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Solving Nonlinear Partial Differential Equations by Using Adomian Decomposition Method, Modified Decomposition Method and Laplace Decomposition Method

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Abstract: *This paper considers mostly the inhomogeneous Fitzhugh-Nagumo-Huxley equation with its initial value. Adomian Decomposition Method (ADM), Modified Decomposition Method (MDM) and Laplace Decomposition Method (LDM) have been applied to this equation in order to obtain the solution that satisfies the given initial condition. It can be easily seen that each method gives the exactly same solution. And then, the inhomogeneous Boussinesq equation and another nonlinear partial differential equation, subject to given initial values, have been solved by using LDM. Application of the given methods has demonstrated that the solution is obtained with a fast convergence by using the advantage of the noise terms. Moreover, there is not any necessity to turn the nonlinear terms into linear ones since Adomian polynomials is used. Therefore, it is deduced that these methods are, indeed, effective and suitable for the nonlinear partial differential equations with the initial values.*

Keywords: *Laplace Decomposition Method, Adomian Decomposition Method, Modified Decomposition Method, Fitzhugh- Nagumo-Huxley Equation and The Noise terms*

Adomian Ayrışma Metodu, Değistirilmis Ayrışma Metodu ve Laplace Ayrışma Metodunu Kullanarak Lineer Olmayan Kısmi Türevli Diferansiyel Denklemleri Çözme

Özet: *Bu makalede çoğunlukla başlangıç değeri verilen homojen olmayan Fitzhugh-Nagumo-Huxley denklemi ele aldık. Bu denklemin verilen başlangıç değerini sağlayan çözümünü elde etmek için Adomian ayrışma, değiştirilmiş ayrışma ve laplace ayrışma metotları bu denkleme uygulanmıştır. Kolaylıkla görülebilir ki, her bir metot tamamen aynı sonucu vermektedir. Daha sonra, verilen başlangıç değerlerine tabi tutulan homojen olmayan Boussinesq denklemi ve bir başka lineer olmayan kısmi türevli diferansiyel denklem laplace ayrışma metodu kullanarak çözülmüştür. Metotların uygulamalarında parazit terimlerin avantajını kullanarak çözümüm hızlı bir yakınsama ile elde edildiği gösterildi. Bunun yanında, Adomian polinomları kullanıldığından dolayı lineer olmayan terimleri lineer olanlara dönüştürmeye ihtiyacımız yoktur. Bu nedenle, lineer olmayan kısmi türevli diferansiyel denklemlerinde bu metotların gerçekten etkili ve uygun olduğu sonucuna varıldı.*

Anahtar kelimeler: *Adomian ayrışma Metodu, değiştirilmiş ayrışma Metodu, laplace ayrışma Metodu, Fitzhugh-Nagumo-Huxley denklemi and parazit terimler*

INTRODUCTION

In almost all fields of science and engineering, most problems can be represented by linear or nonlinear partial or ordinary differential equations (shortly, N/LPDEs or N/LODEs). To go forward in these fields, it is necessary to solve these kind of problems. Solving these equations provides us with information about the scientific problems. So, finding solutions to these type of equations is of a great significance in modern science, including engineering. In this respect, numereous researches have been done to find the ways to obtain reliable solutions of these equations. Eventually, many analytical and numerical methods have been established and used to achieve this goal. Some of the methods for solving N/LPDEs or N/LODEs are the perturbation method [1-5], the homotopy perturbation method [3-7], the delta perturbative method [8], the Adomian Decomposition method [9-19], the Modified Decomposition method [10, 13, 20-27] and the Laplace Decomposition method [28-35] and others.

This study pursues two goals. The first one is to solve the nonlinear Fitzhugh-Nagumo-Huxley equation by ADM, MDM and LDM. And then, it is shown that the same solution is gained by all methods. The second goal is to apply the LDM to some inhomegeneous NLPDEs with initial conditions. These applications demonstrate the efficiency, effectiveness, usefulness and simplicity of the methods in solving NLPDEs.

