MJEN

MANAS Journal of Engineering

Volume 3 (Issue 2) (2015) Pages 48-58

Variational formulation of a boundary-value problem corresponding to forced vibration of an imperfectly bonded bi-layered plate-strip resting on a rigid foundation

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Received: 01.07.2015; Accepted: 16.10.2015

Abstract In the present study, a boundary-value problem corresponding to forced vibration of an imperfectly bonded bi-layered plate-strip resting on a rigid foundation is considered. In the framework of three-dimensional linearized theory of elastic waves in initially stressed bodies, the mathematical modelling of considered problem is given. Then, the variational formulation of the problem considered is obtained in the framework of the principles of calculus of variation. The problem considered differs from the previous studies in the view of imperfect boundary conditions between the layers of the plate-stip and between the plate-strip and the rigid foundation.

Keywords *Forced vibration, plate-strip, variational formulation*

Tam olmayan sınır koşulları için rijit zemin üzerine oturmuş öngerilmeli iki katmanlı şerit-plağın zorlanmış titreşimine karşılık gelen sınır-değer probleminin varyasyonel formülasyonu

Öz Bu çalışmada tam olmayan sınır koşulları için rijit zemin üzerine oturmuş öngerilmeli iki katmanlı bir şerit-plağın zorlanmış titreşimine karşılık gelen bir sınır değer problemi ele alınmıştır. Öncelikle, ilgili problemin öngerilmeli cisimler için elastik dalga yayılımının üç boyutlu lineerleştirilmiş toerisi çerçevesinde matematiksel modeli verilmiştir. Daha sonra incelenen problemin varyasyonel hesap prensipleri çerçevesinde varyasyonel formülasyonu yapılmıştır. İncelenen problem, şerit-plağın katmanları arasında ve şerit-plak ile rijit zemin arasında tam olmayan sınır koşullarını içermesi bakımından önceki çalışmalardan ayrılmaktadır.

Anahtar Sözcükler: *Zorlanmış titreşim, şerit-plak, varyasyonel formülasyon*

INTRODUCTION

One of the major subjects of the computational and applied mathematics is the problems involving non-linear effects in the dynamics of the elastic medium. It is not possible to solve these problems within the framework of the classical linear theory of elastodynamics. The linearized theory of elastodynamics for initially stressed bodies is constructed using the linearization principle from the general nonlinear theory of elasticity or some simplified modifications of it. It is possible to investigate dynamic problems for initially stressed bodies by the linearized equations under certain limitations. Such problems including initially stressed bodies have a wide range of applications. Moreover, wave propagation in abovementioned problems with initially stresses is a very broad and attractive research field. The obtained results of this subject have a wide range of engineering applications such as safety and reliability control of complex structural components by acoustic emission, quantitative nondestructive materials testing by ultrasonics, dynamic fracture mechanics. A large number of theoretical and experimental investigations had been made in this field after 1950s. Eringen and Suhubi give early investigations on the subject [1]. Guz gave an analysis of the obtained results in [2-4], and also, in [5], reviewed the results up to then. Moreover, Akbarov reviewed the investigations on dynamic problems for an elastic body with initial stresses in [6]. Almost all of the investigations in this field were made within the framework of the Three-Dimensional Linearized Theory of Elastic Waves in Initially Stressed Bodies (TDLTEWISB).

Note that the investigations carried out up to now in this field can be divided into two groups: (a) studies related to the wave propagation in the initially stressed bodies, (b) studies related to the time harmonic Lamb's problem for the elastic systems consisting of the pre-stressed half-spaces and with the pre-stressed covering layer. Systematic analyses of results of the investigations in the first group were made in [2-5] by Guz. The review of the investigations obtained before 2007 was made in [6]. Moreover, the review of the recent investigations was detailed in the paper by Akbarov [7]. Investigations of the second group relate to the time harmonic Lamb's problem for the elastic systems consisting of the pre-stressed half-spaces and with the pre-stressed covering layer, as well as to the time harmonic dynamic stress field problem for the system consisting of the pre-stressed two layers resting on a rigid foundation. The subject of the present study regards the time-harmonic dynamic stress field in a bi-layered slab resting on the rigid foundation as in the papers [8-10]. It should be noted that in the papers [8-10], the length and width of the layers in the medium considered are assumed to be infinite. This assumption simplifies the mathematical solutions of the corresponding problems. However, this assumption cannot be applicable for the cases where the thickness and length of the layers in the bi-layered systems are finite. In such cases the corresponding problems for plates with finite length were investigated in [11] for a single-layer plate and in [12] for a bi-layered plate.

Two of the factors on which the dynamical behavior of the layered elastic systems depends significantly are: (a) the imperfectness of the contact on the interface planes between the layers, and also between the plate-strip and the foundation, (b) existing static initial stresses in the layers before the additional dynamical loading. The technological process, as well as constructional requirements caused a defect such as the imperfectness of the contact between layers and between the plate-strip and the foundation. In papers [11-12], the complete contact conditions satisfy that between the layers, as well as between the plate and rigid foundation. However, in most real cases, it is not possible to assume a perfectly bonded interface planes between the constituents of the elastic systems. In order to apply the results of the theoretical investigations related to the forced vibration of the bi-layered plate resting on the rigid foundation to practice real cases, it is necessary to take into account the factor (a) noted above, i.e. the imperfectness of the contacts between the constituents of the considered systems.

