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EXISTENCE RESULTS FOR HYBRID DIFFERENTIAL EQUATION WITH GENERALIZED FRACTIONAL DERIVATIVE

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Abstract. Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary order (non-integer). In recent years, fractional differential equations(FDEs) arise naturally in various fields such as rheology, fractals, chaotic dynamics, modelling and control theory, signal processing, bioengineering and biomedical applications, etc. In this paper, we discuss the existence results for hybrid differential equation with Katugampola fractional derivative. The argument is based upon Dhage fixed point theorem. We also discuss the existence result for hybrid differential equation.

1. Introduction

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary order (non-integer). In recent years, fractional differential equations(FDEs) arise naturally in various fields such as rheology, fractals, chaotic dynamics, modelling and control theory, signal processing, bioengineering and biomedical applications, etc. Detailed study on fractional differential equations can be seen in, see [1, 2, 3, 4, 16]. Theory of fractional hybrid differential equation has been extensively studied by many authors [5, 6, 7, 8, 9, 15]. Recently, U. N. Katugampola [10] introduced generalized fractional derivative and it has been studied extensively by some researchers [11, 12, 13, 14].

Consider the hybrid differential equation involving generalized fractional derivative of the form

$$\begin{cases} {}^{\rho}D^{\alpha}\left(\frac{x(t)}{f(t,x(t))}\right) = g(t,x(t)), \quad t \in J := [0,a], \\ \frac{x(t)}{f(t,x(t))}|_{t=0} = x_0, \end{cases} \tag{1.1}$$

where ${}^{\rho}D^{\alpha}_{a_+}$ is Katugampola fractional derivative of order α and $\rho > 0$. Here $f : J \times R \to R| \{0\}$ and $g : J \times R \to R$ are given continuous function.

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The paper is organized as follows. In Section 2, we present notations and definition used throughout the paper. In Section 3, we discuss the existence result for hybrid differential equation.

2. Preliminary

In this section, we recall some definitions and results from fractional calculus. The following observations are taken from [9, 11]. Throughout this paper, let $C(J)$ a space of continuous functions from J into R with the norm

$$\|x\| = \sup \{|x(t)| : t \in J\}.$$

Definition 2.1. *The generalized left-sided fractional integral* ${}^{\rho}I^{\alpha}_{a^+}f$ *of order* α *is defined by*

$$\left({}^{\rho}I^{\alpha}_{a^+}\right) f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t (t^{\rho} - s^{\rho})^{\alpha-1} s^{\rho-1} f(s) ds, \ t > a, \tag{2.1}$$

if the integral exists.

The generalized fractional derivative, corresponding to the generalized fractional integral (2.1)*, is defined for* $0 \leq a < t$*, by*

$$\left({}^{\rho}D^{\alpha}_{a^+}f\right)(t) = \frac{\rho^{\alpha-n-1}}{\Gamma(n-\alpha)} \left(t^{1-\rho}\frac{d}{dt}\right)^n \int_a^t (t^{\rho} - s^{\rho})^{n-\alpha+1} s^{\rho-1} f(s) ds, \tag{2.2}$$

if the integral exists.

Lemma 2.1. *A function* $x \in C(J)$ *is the solution of fractional initial value problem*

$$\begin{cases} {}^{\rho}D^{\alpha}\left(\frac{x(t)}{f(t,x(t))}\right) = g(t, x(t)), t \in J, \\ \frac{x(t)}{f(t,x(t))}|_{t=0} = x_0, \end{cases}$$

if and only if x *satisfies the following Volterra integral equation*

$$x(t) = f(t, x(t)) \left(x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \left(\frac{t^{\rho} - s^{\rho}}{\rho}\right)^{\alpha-1} s^{\rho-1} g(s, x(s)) ds \right). \tag{2.3}$$

Lemma 2.2. *Let* S *be a non-empty, closed convex and bounded subset of the Banach algebra* X *let* $A : X \to X$ *and* $B : S \to X$ *be two operators such that*

1. A *is Lipschitzian with a Lipschitz constant* k*,*
2. B *is completely continuous,*
3. $x = AxBy \Rightarrow x \in S$ *for all* $y \in S$*, and*
4. $Mk < 1$*, where* $M = \|B(S)\| = \sup\{\|B(x)\| : x \in S\}$*,*

then the operators has a solution.

3. Existence results

We make the following hypotheses to prove our main results.

(H1) The function $f : J \times R \to R|\{0\}$, there exixts a constant $L > 0$, such that

$$|f(t, x(t)) - f(t, y(t))| \leq L(|x(t) - y(t)|),$$

for $t \in J$ and for all $x, y \in R$.

(H2) There exists a function $h : J \to R$, such that

$$|g(t, x(t))| \leq h(t), \ \forall \ t \in J, \ x \in R.$$

(H3)

$$r \geq K\left(x_0 + \left(\frac{a^\rho}{\rho}\right)^\alpha \frac{1}{\Gamma(\alpha+1)}\|h\|_C\right) \tag{3.1}$$

where $|f(t,x)| \leq K,\ \forall\ t \in J,\ x \in R$.

$$L\left(\frac{x_0}{\Gamma(\gamma)} + \left(\frac{a^\rho}{\rho}\right)^\alpha \frac{B(\gamma,\alpha)}{\Gamma(\alpha)}\|h\|_C\right) < 1.$$

Theorem 3.1. *Assume that [H1]-[H3] are satisfied. Then,* (1.1) *has solution on* J.

