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### 3×3 Dimensional Special Matrices Associated with Fibonacci and Lucas Numbers

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#### ABSTRACT

In the study, it has been developed a method for deriving special matrices of 3×3 dimensions, whose powers are related to Fibonacci and Lucas numbers, and some special matrices have been found via the method developed.

**Keywords:** Fibonacci numbers, Lucas numbers, Matrices

#### 1. INTRODUCTION

Fibonacci numbers and the golden ratio have attracted the attention of many mathematicians, physicists, philosophers, painters, architects and musicians since ancient times. The golden ratio, which is related to Fibonacci numbers, is encountered in many art works. The most known of them is the ratio which is seen in the table Mona Lisa of Leonardo Da Vinci. Again, it is possible to see the golden ratio in Egyptian pyramids. Also, it is known that the ratio sequence of numbers obtained by dividing each number in the Fibonacci sequence by previous number in the sequence converges to the golden ratio. Fibonacci numbers and golden ratio are seen in many places in nature [see, e.g., 1,2].

There are also special number sequences other than Fibonacci number sequence. One of them is the Lucas numbers sequence. Fibonacci numbers, Lucas numbers and golden ratio have been involved in many mathematical studies for many years [see, e.g., 3-8].

Fibonacci and Lucas numbers are also correlated with matrices. It is known that some properties of Fibonacci and Lucas numbers can be proved by using matrices [see, e.g., 1,2].

This study has two-stages: First, an approach to the derivation of 3×3 dimensional matrix whose powers are related to Fibonacci and Lucas numbers is presented. Then, based on this approach, some special 3×3 dimensional matrices are obtained and some related identities are given.

#### 2. PRELIMINARIES

In this section, some basic concepts and properties, which will be used in the study, related to Fibonacci and Lucas numbers sequences, are given.

**Definition 2.1.** Fibonacci sequence  $\{F_n\}$  is defined by  $F_n = F_{n-1} + F_{n-2}$  for all integers  $n \geq 2$  with initial conditions  $F_0 = 0$  and  $F_1 = 1$  [1].

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**Definition 2.2.** Lucas sequence  $\{L_n\}$  is defined by  $L_n = L_{n-1} + L_{n-2}$  for all integers  $n \geq 2$  with initial conditions  $L_0 = 2$  and  $L_1 = 1$  [1].

**Definition 2.3.** The special number  $\alpha = \frac{1+\sqrt{5}}{2}$  is called as golden ratio [1].

The number  $\alpha$  is the positive root of the equation  $x^2 - x - 1 = 0$ . The negative root of this equation is the number  $\beta = \frac{1-\sqrt{5}}{2}$  [1].

Some of the properties related to Fibonacci numbers, Lucas numbers and golden ratio are as follows:

**Theorem 2.4. (Binet Formula)** For all non-negative integer  $n$ , the identity  $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$  holds [1].

It can be derived Fibonacci sequence with negative subscripts from the Fibonacci sequence considering the fact that every term is the sum of the two terms preceding it:

$$\begin{aligned} F_{-1} &= F_1 - F_0 = 1 - 0 = 1, \\ F_{-2} &= F_0 - F_{-1} = 0 - 1 = -1, \\ F_{-3} &= F_{-1} - F_{-2} = 1 - (-1) = 2, \\ F_{-4} &= F_{-2} - F_{-3} = -1 - 2 = -3, \end{aligned}$$

and so on.

Similarly, it can be derived Lucas sequence with negative subscripts from Lucas sequence:

$$\begin{aligned} L_{-1} &= L_1 - L_0 = 1 - 2 = -1, \\ L_{-2} &= L_0 - L_{-1} = 2 - (-1) = 3, \\ L_{-3} &= L_{-1} - L_{-2} = -1 - 3 = -4, \\ L_{-4} &= L_{-2} - L_{-3} = 3 - (-4) = 7, \end{aligned}$$

and continue like this.