Taking the foregoing discussions into account, in the present paper the forced vibration of the imperfectly bonded, pre-stressed bi-layered plate with finite length resting on the rigid foundation is studied within the scope of the piecewise homogeneous body model with the use of the TDLTEWISB. Variational formulation of the considered problem is obtained and the validity of the obtained energy functional is checked under the principle of virtual work. Under this study, it is assumed that the shear-spring type

imperfect contact conditions satisfy both between the layers and between the plate and the rigid foundation.
Note that, the investigations carried out in this paper can be considered as a development of the investigations carried out in [13] for the case where the shear-spring type imperfect contact condition are satisfied between the layers of the plate and between the plate and the rigid foundation.

Statement Of The Problem
Let the bi-layered plate-strip with geometries shown in Fig.1 be given. The positions of the points of that is determined by the Lagrangian coordinates in the Cartesian system of coordinates $Ox_1x_2x_3$. The length of the plate is assumed to be infinite in the direction of Ox_3 and all investigations are made for plane-strain state in the Ox_1x_2 plane. The layers of the plate occupy the regions

$$\Omega_1 = \{-a \le x_1 \le +a; -h_1 < x_2 \le 0\}$$
$$\Omega_2 = \{-a \le x_1 \le +a; -h \le x_2 \le -h_1\} \quad (1)$$

The values related to the upper (lower) layer occupying the region Ω1 (Ω2) will be indicated by an upper index (1) ((2)).

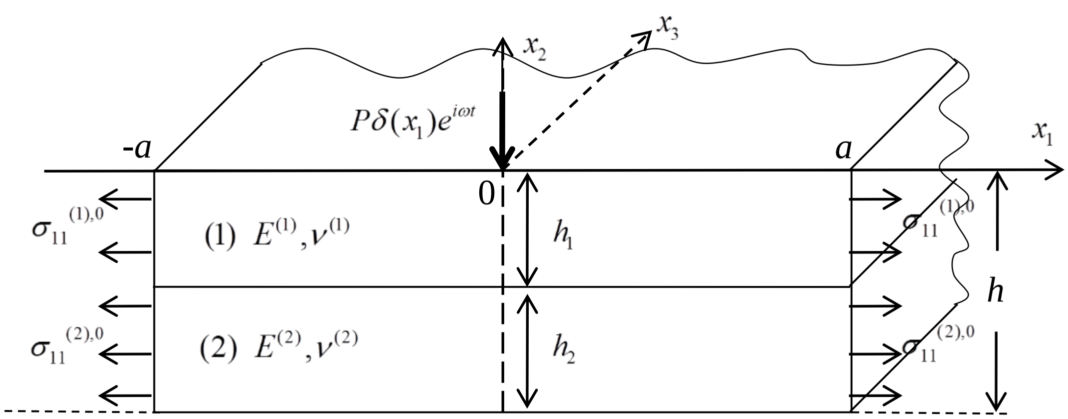

Figure 1. The geometry of the bi-layered plate-strip for the considered problem**.**

Before compounding with one another and with the rigid foundation, the layers are loaded separately with uniformly distributed normal forces acting at the ends of those, as a result of which a uniaxial homogeneous initial stress state appears in each of them. The values related to this initial state will be indicated with additional upper index 0. Assume that the layers' materials are moderately rigid and foregoing initial stress state in the layers is determined within the scope of the classical linear theory of elasticity as follows

$$\sigma_{11}^{(m),0} = c_m; m = 1,2 \quad ,\sigma_{ij}^{(m),0} = 0 \ \text{ for } \ ij \ne 11 \quad (2)$$

where c_m is known constant for each layer. As we consider the case where the initial stress state in the layers is determined by the classical linear theory of elasticity, the distinction between the coordinates regarding the natural and the initial states is so slight that it need not be taken into account.

Thus, given the statements above, we assume that on the upper free face of the upper layer line-located time-harmonic dynamical force acts as shown in Fig.1. This is required to determine the dynamical response of the considered system to this load under the plane-strain state in Ox_1x_2 plane.
According to [4], the equations of motion of TLTEWISB for the small initial deformation considered are

$$\frac{\partial \sigma_{ij}^{(m)}}{\partial x_j} + \sigma_{11}^{(m),0} \frac{\partial^2 u_i^{(m)}}{\partial x_1^2} = \rho^{(m)} \frac{\partial^2 u_i^{(m)}}{\partial t^2} \quad , i; j; m = 1, 2. \tag{3}$$

The materials of the layers are assumed to be isotropic and mechanical relations for those are written as follows

$$\sigma_{ij}^{(m)} = \lambda^{(m)} \theta^{(m)} \delta_{ij} + 2\mu^{(m)} \varepsilon_{ij}^{(m)} \ , \ m = 1, 2 \tag{4}$$

where

$$\varepsilon_{ij}^{(m)} = \frac{1}{2}\left(\frac{\partial u_i^{(m)}}{\partial x_j} + \frac{\partial u_j^{(m)}}{\partial x_i} \right), \ m = 1, 2 \tag{5}$$

In equations (3)-(5) and below conventional notation is used. According to the foregoing discussions, on the upper plane of the upper layer and on the ends of the plate the following boundary conditions satisfy

$$\sigma_{12}^{(1)} \big|_{x_2=0} = 0 \ \ , \ \sigma_{22}^{(1)} \big|_{x_2=0} = -P\delta(x_1) e^{i\omega t} \ , \tag{6}$$

$$\left(\sigma_{11}^{(m),0} \frac{\partial u_i^{(m)}}{\partial x_1} + \sigma_{1j}^{(m)} \right)\Bigg|_{x_1=\pm a} = 0 \ \ , \ \sigma_{12}^{(m)} \big|_{x_1=\pm a} = 0 \ , m; j = 1, 2 \tag{7}$$

In Eq. (6), $\delta(x_1)$ denotes the Dirac’s delta function.

