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Fixed Point Iteration Method

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Abstract We discuss the problem of finding approximate solutions of the equation

$$f(x) = 0 \qquad (1)$$

In some cases it is possible to find the exact roots of the equation (1) for example when $f(x)$ is a quadratic on cubic polynomial otherwise, in general, is interested in finding approximate solutions using some numerical methods. Here, we will discuss a method called fixed point iteration method and a particular case of this method called Newton's method

Keywords:

1. INTRODUCTION

In this section we consider methods for determining the solution to an equation expressed, for some functions g .in the form

$$g(x) = x \quad (2)$$

A solution to such an equation is said to be a fixed point of the function g. Let's we found a fixed point for any given g. Then every root finding problem could also be solved for example. The root finding problem $f(x) = 0$ has solutions that correspond precisely to the fixed points of $g(x) = x$ when $g(x) = x - f(x)$. The first task, then, is to decide when a function will have a fixed point and how the fixed points can be determined. (In numerical analysis, "determined" generally means approximated to a sufficient degree of accuracy.)

EXAMPLE 1.

(a) The function $g(x) = x, 0 \le x \le 1$ has a fixed point at each x in $[0,1]$.

(b) The function $g(x) = x - \sin \pi$ has exactly two fixed points in $[0,1]$. x = 0 and x = 1. (see figure 1.1)

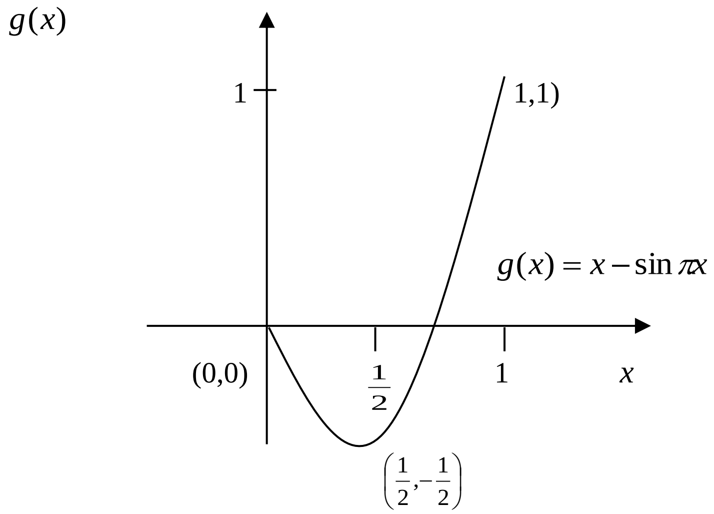

Figure 1.1.

The following theorem gives sufficient conditions for the existence and uniqueness of a fixed point.

Theorem 1.1.

If $g \in [a,b]$ and $g(x) \in [a,b]$. then g has a fixed point in $[a,b]$. Further, suppose $g'(x)$ exists on $[a,b]$ and then a positive constant $k < 1$ exists with

(1.1) $$|g'(x)| \le k < 1 \qquad \text{for all } x \in (a,b).$$

Then g has a unique fixed point p in $[a,b]$. (see figure 1.1)

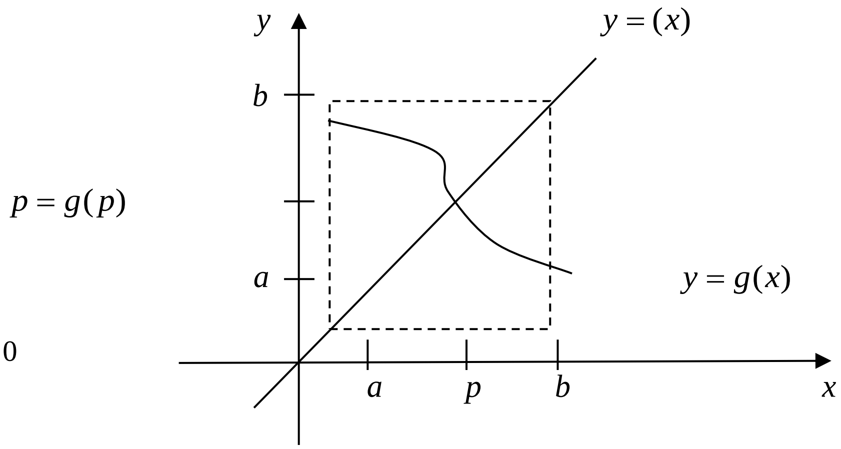

Figure 1.1.