**Theorem 2.5.** The identities  $F_{-n} = (-1)^{n+1} F_n$  and  $L_{-n} = (-1)^n L_n$  hold for all integers  $n \geq 1$  [1].

**Theorem 2.6.** The identity  $L_n = F_{n-1} + F_{n+1}$  holds for all integers  $n \geq 1$  [1].

**Theorem 2.7.** The equalities  $\alpha^n = F_n \alpha + F_{n-1}$  and  $\beta^n = F_n \beta + F_{n-1}$  hold for all integers  $n \geq 0$  [1].

### 3. SOME SPECIAL MATRICES

In this section, as indicated in the Introduction section, by giving an approach for derivating  $3 \times 3$  dimensional matrices associated with Fibonacci and Lucas numbers, based on this approach, special matrices are obtained.

Suppose that  $S = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$  is any  $3 \times 3$  matrix

such that the eigenvalues of it are  $\alpha = \frac{1+\sqrt{5}}{2}$ ,

$\beta = \frac{1-\sqrt{5}}{2}$ , and  $\gamma = 0$ , and all the entries of it are

integers. Represent the eigenvectors corresponding the eigenvalues  $\alpha$ ,  $\beta$ , and  $\gamma$ ,

respectively as  $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ ,  $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ , and  $\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$

For the eigenvalue-eigenvector pairs  $(\alpha, \mathbf{x})$ ,  $(\beta, \mathbf{y})$ , and  $(\gamma, \mathbf{z})$ , respectively, it is obtained the linear equations systems

$$\begin{aligned} ax_1 + bx_2 + cx_3 &= \alpha x_1 \\ dx_1 + ex_2 + fx_3 &= \alpha x_2, \\ gx_1 + hx_2 + ix_3 &= \alpha x_3 \end{aligned} \tag{3.1}$$

$$\begin{aligned} ay_1 + by_2 + cy_3 &= \beta y_1 \\ dy_1 + ey_2 + fy_3 &= \beta y_2, \\ gy_1 + hy_2 + iy_3 &= \beta y_3 \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} az_1 + bz_2 + cz_3 &= 0 \\ dz_1 + ez_2 + fz_3 &= 0. \\ gz_1 + hz_2 + iz_3 &= 0 \end{aligned} \quad (3.3)$$

The matrix  $S$  is diagonalizable since it has different eigenvalues. In this case, without loss of the generality, it can be written  $S = P\Lambda P^{-1}$ , where

$$\Lambda = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and  $P$  is a nonsingular matrix. From this, we get  $S^n = P\Lambda^n P^{-1}$  for all integers  $n \geq 1$ . Considering Theorem 2.7, it is obtained

$$\begin{aligned} S^n &= P \begin{pmatrix} \alpha^n & 0 & 0 \\ 0 & \beta^n & 0 \\ 0 & 0 & 0 \end{pmatrix} P^{-1} \\ &= P \begin{pmatrix} F_n \alpha + F_{n-1} & 0 & 0 \\ 0 & F_n \beta + F_{n-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} P^{-1} \\ &= P(F_n \Lambda + F_{n-1} I - F_{n-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - F_{n-1} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}) P^{-1} \\ &= P(F_n \Lambda + F_{n-1} I - F_{n-1} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}) P^{-1} \\ &= F_n (P\Lambda P^{-1}) + F_{n-1} (PIP^{-1}) - F_{n-1} (P \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} P^{-1}), \end{aligned}$$

that is,

$$S^n = F_n S + F_{n-1} I - F_{n-1} (P \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} P^{-1}). \quad (3.4)$$

Thus, it is seen that the power of the matrix  $S$  is associated with Fibonacci numbers.