Now we consider the formulation of the imperfect contact conditions on the interface plane between the lower and upper layers and between the lower layer and the foundation. It should be noted that, in general, the imperfectness of the contact conditions is identified by discontinuities of the displacements and forces across the mentioned interfaces. Some comments on the various types of incomplete contact conditions for elastodynamics problems has been given in [13]. According to the comments given in [13], we also use the same model for the mathematical formulation of the imperfectness of the contact conditions mentioned there. Consequently, these conditions are written as follows

$$\sigma_{i2}^{(1)} \big|_{x_2=-h_1} = \sigma_{i2}^{(2)} \big|_{x_2=-h_1} \ \ , \ i = 1, 2, \quad u_2^{(1)} \big|_{x_2=-h_1} = u_2^{(2)} \big|_{x_2=-h_1} \tag{8}$$

$$u_1^{(1)} \big|_{x_2=-h_1} - u_1^{(2)} \big|_{x_2=-h_1} = F_1 \frac{h_1}{\mu^{(1)}} \sigma_{12}^{(1)} \big|_{x_2=-h_1}, F_1 > 0 \tag{9}$$

Moreover, the incomplete contact conditions between the lower layer of the plate and the rigid foundation are given as

$$u_1^{(2)} \Big|_{x_2=-h} = F_2 \frac{h}{\mu^{(2)}} \sigma_{12}^{(2)} \Big|_{x_2=-h}, \ \ u_2^{(2)} \Big|_{x_2=-h} = 0, F_2 > 0 \tag{10}$$

where $h = h_1 + h_2$.

We will estimate below the degree of the shear-spring type imperfectness of the contact conditions through the parameter $F_k, k=1,2$ in Eqs. (9) and (10). Note that the case where $F_k=0$ corresponds the full contact of interface planes, but the case where $F_k=\infty$ to the full slipping contact of the interface planes of the given system. Thus, formulation of the considered problem is completed.
varıatıonal formulatıon
Since the applied lineal located load is time-harmonic and the steady state is considered, all the dependent variables are also time-harmonic and can be represented as

$$\{u_i^{(m)},\varepsilon_{ij}^{(m)},\sigma_{ij}^{(m)}\}=\{\bar{u}_i^{(m)},\bar{\varepsilon}_{ij}^{(m)},\bar{\sigma}_{ij}^{(m)}\}e^{i\omega t} \tag{11}$$

where the superposed bar denotes the amplitude of the corresponding quantity. Hereafter the bars will be omitted. Substituting expression (11) into the foregoing equations and conditions, with the change $(\partial^2 u_j^{(m)}/\partial t^2)$ and $P\delta(x_1)e^{i\omega t}$ by $(-\omega^2 u_j^{(m)})$ and $P\delta(x_1)$ respectively, we obtain the same equations and conditions for the amplitude of the sought values. It is impossible to find the analytical solution of the formulated problem, therefore some numerical methods (such as Finite Element Method (FEM)) should be employed to find an approximate solution to this problem. Here, we give the variational formulation of the considered problem.

First, we introduce the dimensionless coordinate system by the following transformation

$$\hat{x}_1=\frac{x_1}{h} \quad , \quad \hat{x}_2=\frac{x_2}{h}. \tag{12}$$

Let a^* and h^* is given by

$$a^*=\frac{a}{h} \quad , \quad h^*=\frac{h_1}{h}. \tag{13}$$

Substituting (11) and (12) into Eqs. (3) and then multiplying both sides of the equations by h^2 we get

$$h\frac{\partial\sigma_{11}^{(m)}}{\partial\hat{x}_1}+h\frac{\partial\sigma_{12}^{(m)}}{\partial\hat{x}_2}+\sigma_0^{(m)}\frac{\partial^2 u_1^{(m)}}{\partial\hat{x}_1^{\,2}}=-\rho\omega^2h^2u_1^{(m)}, \tag{14}$$

$$h\frac{\partial\sigma_{21}^{(m)}}{\partial\hat{x}_1}+h\frac{\partial\sigma_{22}^{(m)}}{\partial\hat{x}_2}+\sigma_0^{(m)}\frac{\partial^2 u_2^{(m)}}{\partial\hat{x}_1^{\,2}}=-\rho\omega^2h^2u_2^{(m)}. \tag{15}$$

The mechanical relations in terms of dimensionless coordinates are as follows

$$\sigma_{11}^{(m)}=(\lambda^{(m)}+2\mu^{(m)})\frac{\partial u_1^{(m)}}{\partial\hat{x}_1}\frac{\partial\hat{x}_1}{\partial x_1}+\lambda^{(m)}\frac{\partial u_2^{(m)}}{\partial\hat{x}_2}\frac{\partial\hat{x}_2}{\partial x_2}=\frac{1}{h}\left[(\lambda^{(m)}+2\mu^{(m)})\frac{\partial u_1^{(m)}}{\partial\hat{x}_1}+\lambda^{(m)}\frac{\partial u_2^{(m)}}{\partial\hat{x}_2}\right], \tag{16}$$

$$\sigma_{22}^{(m)}=\lambda^{(m)}\frac{\partial u_1^{(m)}}{\partial\hat{x}_1}\frac{\partial\hat{x}_1}{\partial x_1}+(\lambda^{(m)}+2\mu^{(m)})\frac{\partial u_2^{(m)}}{\partial\hat{x}_2}\frac{\partial\hat{x}_2}{\partial x_2}=\frac{1}{h}\left[\lambda^{(m)}\frac{\partial u_1^{(m)}}{\partial\hat{x}_1}+(\lambda^{(m)}+2\mu^{(m)})\frac{\partial u_2^{(m)}}{\partial\hat{x}_2}\right], \tag{17}$$