Proof: if $g(a) = a$ or $g(b) = b$, the existence of a fixed point is obvious. Suppose not; then it must be true that $g(a) > a$ and $g(b) < b$. Decline $h(x) = g(x) - x$. Then his continuous on $[a,b]$ and

$h(a) = g(a) - a > 0, h(b) = g(b) - b < 0$

The intermediate value theorem implies that there exists $p \in (a,b)$ for which $h(p) = 0$ thus, $g(p) - p = 0$ and p is a fixed point of g.

Suppose in addition that inequality (1.1) holds and that p and q are both fixed points in $[a,b]$ with $p \neq q$. by the mean value theorem a number ξ exists between p and q. And hence in $[a,b]$ with.

$|p-q| = |g(p) - g(q)| = |g'(f)|\ |p-q| \leq k|p-q| < |p-q|$

Which is a contradiction this contradiction must come from the only supposition $p \neq q$ .hence $p = q$ and the fixed point in $[a,b]$ is unique

EXAMPLE 2.

(a) Let $g(x) = (x^2\text{-}1)/3$ on [-1, 1] using the extreme value theorem, it is easy to show that the absolute minimum or g occurs at $x = 0$ and $g(0) = -\frac{1}{3}$. Similarly. The absolute maximum of g occurs at $x = \pm 1$ and has the value $g(\pm 1) = 0$.moreover. g is continuous and

$|g'(x)| = \left|\frac{2x}{3}\right| \leq \frac{2}{3}$ for all $x \in [-1,1]$.

So g satisfies the hypotheses of theorem 1.1 and has a unique fixed in [-1, 1].

In this example the unique fixed point p in the interval [-1, 1] can be determined exactly. If

$P = g(p) = \frac{p^2 - 1}{3}$, then $p^2 - 3_p - 1 = 0$

Which by the quadratic Formula implies that?

$p = \frac{3 - \sqrt{13}}{2}$.

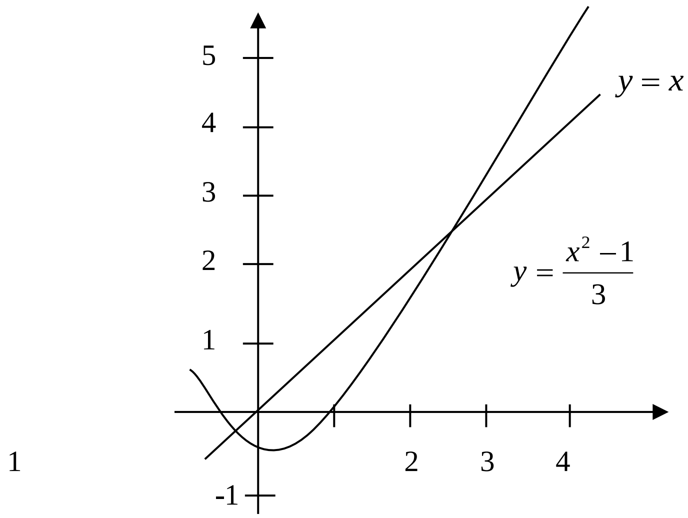

Figure 1.2.

That g also has a unique fixed point p = (3 + $\sqrt{(13)}$ / 2 for interval [3,4] forever $g(4) = 5$ and $g'(4) = \frac{1}{3} > 1 : so$ g does not satisfy their hypotheses of theorem 1.1 this shows that the hypotheses of theorem 1.1 sufficient guarantee a unique fixed point, but are not necessary. (see figure 1.2).