Now, if we write

$$D = P \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} P^{-1},$$

then we get

$$S^n = F_n S + F_{n-1} (I - D). \quad (3.5)$$

Since

$$D = \begin{pmatrix} 0 & 0 & z_1 \\ 0 & 0 & z_2 \\ 0 & 0 & z_3 \end{pmatrix} P^{-1}$$

and

$$P^{-1} = \frac{1}{|P|} \begin{pmatrix} y_2 z_3 - z_2 y_3 & z_1 y_3 - y_1 z_3 & y_1 z_2 - z_1 y_2 \\ z_2 x_3 - x_2 z_3 & x_1 z_3 - z_1 x_3 & z_1 x_2 - x_1 z_2 \\ x_2 y_3 - y_2 x_3 & y_1 x_3 - x_1 y_3 & x_1 y_2 - y_1 x_2 \end{pmatrix},$$

it is obtained

$$D = \frac{1}{|P|} \begin{pmatrix} z_1(x_2 y_3 - y_2 x_3) & z_1(y_1 x_3 - x_1 y_3) & z_1(x_1 y_2 - y_1 x_2) \\ z_2(x_2 y_3 - y_2 x_3) & z_2(y_1 x_3 - x_1 y_3) & z_2(x_1 y_2 - y_1 x_2) \\ z_3(x_2 y_3 - y_2 x_3) & z_3(y_1 x_3 - x_1 y_3) & z_3(x_1 y_2 - y_1 x_2) \end{pmatrix}. \quad (3.6)$$

A writing of the determinant of the matrix  $P$  is

$$|P| = z_1(x_2 y_3 - y_2 x_3) + z_2(y_1 x_3 - x_1 y_3) + z_3(x_1 y_2 - y_1 x_2).$$

Hence, it can be provided that the entries of the matrix  $D$  are also integers with the appropriate choices of the terms  $(x_2 y_3 - y_2 x_3)$ ,  $(y_1 x_3 - x_1 y_3)$ , and  $(x_1 y_2 - y_1 x_2)$ .

Now, it is given some special matrices  $S$  which occur for the special cases of the eigenvectors  $\mathbf{x}$ ,  $\mathbf{y}$ , and  $\mathbf{z}$  and some identities associated with these.

If it is chosen

$$\mathbf{x}_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ -1 \end{pmatrix}, \quad \mathbf{y}_1 = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \\ -1 \end{pmatrix},$$

then, it is necessary to hold the systems (3.1), (3.2), and (3.3) for being these vectors are eigenvectors associated with the eigenvalues  $\alpha$ ,  $\beta$ , and  $\gamma$ , respectively, of the matrix  $S$ . From these, the following equations must be hold:

$$\begin{cases} a+b-2c=3 \\ a-b=1 \end{cases}, \quad (3.7)$$

$$\begin{cases} d+e-2f=-2 \\ e-d=0 \end{cases}, \quad (3.8)$$

and

$$\begin{cases} g+h-2i=-1 \\ g-h=-1 \end{cases}. \quad (3.9)$$

Now, let us consider the different choices of the vector  $\mathbf{z}$ .

For example, if it is chosen as  $z_1 = k$ ,  $z_2 = -k$ , and  $z_3 = k$  where  $k \in \mathbb{Z} \setminus \{0\}$  is arbitrary, then the matrix  $D$  in (3.6) is obtained as

$$D = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix},$$

and also, the equation (3.3) turns to the system

$$\begin{cases} a-b+c=0 \\ d-e+f=0 \\ g-h+i=0 \end{cases}. \quad (3.10)$$

From the solutions of the equations systems (3.7), (3.8), (3.9), and (3.10), we get

$$S = \begin{pmatrix} 1 & 0 & -1 \\ -1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}. \quad (3.11)$$