$$\sigma_{12}{}^{(m)} = \mu^{(m)}\left(\frac{\partial u_1{}^{(m)}}{\partial \hat{x}_2}\frac{\partial \hat{x}_2}{\partial x_2} + \frac{\partial u_2{}^{(m)}}{\partial \hat{x}_1}\frac{\partial \hat{x}_1}{\partial x_1}\right) = \frac{1}{h}\mu^{(m)}\left(\frac{\partial u_1{}^{(m)}}{\partial \hat{x}_2} + \frac{\partial u_2{}^{(m)}}{\partial \hat{x}_1}\right). \tag{18}$$

The relations above can be written in compact form as

$$\sigma_{ij}{}^{(m)} = \frac{1}{h}\hat{\sigma}_{ij}{}^{(m)}, \quad i; j = 1,2. \tag{19}$$

The boundary and contact conditions (6-10) under the transformation (12) can be written as

$$\sigma_{12}{}^{(1)}\,|_{x_2=0} = 0 \;, \; \sigma_{22}{}^{(1)}\,|_{x_2=0} = -P\delta(h\hat{x}_1), \tag{20}$$

$$\left.\left(\sigma_{11}{}^{(m),0}\frac{\partial u_i{}^{(m)}}{\partial x_1} + \sigma_{1j}{}^{(m)}\right)\right|_{x_1=\pm a^*} = 0 \;, \; \sigma_{12}{}^{(m)}\,|_{x_1=\pm a^*} = 0 \,, m; j = 1,2, \tag{21}$$

$$\sigma_{i2}{}^{(1)}\,|_{x_2=-h^*} = \sigma_{i2}{}^{(2)}\,|_{x_2=-h^*} \;, \; i = 1,2, \quad u_2{}^{(1)}\,|_{x_2=-h^*} = u_2{}^{(2)}\,|_{x_2=-h^*}, \tag{22}$$

$$u_1{}^{(1)}\,|_{x_2=-h^*} - u_1{}^{(2)}\,|_{x_2=-h^*} = F_1\frac{h^*}{\mu^{(1)}}\sigma_{12}{}^{(1)}\,|_{x_2=-h^*}, F_1 > 0, \tag{23}$$

$$u_1{}^{(2)}\Big|_{x_2=-1} = F_2\frac{1}{\mu^{(2)}}\sigma_{12}{}^{(2)}\Big|_{x_2=-1}, \quad u_2{}^{(2)}\Big|_{x_2=-1} = 0, F_2 > 0. \tag{24}$$

Next, we multiply Eqs. (14) and (15) by the test functions $v_1{}^{(m)} = v_1{}^{(m)}(\hat{x}_1 \,,\, \hat{x}_2)$ and $v_2{}^{(m)} = v_2{}^{(m)}(\hat{x}_1 \,,\, \hat{x}_2)$; $m=1$,2, respectively. Adding the obtained equations we get

$$h\frac{\partial \sigma_{11}{}^{(m)}}{\partial \hat{x}_1}v_1{}^{(m)} + h\frac{\partial \sigma_{21}{}^{(m)}}{\partial \hat{x}_1}v_2{}^{(m)} + h\frac{\partial \sigma_{12}{}^{(m)}}{\partial \hat{x}_2}v_1{}^{(m)} + h\frac{\partial \sigma_{22}{}^{(m)}}{\partial \hat{x}_2}v_2{}^{(m)} + \\ + \sigma_0{}^{(m)}\left[\frac{\partial^2 u_1{}^{(m)}}{\partial \hat{x}_1{}^2}v_1{}^{(m)} + \frac{\partial^2 u_2{}^{(m)}}{\partial \hat{x}_1{}^2}v_2{}^{(m)}\right] = -\rho\omega^2 h^2 (u_1{}^{(m)}v_1{}^{(m)} + u_2{}^{(m)}v_2{}^{(m)}).$$

Integrating the last equation over the domains

$$\hat{\Omega}_1 = \{(\hat{x}_1, \hat{x}_2) \;:\; -a^* \le \hat{x}_1 \le a^*, 0 \le \hat{x}_2 \le h^*\}, \tag{25}$$

$$\hat{\Omega}_2 = \{(\hat{x}_1, \hat{x}_2) \;:\; -a^* \le \hat{x}_1 \le a^*, h^* \le \hat{x}_2 \le h\}, \tag{26}$$

we get the following equation

$$\sum_m \iint_{\hat{\Omega}_m}\left[h\frac{\partial \sigma_{11}{}^{(m)}}{\partial \hat{x}_1}v_1{}^{(m)} + h\frac{\partial \sigma_{21}{}^{(m)}}{\partial \hat{x}_1}v_2{}^{(m)} + h\frac{\partial \sigma_{12}{}^{(m)}}{\partial \hat{x}_2}v_1{}^{(m)} + h\frac{\partial \sigma_{22}{}^{(m)}}{\partial \hat{x}_2}v_2{}^{(m)} \right. \\ \left. + \sigma_0{}^{(m)}\left(\frac{\partial^2 u_1{}^{(m)}}{\partial \hat{x}_1{}^2}v_1{}^{(m)} + \frac{\partial^2 u_2{}^{(m)}}{\partial \hat{x}_1{}^2}v_2{}^{(m)}\right)\right]d\hat{x}_1 d\hat{x}_2 \\ = -\sum_m \iint_{\hat{\Omega}_m}\rho\omega^2 h^2(\,u_1{}^{(m)}v_1{}^{(m)} + u_2{}^{(m)}v_2{}^{(m)})d\hat{x}_1 d\hat{x}_2. \tag{27}$$