$G(x) =$ 3^{-x}. since $g'(x) = -3^{-x} \ln 3 < 0 = on[.0.1]$, the function this decreasing [0,1] hence g (1) = $\frac{1}{3} \leq g(x) \leq 1 = g(0)\, for\, 0 \leq x \leq 1$. this for $x \in [0,1]$ $g(x)$ $\in [0,1]$ therefore, g has a fixed point in [0,1] since

g' (0) = - in 3 = -1.098612289

$f(x) \not\leq 1$ on [0, 1] theorem 1.1 cannot be used determinant unequation forever g is decreasing so it is clear that the fixed point must the unique (see figure 1.3)

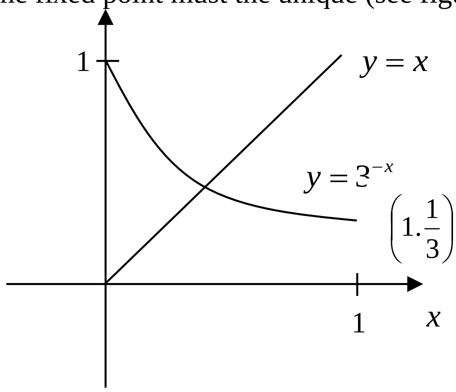

Figure 1.3.

Approximate point of a function g,we choose an initial information p and sequence $\{p_n\}^1{}_n = 0$ by letting $p_n = q(p_{n-1})$ h $n \geq 1$ if the for p and g is continuous then by

Theorem 1.2

$$p = \lim_{n\to\infty} p_n = \lim_{n\to\infty} g(p_{n-1}) = g(\lim_{n\to\infty} p_{n-1}) = g(p)$$

and a solution to $x = g(x)$ is obtained this technique is called fixed – point or functional iteration the procedure is detailed in algorithm 1.2 and described in figure 1.4

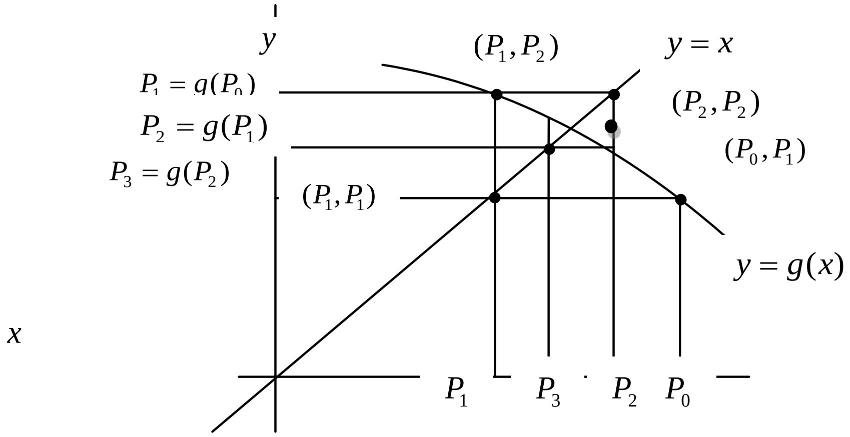

Figure 1.4

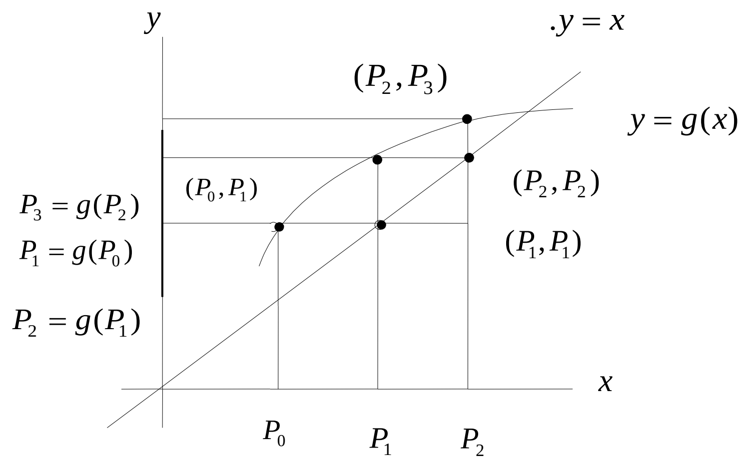

Figure 1.5

FIXED – POINT ALGORITHM 1

To find a solution to $p = g(p)$ given an initial approximation p_0 : INPUT initial approximation p_0; tolerance TOL; maximum number of iterations no: OUTPUT approximate solution p or message failure.