The nonlinear Fitzhugh-Nagumo-Huxley equation is an important model in the work of neuron axon [36]. The equation is in the form of

$$\boldsymbol{u_t - u_{xx} + u(1-u)(\alpha - u) = 0}, \qquad \boldsymbol{\alpha = const}. \tag{1}$$

In this study, we first consider the following inhomogeneous form of the equation (1)

$$\boldsymbol{u_t(x,t) - u_{xx}(x,t) + \alpha u(x,t) + F(u(x,t)) = h(x,t)} \tag{2}$$

with the initial condition

$$\boldsymbol{u(x,0) = f(x)} \tag{3}$$

where $\boldsymbol{\alpha}$ is a constant, $\boldsymbol{h(x,t)}$ is the inhomogeneous part and $\boldsymbol{F(u(x,t))}$ is a nonlinear function of $\boldsymbol{u(x,t)}$. And then, we take $\boldsymbol{\alpha = -1}$, $\boldsymbol{h(x,t) = x^3t^3 + 3x^2t^2 + 2xt + x}$ and $\boldsymbol{u(x,0) = 1}$. So, we get the following equation

$$\mathbf{u_t(x,t) - u_{xx}(x,t) - u(x,t)} + \boldsymbol{u^3}\mathbf{(x,t)} = \boldsymbol{x^3t^3 + 3x^2t^2 + 2xt + x}, \qquad \boldsymbol{u(x,0) = 1} \tag{4}$$

and solve this equation by ADM, MDM and LDM.

In this application, we take the advantage of the noise terms [37] as they give the opportunity to see a fast convergence of the solution. The noise terms phenomenon may show up in only inhomegeneous PDEs. And also, this is applicable to all kinds of inhomegeneous PDEs with any order. The noise terms usually provide us with the solution after two consecutive iterations if they exist in the components $\boldsymbol{u_0}$ and $\boldsymbol{u_1}$.

The study has the following structure: In chapter 2 the decomposition methods, mentioned above, are described for the general form of equation (2) and its initial value (3) is considered. Then, these methods are applied to the equation (4) in chapter 3. In the final chapter, a short conclusion of the study is provided.

OUTLINES OF THE METHODS

Here, the use of the methods is shown for the general equation (2) with the condition (3).

Adomian Decomposition Method (ADM)

First, the equation (2) should be written as follows:

$$u_t(x,t) = u_{xx}(x,t) - \alpha u(x,t) - F(u(x,t)) + h(x,t) \tag{5}$$

and the initial condition $u(x,0) = f(x)$.

And also, it can be written in an operator form as in the following

$$\mathrm{L_t} u(\mathrm{x,t}) = \mathrm{L_{xx}} u(\mathrm{x,t}) - \alpha \mathrm{u}(\mathrm{x,t}) - \mathrm{F}(\mathrm{u}(\mathrm{x,t})) + \mathrm{h}(\mathrm{x,t}), \qquad u(x,0) = f(x) \tag{6}$$

where

$$L_t = \frac{\partial}{\partial t}, \qquad L_{xx} = \frac{\partial^2}{\partial x^2}, \qquad {L_t}^{-1} = \int_0^t (.)\ dt. \tag{7}$$

To get $u(x,t)$ alone in the left side, ${L_t}^{-1}$ is applied to both sides of (6). Using the inital conditon, we get

$$u(x,t) = f(x) + {L_t}^{-1}(u_{xx}(x,t) - \alpha u(x,t)) - {L_t}^{-1}\left(F(u(x,t))\right) + {L_t}^{-1}(h(x,t)). \tag{8}$$

For the linear term, the decomposition series are used, which is given by

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \tag{9}$$

For the nonlinear term $F(u(x,t))$, the infinite series of Adomian polynomials are given by

$$F(u(x,t)) = \sum_{n=0}^{\infty} A_n(x,t). \tag{10}$$

Here, $A_n(x,t)$ there are Adomian polynomials that are found by the following formula