Performing integration by parts in Eq. (27) and then collecting terms we get

$$\begin{aligned}\sum_{m=1}^{2}\int_{\partial\hat{\Omega}_m}[&\left\{h\sigma_{11}^{(m)}v_1^{(m)}+h\sigma_{21}^{(m)}v_2^{(m)}+\sigma_0^{(m)}\left(\frac{\partial u_1^{(m)}}{\partial\hat{x}_1}v_1^{(m)}+\frac{\partial u_2^{(m)}}{\partial\hat{x}_1}v_2^{(m)}\right)\right\}\cos(\vec{n},\hat{x}_1)\\
&+\left\{h\sigma_{12}^{(m)}v_1^{(m)}+h\sigma_{22}^{(m)}v_2^{(m)}\right\}\cos(\vec{n},\hat{x}_2)]ds\\
&-\sum_{m=1}^{2}\iint_{\hat{\Omega}_m}\left[h\sigma_{11}^{(m)}\frac{\partial v_1^{(m)}}{\partial\hat{x}_1}+h\sigma_{21}^{(m)}\frac{\partial v_2^{(m)}}{\partial\hat{x}_1}+h\sigma_{12}^{(m)}\frac{\partial v_1^{(m)}}{\partial\hat{x}_2}+\right.\\
&\left.+h\sigma_{22}^{(m)}\frac{\partial v_2^{(m)}}{\partial\hat{x}_2}+\sigma_0^{(m)}\left(\frac{\partial u_1^{(m)}}{\partial\hat{x}_1}\frac{\partial v_1^{(m)}}{\partial\hat{x}_1}+\frac{\partial u_2^{(m)}}{\partial\hat{x}_1}\frac{\partial v_2^{(m)}}{\partial\hat{x}_1}\right)\right]d\hat{x}_1d\hat{x}_2\\
&=-\sum_{m=1}^{2}\iint_{\hat{\Omega}_m}\rho\omega^2h^2(u_1^{(m)}v_1^{(m)}+u_2^{(m)}v_2^{(m)})d\hat{x}_1d\hat{x}_2.\end{aligned}\qquad(28)$$

In Eq. (28), the boundaries of the domains $\hat{\Omega}_1$ and $\hat{\Omega}_2$ are denoted by $\partial\hat{\Omega}_1$ and $\partial\hat{\Omega}_2$, respectively. Taking the boundary conditions (20-24) into account, we perform the integration along the boundaries according to Fig.2 and well-known properties of mechanics. Consequently, we get the following equation

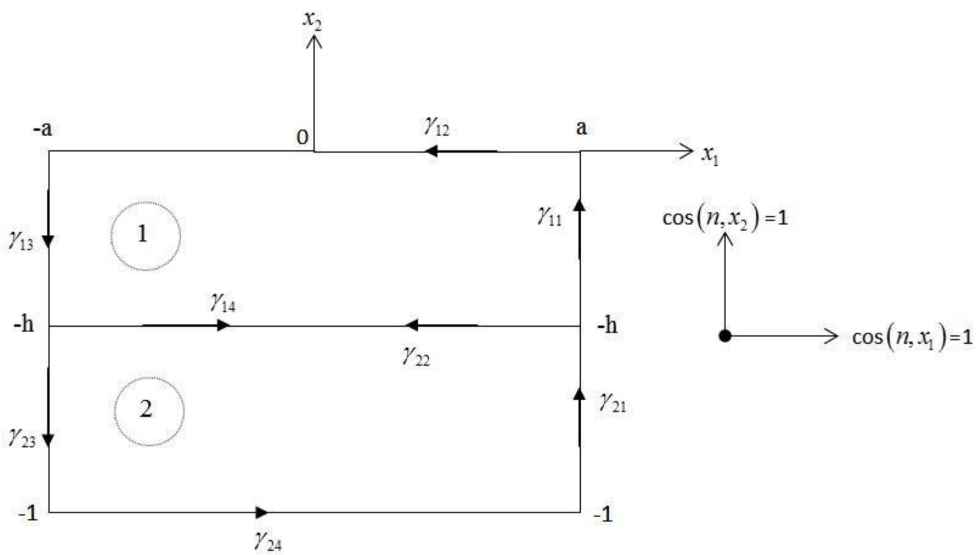

Figure 2. The direction cosines for the boundaries**.**

$$\begin{aligned}\sum_{m=1}^{2}\iint_{\hat{\Omega}_m}&\left[h\sigma_{11}^{(m)}\frac{\partial v_1^{(m)}}{\partial\hat{x}_1}+h\sigma_{21}^{(m)}\frac{\partial v_2^{(m)}}{\partial\hat{x}_1}+h\sigma_{12}^{(m)}\frac{\partial v_1^{(m)}}{\partial\hat{x}_2}+h\sigma_{22}^{(m)}\frac{\partial v_2^{(m)}}{\partial\hat{x}_2}\right.\\
&\left.+\sigma_0^{(m)}\left(\frac{\partial u_1^{(m)}}{\partial\hat{x}_1}\frac{\partial v_1^{(m)}}{\partial\hat{x}_1}+\frac{\partial u_2^{(m)}}{\partial\hat{x}_1}\frac{\partial v_2^{(m)}}{\partial\hat{x}_1}\right)-\rho\omega^2h^2(u_1^{(m)}v_1^{(m)}+u_2^{(m)}v_2^{(m)})\right]d\hat{x}_1d\hat{x}_2\\
&=-\int_{-a^*}^{a^*}\left[P\delta(\hat{x}_1)v_2^{(1)}\Big|_{x_2=0}+\frac{F_1}{\mu^{(1)}}h_1\sigma_{12}^{*}\sigma_{12}\Big|_{x_2=-h^*}+\frac{F_2}{\mu^{(2)}}h\sigma_{12}^{*}\sigma_{12}\Big|_{x_2=-1}\right]dx_1.\end{aligned}\qquad(29)$$