Proof. We define a subset S of X by

$$S = \{x \in X : \|x\|_C \leq r\}$$

where r satisfies inequality

$$r \geq K\left(x_0 + \left(\frac{a^\rho}{\rho}\right)^\alpha \frac{1}{\Gamma(\alpha+1)}\|h\|_C\right),$$

where $|f(t,x)| \leq K$.
Clearly S is closed, convex and bounded subset of the Banach space X. By Lemma 2.1 the initial value problem (1.1)

$$x(t) = f(t,x(t))\left(x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} g(s,x(s))ds\right). \tag{3.2}$$

Define two operators $A : X \to X$ by

$$Ax(t) = f(t,x(t)), \tag{3.3}$$

and $B : S \to X$ by

$$Bx(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} g(s,x(s))ds. \tag{3.4}$$

Then $x = AxBx$. We shall show that the operators A and B satisfy all the condition of Lemma 2.2. We split the proof into a sequence of steps.
Step 1. The operator A is a Lipschitz on X.

$$\begin{aligned} |(Ax(t) - Ay(t))| &= |f(t,x(t)) - f(t,y(t))| \\ &\leq L\,|(x(t) - y(t))| \\ &\leq L\,\|x - y\|_C, \end{aligned}$$

which implies

$$\|Ax - Ay\| \leq L\,\|x - y\|_C.$$

Step 2. The Operator B is completely continuous on S.
First, we show that B is continuous on S. Let $\{x_n\}$ be a sequence in S convergent

to a point $x \in S$. Then by Lebesgue dominated convergence theorem,

$$\begin{aligned}
\lim_{n\to\infty} Bx_n(t) &= \lim_{n\to\infty}\left(x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} g(s, x_n(s))ds\right)\\
&= x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} \lim_{n\to\infty} g(s, x_n(s))ds\\
&= x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} g(s, x(s))ds\\
&= Bx(t).
\end{aligned}$$

This shows that B is continuous on S. It is sufficient to show that B(S) is uniformly bounded and equicontinuous set in X. First we note that

$$\begin{aligned}
|Bx(t)| &= \left|x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} g(s, x(s))ds\right|\\
&\leq x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} \left|g(s, x(s))\right| ds\\
&\leq x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} \left|h(s)\right| ds\\
&\leq x_0 + \left(\frac{a^\rho}{\rho}\right)^{\alpha} \frac{1}{\Gamma(\alpha+1)} \left\|h\right\|_C,
\end{aligned}$$

for all $t \in J$.

$$\left\|Bx\right\|_C \leq x_0 + \left(\frac{a^\rho}{\rho}\right)^{\alpha} \frac{1}{\Gamma(\alpha+1)} \left\|h\right\|_C.$$

This shows that B is uniformly bounded on S.
Next, we show that B is an equicontinuous set in X. Let $t_1, t_2 \in J$ with $t_1 < t_2$ and $x \in S$. Then we have

$$|Bx(t_1) - Bx(t_2)| \leq \frac{1}{\Gamma(\alpha+1)} \left\|h\right\|_C \left(\left(\frac{t_1^\rho}{\rho}\right)^{\alpha} - \left(\frac{t_2^\rho}{\rho}\right)^{\alpha}\right).$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in S$ as $t_1 - t_2 \to 0$. Therefore, it follows fom the Arzela-Ascoli theorem that B is a completely continuous operator on S.
Step 3. Next we prove that (3) of Lemma 2.2.

$$\begin{aligned}
|x(t)| &= \left|f(t, x(t))\left(x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} g(s, x(s))ds\right)\right|\\
&\leq |f(t, x(t))|\left(x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} \left|g(s, x(s))\right| ds\right)\\
&\leq K\left(x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} \left|h(s)\right| ds\right)\\
&\leq K\left(x_0 + \left(\frac{a^\rho}{\rho}\right)^{\alpha} \frac{1}{\Gamma(\alpha+1)} \left\|h\right\|_C\right).
\end{aligned}$$

Thus, we obtain

$$\|x\|_C \leq K\left(x_0 + \left(\frac{a^\rho}{\rho}\right)^\alpha \frac{1}{\Gamma(\alpha+1)}\|h\|_C\right) \leq r.$$

Step 4. Now, we show that $Mk < 1$, that is (4) of Lemma 2.2 holds.
Thus we have

$$M = \|B(s)\| = \sup\{\|Bx : x \in S\|\} \leq x_0 + \left(\frac{a^\rho}{\rho}\right)^\alpha \frac{1}{\Gamma(\alpha+1)}\|h\|_C \leq r,$$

and $k = L$. Thus, all the conditions of Lemma 2.2 are satisfied and hence the operator equation $x = AxBx$ has a solution in S. In consequence, the problem (1.1) has a solution on J. This complete the proof. □

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