If we use the equality (3.5) for the matrix  $S$  in (3.11), then we obtain

$$\begin{aligned} S^n &= F_n S + F_{n-1} (I - D) \\ &= \begin{pmatrix} F_n & -F_{n-1} & -F_n - F_{n-1} \\ -F_n + F_{n-1} & -F_n + 2F_{n-1} & F_{n-1} \\ -F_{n-1} & F_n - F_{n-1} & F_n \end{pmatrix} \\ &= \begin{pmatrix} F_n & -F_{n-1} & -F_{n+1} \\ -F_{n-2} & F_{n-3} & F_{n-1} \\ -F_{n-1} & F_{n-2} & F_n \end{pmatrix} \end{aligned}$$

for all integers  $n \geq 1$ . Thus, we have been proved the following theorem:

**Theorem 3.1.** If  $S = \begin{pmatrix} 1 & 0 & -1 \\ -1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ , then,

$$S^n = \begin{pmatrix} F_n & -F_{n-1} & -F_{n+1} \\ -F_{n-2} & F_{n-3} & F_{n-1} \\ -F_{n-1} & F_{n-2} & F_n \end{pmatrix} \quad \text{for all integers } n \geq 1.$$

Since one of the eigenvalue of the matrix  $S$  is zero, the matrix  $S$  is singular. So, the result obtained is valid for only all integers  $n \geq 1$ .

Taking different choices of the eigenvectors  $\mathbf{x}$ ,  $\mathbf{y}$ , and  $\mathbf{z}$ , and progressing similarly to the above, it can be given different matrices  $S$  and related results. Now, it is presented, without proof, some of these kind of results, for clarification.

For the choice of  $(\alpha, \begin{pmatrix} \beta \\ -1 \end{pmatrix})$ ,  $(\beta, \begin{pmatrix} \alpha \\ -1 \end{pmatrix})$ , and

$$(0, \begin{pmatrix} k \\ k \\ -k \end{pmatrix}):$$

**Theorem 3.2.**  $S = \begin{pmatrix} -1 & -2 & -3 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}$  and

$$S^n = \begin{pmatrix} -F_n & -F_{n+2} & -L_{n+1} \\ F_{n-2} & F_n & L_{n-1} \\ F_{n-1} & F_{n+1} & L_n \end{pmatrix} \quad \text{for all integers } n \geq 1.$$

For the choice of  $(\alpha, \begin{pmatrix} \beta \\ -1 \end{pmatrix})$ ,  $(\beta, \begin{pmatrix} \alpha \\ -1 \end{pmatrix})$ , and

$$(0, \begin{pmatrix} -k \\ k \\ k \end{pmatrix}):$$

**Theorem 3.3**  $S = \begin{pmatrix} 3 & 2 & 1 \\ -1 & -1 & 0 \\ -2 & -1 & -1 \end{pmatrix}$  and For the choice of  $(\alpha, \begin{pmatrix} \alpha \\ -\beta \\ 1 \end{pmatrix})$ ,  $(\beta, \begin{pmatrix} \beta \\ -\alpha \\ 1 \end{pmatrix})$ , and  $(0, \begin{pmatrix} k \\ -k \\ k \end{pmatrix})$ :

$$S^n = \begin{pmatrix} F_{n+3} & F_{n+2} & F_{n+1} \\ -F_{n+1} & -F_n & -F_{n-1} \\ -F_{n+2} & -F_{n+1} & -F_n \end{pmatrix}$$
 for all integers  $n \geq 1$ .

For the choice of  $(\alpha, \begin{pmatrix} \alpha \\ -\beta \\ 1 \end{pmatrix})$ ,  $(\beta, \begin{pmatrix} \beta \\ -\alpha \\ 1 \end{pmatrix})$ , and  $(0, \begin{pmatrix} k \\ k \\ k \end{pmatrix})$ :

**Theorem 3.6.**  $S = \begin{pmatrix} -1 & 2 & 3 \\ -1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}$  and

$$S^n = \begin{pmatrix} -F_n & F_{n+2} & L_{n+1} \\ -F_{n-2} & F_n & L_{n-1} \\ -F_{n-1} & F_{n+1} & L_n \end{pmatrix}$$
 for all integers  $n \geq 1$ .