Using the relations (16)-(18), Eq. (29) can be written as

$$\sum_{m=1}^{2}\iint_{\hat{\Omega}_m}\left[\begin{array}{c}\left((\lambda^{(m)}+2\mu^{(m)})\frac{\partial u_1^{(m)}}{\partial \hat{x}_1}+\lambda^{(m)}\frac{\partial u_2^{(m)}}{\partial \hat{x}_2}\right)\frac{\partial v_1^{(m)}}{\partial \hat{x}_1}\\ +\mu^{(m)}\left(\frac{\partial u_1^{(m)}}{\partial \hat{x}_2}+\frac{\partial u_2^{(m)}}{\partial \hat{x}_1}\right)\frac{\partial v_2^{(m)}}{\partial \hat{x}_1}+\mu^{(m)}\left(\frac{\partial u_1^{(m)}}{\partial \hat{x}_2}+\frac{\partial u_2^{(m)}}{\partial \hat{x}_1}\right)\frac{\partial v_1^{(m)}}{\partial \hat{x}_2}\\ +\left(\lambda^{(m)}\frac{\partial u_1^{(m)}}{\partial \hat{x}_1}+(\lambda^{(m)}+2\mu^{(m)})\frac{\partial u_2^{(m)}}{\partial \hat{x}_2}\right)\frac{\partial v_2^{(m)}}{\partial \hat{x}_2}\\ +\sigma_0^{(m)}\left(\frac{\partial u_1^{(m)}}{\partial \hat{x}_1}\frac{\partial v_1^{(m)}}{\partial \hat{x}_1}+\frac{\partial u_2^{(m)}}{\partial \hat{x}_1}\frac{\partial v_2^{(m)}}{\partial \hat{x}_1}\right)-\rho\omega^2h^2(u_1^{(m)}v_1^{(m)}+u_2^{(m)}v_2^{(m)})\end{array}\right]d\hat{x}_1d\hat{x}_2$$

$$=-\int_{-a^*}^{a^*}\left[\begin{array}{c}F_1\mu^{(1)}\frac{h^*}{h}\left(\frac{\partial v_1^{(1)}}{\partial \hat{x}_2}+\frac{\partial v_2^{(1)}}{\partial \hat{x}_1}\right)\left(\frac{\partial u_1^{(1)}}{\partial \hat{x}_2}+\frac{\partial u_2^{(1)}}{\partial \hat{x}_1}\right)\Big|_{x_2=-h^*}\\ +F_2\frac{\mu^{(2)}}{h}\left(\frac{\partial v_1^{(2)}}{\partial \hat{x}_2}+\frac{\partial v_2^{(2)}}{\partial \hat{x}_1}\right)\left(\frac{\partial u_1^{(2)}}{\partial \hat{x}_2}+\frac{\partial u_2^{(2)}}{\partial \hat{x}_1}\right)\Big|_{x_2=-1}+P\delta(\hat{x}_1)v_2^{(1)}\Big|_{x_2=0}\end{array}\right]d\hat{x}_1. \tag{30}$$

Multiplying both sides of Eq. (30) by $1/\mu^{(m)}$ and collecting terms we get

$$\sum_{m=1}^{2}\iint_{\hat{\Omega}_m}\left[\begin{array}{c}\left((\frac{\lambda^{(m)}+2\mu^{(m)}+\sigma_0^{(m)}}{\mu^{(m)}})\frac{\partial u_1^{(m)}}{\partial \hat{x}_1}+\frac{\lambda^{(m)}}{\mu^{(m)}}\frac{\partial u_2^{(m)}}{\partial \hat{x}_2}\right)\frac{\partial v_1^{(m)}}{\partial \hat{x}_1}\\ +\left(\frac{\lambda^{(m)}}{\mu^{(m)}}\frac{\partial u_1^{(m)}}{\partial \hat{x}_1}+\frac{\lambda^{(m)}+2\mu^{(m)}}{\mu^{(m)}}\frac{\partial u_2^{(m)}}{\partial \hat{x}_2}\right)\frac{\partial v_2^{(m)}}{\partial \hat{x}_2}\\ +\left\{\frac{\partial u_1^{(m)}}{\partial \hat{x}_2}+\frac{\mu^{(m)}+\sigma_0^{(m)}}{\mu^{(m)}}\frac{\partial u_2^{(m)}}{\partial \hat{x}_1}\right\}\frac{\partial v_2^{(m)}}{\partial \hat{x}_1}\\ +\left(\frac{\partial u_1^{(m)}}{\partial \hat{x}_2}+\frac{\partial u_2^{(m)}}{\partial \hat{x}_1}\right)\frac{\partial v_1^{(m)}}{\partial \hat{x}_2}-\frac{\rho\omega^2h^2}{\mu^{(m)}}(u_1^{(m)}v_1^{(m)}+u_2^{(m)}v_2^{(m)})\end{array}\right]d\hat{x}_1d\hat{x}_2$$

