2}} e^v \mathbf{F}\Big[1 + t^{-1}a, -a; \frac{1}{v}\Big] \quad \Big(\Big|\frac{1}{v}\Big| < 1\Big). \tag{3.12}$$

Proof. We first apply Eq. (2.1) to Eq. (3.10), and so,

$$F(v) = Av^{-(a+\frac{\lambda}{2})} \sum_{n=0}^{\infty} \frac{\Gamma(-t^{-1}a)}{\Gamma(-t^{-1}a - n)n!}(v^a)_n (e^v)_{-(1+t^{-1}a+n)}. \tag{3.13}$$

And, the following form is written by means of (2.2) and (2.4):

$$F(v) = Av^{-\frac{\lambda}{2}} e^v \sum_{n=0}^{\infty} \frac{\Gamma(n + 1 + t^{-1}a)}{\Gamma(1 + t^{-1}a)} \frac{\Gamma(n - a)}{\Gamma(-a)} \frac{1}{n!}\Big(\frac{1}{v}\Big)^n.$$

Finally, we have

$$F(v) = Av^{-\frac{\lambda}{2}} e^v \sum_{n=0}^{\infty} (1 + t^{-1}a)_n (-a)_n \frac{1}{n!} \Big(\frac{1}{v}\Big)^n,$$

and,

$$F(v) = Av^{-\frac{\lambda}{2}} e^v \mathbf{F}\Big[1 + t^{-1}a, -a; \frac{1}{v}\Big].$$

□

Theorem 4. *Let* $\mathbf{F}$ *be the Gauss hypergeometric function and* $|(v^b)_n| < \infty$ $(n \in \mathbf{N}, v \neq 0)$*. Thus, function* $F(v)$ *in Eq. (3.11) is written as*

$$F(v) = Bv^{-\frac{\lambda}{2}} e^v \mathbf{F}\Big[-a, 1 + t^{-1}a; \frac{1}{v}\Big] \quad \Big(\Big|\frac{1}{v}\Big| < 1\Big). \tag{3.14}$$

Conclusion

We first transform radial part of the Schrödinger equation that is the most important equation of quantum physics into a singular differential equation in order to apply the nabla discrete fractional calculus operator. After, we obtain hypergeometric forms of the discrete fractional solutions by using some properties of the fractional calculus. Thus, we introduce a new method for these kind of equations and exhibit different forms of the solutions.

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