**Theorem 3.4.**  $S = \begin{pmatrix} 3 & -2 & -1 \\ 1 & -1 & 0 \\ 2 & -1 & -1 \end{pmatrix}$  and For the choice of  $(\alpha, \begin{pmatrix} -\alpha \\ \beta \\ 1 \end{pmatrix})$ ,  $(\beta, \begin{pmatrix} -\beta \\ \alpha \\ 1 \end{pmatrix})$ , and  $(0, \begin{pmatrix} k \\ k \\ k \end{pmatrix})$ :

$$S^n = \begin{pmatrix} F_{n+3} & -F_{n+2} & -F_{n+1} \\ F_{n+1} & -F_n & -F_{n-1} \\ F_{n+2} & -F_{n+1} & -F_n \end{pmatrix}$$
 for all integers  $n \geq 1$ .

For the choice of  $(\alpha, \begin{pmatrix} \alpha \\ -\beta \\ 1 \end{pmatrix})$ ,  $(\beta, \begin{pmatrix} \beta \\ -\alpha \\ 1 \end{pmatrix})$ , and  $(0, \begin{pmatrix} k \\ k \\ -k \end{pmatrix})$ :

**Theorem 3.7.**  $S = \begin{pmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$  and

$$S^n = \begin{pmatrix} F_n & F_{n-1} & -F_{n+1} \\ F_{n-2} & F_{n-3} & -F_{n-1} \\ -F_{n-1} & -F_{n-2} & F_n \end{pmatrix}$$
 for all integers  $n \geq 1$ .

**Theorem 3.5.**  $S = \begin{pmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$  and For the choice  $(\alpha, \begin{pmatrix} -\alpha \\ \beta \\ 1 \end{pmatrix})$ ,  $(\beta, \begin{pmatrix} -\beta \\ \alpha \\ 1 \end{pmatrix})$ , and  $(0, \begin{pmatrix} k \\ k \\ -k \end{pmatrix})$ :

$$S^n = \begin{pmatrix} F_n & F_{n-1} & F_{n+1} \\ F_{n-2} & F_{n-3} & F_{n-1} \\ F_{n-1} & F_{n-2} & F_n \end{pmatrix}$$
 for all integers  $n \geq 1$ .

**Theorem 3.8.**  $S = \begin{pmatrix} 3 & -2 & 1 \\ 1 & -1 & 0 \\ -2 & 1 & -1 \end{pmatrix}$  and

$S^n = \begin{pmatrix} F_{n+3} & -F_{n+2} & F_{n+1} \\ F_{n+1} & -F_n & F_{n-1} \\ -F_{n+2} & F_{n+1} & -F_n \end{pmatrix}$  for all integers  
 $n \geq 1$ .

For the choice  $(\alpha, \begin{pmatrix} -\alpha \\ \beta \\ 1 \end{pmatrix})$ ,  $(\beta, \begin{pmatrix} -\beta \\ \alpha \\ 1 \end{pmatrix})$ , and

$(0, \begin{pmatrix} -k \\ k \\ k \end{pmatrix})$  :

**Theorem 3.9.**  $S = \begin{pmatrix} -1 & 2 & -3 \\ -1 & 1 & -2 \\ 0 & -1 & 1 \end{pmatrix}$  and

$S^n = \begin{pmatrix} -F_n & F_{n+2} & -L_{n+1} \\ -F_{n-2} & F_n & -L_{n-1} \\ F_{n-1} & -F_{n+1} & L_n \end{pmatrix}$  for all integers  
 $n \geq 1$ .

Note that Teorem 2.6 has been used in Teorem 3.2, Teorem 3.6, and Teorem 3.9.

Based on the approach given here, it is seen that it can be written any finite dimensional matrix and

related results. Although the approach is simple, we believe that it is important in terms of its role and useful in the study related to these subjects.

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