$$=-\int_{-a^*}^{a^*}\left[\begin{array}{c}\frac{P}{\mu^{(1)}}\delta(\hat{x}_1)v_2^{(1)}\Big|_{x_2=0}+F_1\frac{h^*}{h}\left(\frac{\partial v_1^{(1)}}{\partial \hat{x}_2}+\frac{\partial v_2^{(1)}}{\partial \hat{x}_1}\right)\left(\frac{\partial u_1^{(1)}}{\partial \hat{x}_2}+\frac{\partial u_2^{(1)}}{\partial \hat{x}_1}\right)\Big|_{x_2=-h^*}\\ +F_2h\left(\frac{\partial v_1^{(2)}}{\partial \hat{x}_2}+\frac{\partial v_2^{(2)}}{\partial \hat{x}_1}\right)\left(\frac{\partial u_1^{(2)}}{\partial \hat{x}_2}+\frac{\partial u_2^{(2)}}{\partial \hat{x}_1}\right)\Big|_{x_2=-1}\end{array}\right]d\hat{x}_1. \tag{31}$$

Let $c_1^{(m)}$ denotes the speed of dilatation waves, $c_2^{(m)}$ denotes the speed of distortion waves, Ω denotes the dimensionless frequency, and $\eta_2^{(m)}$ denotes the parameter related to the pre-stress intensities. These quantities are given as

$$c_1^{(m)} = \sqrt{\lambda^{(m)} + 2\mu^{(m)} / \rho^{(m)}},\ c_2^{(m)} = \sqrt{\mu^{(m)} / \rho^{(m)}},\ \Omega^{(m)} = \frac{\omega h}{c_2^{(m)}},\ \eta_2^{(m)} = \sigma_0^{(m)} / \mu^{(m)}. \quad (32)$$

Using these notation in Eq. (31) and collecting terms we get the following equation

$$\sum_{m=1}^{2} \iint_{\hat{\Omega}_m} \left[\begin{aligned} &\left(\left(\frac{c_1^{(m)}}{c_2^{(m)}} \right)^2 + \eta_2^{(m)} \right) \frac{\partial u_1^{(m)}}{\partial \hat{x}_1} \frac{\partial v_1^{(m)}}{\partial \hat{x}_1} + \frac{\partial u_1^{(m)}}{\partial \hat{x}_2} \frac{\partial v_1^{(m)}}{\partial \hat{x}_2} + \frac{\partial u_1^{(m)}}{\partial \hat{x}_2} \frac{\partial v_2^{(m)}}{\partial \hat{x}_1} \\ &+ \frac{\lambda^{(m)}}{\mu^{(m)}} \frac{\partial u_1^{(m)}}{\partial \hat{x}_1} \frac{\partial v_2^{(m)}}{\partial \hat{x}_2} + \frac{\lambda^{(m)}}{\mu^{(m)}} \frac{\partial u_2^{(m)}}{\partial \hat{x}_2} \frac{\partial v_1^{(m)}}{\partial \hat{x}_1} + \left(1 + \eta_2^{(m)}\right) \frac{\partial u_2^{(m)}}{\partial \hat{x}_1} \frac{\partial v_2^{(m)}}{\partial \hat{x}_1} \\ &+ \left(\frac{c_1^{(m)}}{c_2^{(m)}} \right)^2 \frac{\partial u_2^{(m)}}{\partial \hat{x}_2} \frac{\partial v_2^{(m)}}{\partial \hat{x}_2} + \frac{\partial u_2^{(m)}}{\partial \hat{x}_1} \frac{\partial v_1^{(m)}}{\partial \hat{x}_2} - \left(\Omega^{(m)}\right)^2 (u_1^{(m)} v_1^{(m)} + u_2^{(m)} v_2^{(m)}) \end{aligned} \right] d\hat{x}_1 d\hat{x}_2$$

$$+ \int_{-a^*}^{a^*} \left[\begin{aligned} &F_1 \frac{h^*}{h} \left(\frac{\partial v_1^{(1)}}{\partial \hat{x}_2} + \frac{\partial v_2^{(1)}}{\partial \hat{x}_1} \right) \left(\frac{\partial u_1^{(1)}}{\partial \hat{x}_2} + \frac{\partial u_2^{(1)}}{\partial \hat{x}_1} \right) \Bigg|_{x_2 = -h^*} \\ &+ F_2 h \left(\frac{\partial v_1^{(2)}}{\partial \hat{x}_2} + \frac{\partial v_2^{(2)}}{\partial \hat{x}_1} \right) \left(\frac{\partial u_1^{(2)}}{\partial \hat{x}_2} + \frac{\partial u_2^{(2)}}{\partial \hat{x}_1} \right) \Bigg|_{x_2 = -1} \end{aligned} \right] d\hat{x}_1 \quad (33)$$

$$= - \int_{-a/h}^{a/h} \frac{P}{\mu^{(1)}} \delta(\hat{x}_1) v_2^{(1)} \Big|_{x_2 = 0} d\hat{x}_1.$$