(11)

$$A_n = \frac{1}{n!}\frac{d^n}{d\alpha^n}[F\left(\sum_{n=0}^{\infty}\alpha^i u_i\right)]_{\alpha=0}, \quad n = 0, 1, 2, 3, \ldots$$

The formula, given in (11), for the first time was intorduced by Adomian and Rach in 1983 [38]. Here, we present the first four Adomian polynomials, which look as follows:

$$A_0 = F(u_0) \tag{12}$$

$$A_1 = F'(u_0)u_1 \tag{13}$$

$$A_2 = F'(u_0)u_2 + F''(u_0)\frac{{u_1}^2}{2!} \tag{14}$$

$$A_3 = F'(u_0)u_3 + F''(u_0)u_1u_2 + F'''(u_0)\frac{{u_1}^3}{3!} \tag{15}$$

By putting (9) and (10) into the equation (8), we have obtained the following sequential relation;

$$u_0(x,t) = f(x) + {L_t}^{-1}\big(h(x,t)\big). \tag{16}$$

$$u_{k+1}(x,t) = {L_t}^{-1}\left(u_{k_{xx}}(x,t) - \alpha u_k(x,t)\right) - {L_t}^{-1}\big(A_k(x,t)\big), \qquad k \geq 0$$

For $k = 0$, we get

$$u_1(x,t) = {L_t}^{-1}\left(u_{0_{xx}}(x,t) - \alpha u_0(x,t)\right) - {L_t}^{-1}(A_0(x,t)). \tag{17}$$

For $k = 1$, we get

$$u_2(x,t) = {L_t}^{-1}\left(u_{1_{xx}}(x,t) - \alpha u_1(x,t)\right) - {L_t}^{-1}(A_1(x,t)) \tag{18}$$

and so on for other values of k.

The formulae, mentioned above (16-17-18), provide us with some components of the solution. By putting these components in the expansion (9) and using the advantage of the noise terms, the solution has been obtained quickly.

Modified Decomposition Method (MDM)

The equation given in the operator form (6) is used in this method, as well.

$$L_t u(x,t) = L_{xx}u(x,t) - \alpha u(x,t) - F\big(u(x,t)\big) + h(x,t), \qquad u(x,0) = f(x).$$

L_t, L_{xx} and ${L_t}^{-1}$ is given in (7).
This method presupposes the application of the following expansions.

$$u(x,t) = \sum_{m=0}^{\infty} a_m(x)t^m = a_0(x) + a_1(x)t + a_2(x)t^2 + \cdots \tag{19}$$

$$h(x,t)=\sum_{m=0}^{\infty} r_m(x)t^m = r_0(x)+r_1(x)t+r_2(x)t^2+\cdots \tag{20}$$

$$F(u(x,t))=\sum_{m=0}^{\infty} A_m(x)t^m = A_0(x)+A_1(x)t+A_2(x)t^2+\cdots \tag{21}$$

Now, by operating both sides of the given equation with L_t^{-1} and using the above expansions, we have obtained the following equality

$$\sum_{m=0}^{\infty} a_m(x)t^m = f(x)+\sum_{m=0}^{\infty} r_m(x)\frac{t^{m+1}}{m+1}+\frac{\partial^2}{\partial x^2}\left(\sum_{m=0}^{\infty} a_m(x)\frac{t^{m+1}}{m+1}\right)+\sum_{m=0}^{\infty} a_m(x)\frac{t^{m+1}}{m+1} -\sum_{m=0}^{\infty} A_m(x)\frac{t^{m+1}}{m+1}. \tag{22}$$

Placing $m=m-1$ on the right side of (22), we get

$$\sum_{m=0}^{\infty} a_m(x)t^m = f(x)+\sum_{m=1}^{\infty} r_{m-1}(x)\frac{t^{m}}{m}+\frac{\partial^2}{\partial x^2}\left(\sum_{m=1}^{\infty} a_{m-1}(x)\frac{t^{m}}{m}\right)+\sum_{m=1}^{\infty} a_{m-1}(x)\frac{t^{m}}{m} -\sum_{m=1}^{\infty} A_{m-1}(x)\frac{t^{m}}{m}. \tag{23}$$