Step 1 set i = 1.

Step 2 white i $\leq N_0$

Step 3 set $p = g\left(p_0\right)$. (compare p.)

Step 4 if $\left|p - p_0\right| < TOL$ then

OUTPUT (P), (Procedure completed successfully)
STOP.
Step 5 set i = i + 1.
Step 6 set p_0 = p. (Update p_0)
Step 7 OUTPUT (Method failed after N_0 iterations $N_0 = N_0$;
(Procedure completed unsuccessfully.)
STOP.
To illustrate the technique of functional iteration consider the following example.

EXAMPLE 3.

a) Let us take the problem given where $g(x) = \frac{1}{7}(x^3 + 2)$. Then $g:[0,1] \to [0,1]$ and $|g'(x)| < \frac{3}{7}$ for all $x \in [0,1]$. Home by the previous theorem sequence P_n defined by the process $P_{n+1} = \frac{1}{7}(P_n^{\,3} + 2)$ converges to a root of $x^3 - 7x + 2 = 0$

b) Consider $f:[0,2] \to R$ defined by $f(x) = (1+x)^{\frac{1}{5}}$. Observe that f maps [0, 2] onto itself. Moreover $|f'(x)| \le \frac{1}{5} < 1$ for $x \in [0,2]$. By the previous theorem the sequence (P_n) defined by $P_{n+1} = (1+P_n)^{1/5}$ converges to a root of $x^2 - x - 1 = 0$ in the interval [0,2]

In practice, it is often difficult to check the condition $f([a,b] \le [a,b])$ given in the previous theorem. We now present a variant of theorem.

Theorem 1.2. (Fixed point theorem) let $g \in [a,b]$ and suppose that $g(x) \in [a,b]$ for all x in $[a,b]$. further,

Suppose g' exists on $[a,b]$ with

$|g'(x)| \le k < 1$ for all $x \in (a,b)$

If p_0 is any number in $[a,b]$ then the sequence defined by

$p_n = g(p_n - 1)$ $n \ge 1$.

Converges to the unique fixed point p in $[a,b]$

Proof by theorem 1.1 a unique fixed point exist in $[a,b]$ since g maps $[a,b]$ into itself the sequence $\{p_n\}_{n=0}^{\infty}$ is defined for all $n \ge 0$ and $p_n \in [a,b]$ for all n. Using inequality and the mean value theorem.

$$|p_n - p| = |g(p_n - 1) - g(p)| = |g'(\xi)||p_{n-1} - p| \le k|p_{n-1} - p|.$$

Where $\xi \in (a,b)$ applying inequality (1.3) inductively gives:

$$|p_n - p| \le k|p_{n-1} - p| \le k^2|p_{n-2} - p| \le .... \le k^n|p_0 - p|.$$

Since $k < 1$,

$$\lim_{n \to \infty}|p_n - p| \le \lim_{n \to \infty} k^n|p_0 - p| = 0$$

and $\{p_n\}_{n=0}^{\infty}$ converges to p.

Corollary 1.3 If g satisfies the hypotheses of theorem 1.2 a bound for the error involve in using p_n to apporoximate p is given by.

$$|p_n - p| \le k^n \max\{p_0 - a, b - p_0\} \qquad \text{for all } n \ge 1.$$

Proof from inequality,

$$|p_n - p| \le k^n |p_0 - p| \le k^n \max\{p_0 - a, b - p_0\},$$

Since $p \in [a,b]$.