The left hand side and the right hand side of Eq. (33) can be denoted by $\mathrm{L}(\mathbf{u^{(m)}}, \mathbf{v^{(m)}})$ and $\mathrm{l}(\mathbf{v^{(m)}})$, respectively, where $\mathrm{L}(\mathbf{u^{(m)}}, \mathbf{v^{(m)}})$ is a bilinear form and $\mathrm{l}(\mathbf{v^{(m)}})$ is a linear form. So, Eq. (33) is written in the form $\mathrm{L}(\mathbf{u^{(m)}}, \mathbf{v^{(m)}}) = \mathrm{l}(\mathbf{v^{(m)}})$. Here, $\mathbf{u^{(m)}} = \mathbf{u^{(m)}}(u_1^{(m)}, u_2^{(m)})$ and $\mathbf{v^{(m)}} = \mathbf{v^{(m)}}(u_1^{(m)}, u_2^{(m)})$. According to variational principles, using Eq. (33), the energy functional $\mathrm{J}(\mathbf{u^{(m)}}) = \frac{1}{2} \mathrm{L}(\mathbf{u^{(m)}}, \mathbf{u^{(m)}}) - \mathrm{l}(\mathbf{u^{(m)}})$ is obtained as

$$
\begin{aligned}
\mathrm{J}(\mathbf{u}^{(\mathbf{m})}) = \frac{1}{2}\sum_{m=1}^{2}\iint_{\hat{\Omega}_m} & \left[\left(\frac{c_1^{(m)}}{c_2^{(m)}}\right)^2 \left\{ \left(\frac{\partial u_1{}^{(m)}}{\partial \hat{x}_1}\right)^2 + \left(\frac{\partial u_2{}^{(m)}}{\partial \hat{x}_2}\right)^2 \right\} + \left(\frac{\partial u_1{}^{(m)}}{\partial \hat{x}_2}\right)^2 + \left(\frac{\partial u_2{}^{(m)}}{\partial \hat{x}_1}\right)^2 \right. \\
& + \eta_2^{(m)} \left\{ \left(\frac{\partial u_1{}^{(m)}}{\partial \hat{x}_1}\right)^2 + \left(\frac{\partial u_2{}^{(m)}}{\partial \hat{x}_1}\right)^2 \right\} + 2\frac{\lambda^{(m)}}{\mu^{(m)}} \frac{\partial u_1{}^{(m)}}{\partial \hat{x}_1} \frac{\partial u_2{}^{(m)}}{\partial \hat{x}_2} \\
& \left. + 2\frac{\partial u_1{}^{(m)}}{\partial \hat{x}_2} \frac{\partial u_2{}^{(m)}}{\partial \hat{x}_1} - \left(\Omega^{(m)}\right)^2 \left\{ \left(u_1{}^{(m)}\right)^2 + \left(u_2{}^{(m)}\right)^2 \right\} \right] d\hat{x}_1 d\hat{x}_2 \\
& + \underline{\int_{-a^*}^{a^*} \left[F_1 \frac{h^*}{h} \left(\frac{\partial u_1{}^{(1)}}{\partial \hat{x}_2} + \frac{\partial u_2{}^{(1)}}{\partial \hat{x}_1} \right)^2 \Bigg|_{x_2=-h^*} + F_2 h \left(\frac{\partial u_1{}^{(2)}}{\partial \hat{x}_2} + \frac{\partial u_2{}^{(2)}}{\partial \hat{x}_1} \right)^2 \Bigg|_{x_2=-1} \right] d\hat{x}_1} \\
& + \int_{-a^*}^{a^*} \frac{P}{\mu^{(1)}} \delta(\hat{x}_1) u_2^{(1)} \Big|_{x_2=0} d\hat{x}_1 .
\end{aligned}
\qquad (34)
$$

Note that the underlined term in the energy functional (34) characterizes the imperfectness of the contact conditions both between the layers of the plate and between the plate and the rigid foundation.

It is well-known from calculus of variation that equating the first variation of the energy functional $\mathrm{J}(\mathbf{u}^{(\mathbf{m})})$ (denoted by $\delta\mathrm{J}(\mathbf{u}^{(\mathbf{m})})$) to zero we get both the equations of motion (3) and the boundary-contact conditions (20)-(24). The above-mentioned situation can be seen after some mathematical manipulations. A brief explanation is given as: The explicit form of the equation $\delta\mathrm{J}(\mathbf{u}^{(\mathbf{m})}) = 0$ is given as

$$\delta\mathrm{J}(\mathbf{u}^{(\mathbf{m})}) = \delta\mathrm{J}_{u_1{}^{(m)}} + \delta\mathrm{J}_{u_2{}^{(m)}} = 0 . \qquad (38)$$

In Eq. (38), each term on the left hand side must be zero. So, to evaluate the first variation we set the equations

$$\delta\mathrm{J}_{u_1{}^{(m)}} = \frac{d}{d\alpha} J(u_1{}^{(m)} + \alpha\xi^{(m)}, u_2{}^{(m)})\,|_{\alpha=0} = 0 \qquad (39)$$

and

$$\delta\mathrm{J}_{u_2{}^{(m)}} = \frac{d}{d\alpha} J(u_1{}^{(m)}, u_2{}^{(m)} + \alpha\eta^{(m)})\,|_{\alpha=0} = 0 . \qquad (40)$$

Using Eqs. (39)-(40), we get the the equations of motion (3) and the boundary-contact conditions (20)-(24).

some results and dıscussıons

For the FEM modeling of the boundary-value - contact problem considered here which are obtained for the amplitude from equations (3)-(10) we propose the energy functional (34). The effect of the imperfectness of the contact conditions both between the layers of the plate and between the plate and the rigid foundation (the underlined term in the energy functional (34)) will be investigated as a further study.

ACKNOWLEDGEMENTS

This study is supported by grant from Kyrgyzstan-Turkey Manas University under research No. 2015.FBE.02(BAP-004).

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