Corollary 1.4 If g satisfies the hypotheses of theorem 1.2, then

$$|p_n - p| \le \frac{k^n}{1-k}|p_0 - p_1| \text{ for all } n \ge 1$$

Proof for $n \ge 1$ the procedure used in the proof of theorem 1.2 implies that

$$|p_{n+1} - p_n| = |g(p_n) - g(p_{n-1})| \le k|p_n - p_{n-1}| \le .... \le k_n|p_1 - p_0|$$

Thus, for $m > n \ge 1$

$$|p_m - p_n| = |p_m - p_{m-1} + p_{m-1} - ... + p_{n+1} - p_n|$$

$$\le |p_m - p_{m-1}| + |p_{m-1} - p_{m-2}| + ... + |p_{n+1} - p_n|$$

$$\le k^{m-1}|p_1 - p_0| + k^{m-2}|p_1 - p_0| + ... + k^n|p_1 - p_0|$$

$$= k^n(1 + k + k^2 + ... + k^{m-n-}1)|p_1 - p_0|$$

By theorem 1.2, lim. $p_m = p$ so

$m \to \infty$

$$|p - p_n| = \lim|p_m - p_n| \le k^n|p_1 - p_0|\sum_{p=0}^{\infty} k^p = \frac{k^n}{1-k}|p_1 - p_0|$$

$m \to \infty$

Both corollaries relate the rate of convergence to the bound k on the first derivate it is clear that the rate of convergence depends on the factor $k^n(1-k)$ and that the smaller k can be made the faster the convergence the convergence may be very slow if k is close to 1.In the following example the fixed-point methods in example 3 are reconsidered in light of the results described in theorem 1.2.

EXAMPLE 4.

(a) When $g_1(x) = x - x^3 - 4x^2 + 10, g_1'(x) = 1 \ 3x^2 - 8x$. Then is no interval $[a,b]$ containing p for which $|g_1'(x)| < 1$ though theorem (1.2) does not guarantee that the method must fail for this choice of g, there is no reason to expect convergence.

(b) With $g_2(x) = [(10/x) - 4x]^{1/2}$, we can see that p_2 does not map [1,5] into [1,2] and the sequence $\{p_n\}_{n=0}^{\infty}$ is not defined with p =1.5 moreover there is no interval containing such that $|g_2'(x)| < 1$, since $|g_2'(p)| \approx 3.4$

(c) for the function $g_3(x) = \frac{1}{2}(10 - x^3)^{1/2}$

$$g_3(x) = -\frac{3}{4}x^2\left(10 - x^3\right)^{-1/2} < 0 \quad \text{on } [1,2],$$

So g is strictly decreasing on [1,2] however, $\left|g_3'(2)\right| \approx 2.12,$ so inequality (1.2) does not hold on [1,2].A closer examination of the sequence $\{p_0\}_{n=0}^{\infty}$ starting with $p_0 = 1.5$ will show $g_3'(x) < 0$ and g is strictly decreasing but additionally,

$1 < 1.28 \approx g_3(1.5) \le g_3(x) \le g_3(1) = 1.5$

For all $x \in [1,1.5]$ this shows that g_3 maps the interval [1,1.5] into itself. Since it is also true that $\left|g_3'(x)\right| \le \left|g_3'(1.5)\right| \approx 0.66$ on this interval, theorem 1.2 configures the convergence which we were already aware

(c) for $g_4(x) = \left(\frac{10}{4+x}\right)^{1/2}$,

$$\left|g_4'(x)\right| = \left|\frac{-5}{\sqrt{10}(4+x)^{3/2}}\right| < \frac{5}{\sqrt{10}(5)^{3/2}} < 0.15 \quad \text{for all } [1.2]$$

The bound on the magnitude $g_4'(x)$ is much smaller than the bound on the magnitude of $g_3'(x)$ which explains the more rapid convergence using g_4 the other part of example 3 can be handled in a similar manner.

REMARK: If g is invertible then P is a fixed point of g if and only if q is a fixed point of g^{-1}, in view of this fact, sometimes we can apply the fixed point iteration method for g^{-1} instead of g .For understanding, consider $g(x) = 3x - 21$ then $\left|g'(x)\right| = 3$ for all x. So the fixed point iteration method may not work. However, $g^{-1}(x); = \frac{1}{3}x + 7$ and in this case $\left|(g^{-1})'(x)\right| = \frac{1}{3}$ for all x.

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