In equation (23), the coefficients of the same power of t must be equalized to each other. After doing that, the following recurrence relations are found among the coefficients a_m, r_m and A_m.

$$a_0(x)=f(x) \tag{24}$$

$$a_m(x)=\frac{1}{m}[(a_{m-1}(x))_{xx}+a_{m-1}(x)+r_{m-1}(x)-A_{m-1}(x)], \qquad m\geq 1 \tag{25}$$

Having determined the cofficients $a_m(x)$ by the formula given in (25) and putting them into the expansion (19), we obtain the solution $u(x,t)$.

Laplace Decomposition Method (LDM)

The equation (2), subject to the initial condition (3), is considered in this part. The LDM is applied to this equation. Firstly, laplace transform (denoted by L) is applied on both sides of the equation (2). Then, we get

$$L[u_t(x,t)]-L[u_{xx}(x,t)]+\alpha L[u(x,t)]+L[F(u(x,t))]=L[h(x,t)]. \tag{26}$$

Using the laplace property for derivative, that is, $L[u_t(x,t)]=sL[u(x,t)]-u(x,0)$, the equation (26) can be written as

$$sL[u(x,t)] - u(x,0) = L[u_{xx}(x,t)] - \alpha L[u(x,t)] - L[F(u(x,t))] + L[h(x,t)]. \tag{27}$$

Furthermore, since $u(x,0) = f(x)$, we can write this (27) as

$$L[u(x,t)] = \frac{1}{s}(f(x) + L[u_{xx}(x,t)] - \alpha L[u(x,t)] - L[F(u(x,t))] + L[h(x,t)]). \tag{28}$$

Secondly, using the decomposition series for the linear term $u(x,t)$ and the infinite series of Adomian polynomials for the nonlinear term $F(u(x,t))$, provides

$$L\left[\sum_{n=0}^{\infty} u_n(x,t)\right] = \frac{1}{s}f(x) + \frac{1}{s}L\left[\sum_{n=0}^{\infty} u_{n_{xx}}(x,t)\right] - \frac{\alpha}{s}L\left[\sum_{n=0}^{\infty} u_n(x,t)\right] - L\left[\sum_{n=0}^{\infty} A_n(x,t)\right] + \frac{1}{s}L[h(x,t)] \tag{29}$$

where A_n is the Adomian polynomials given in the formula (11). Using the linearity property of the laplace transform, the equation (29) can be written as

$$\sum_{n=0}^{\infty} L[u_n(x,t)] = \frac{1}{s}f(x) + \frac{1}{s}\sum_{n=0}^{\infty} L[u_{n_{xx}}(x,t)] - \frac{\alpha}{s}\sum_{n=0}^{\infty} L[u_n(x,t)] - \sum_{n=0}^{\infty} L[A_n(x,t)] + \frac{1}{s}L[h(x,t)]. \tag{30}$$

This equation (30) provides us with the following sequential relations;

$$L[u_0(x,t)] = \frac{1}{s}f(x) + \frac{1}{s}L[h(x,t)] \tag{31}$$

$$L[u_{k+1}(x,t)] = \frac{1}{s}\sum_{k=0}^{\infty} L[u_{k_{xx}}(x,t)] - \frac{\alpha}{s}\sum_{k=0}^{\infty} L[u_k(x,t)] - \sum_{k=0}^{\infty} L[A_k(x,t)], \qquad k \geq 0 \tag{32}$$

Then, the components of $u(x,t)$ are obtained easily by applying the inverse laplace transform L^{-1}. Putting these components into the expansion given by

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots \tag{33}$$

completes the solution.

APPLICATIONS OF THE METHODS

In this section the decomposition methods, described above, are implemented to the nonlinear Fitzhugh-Nagumo-Huxley equation given in (4), respectively. Then, some other nonlinear inhomogeneous partial differential equations with initial conditions are solved by LDM.

Solving The Fitzhugh-Nagumo-Huxley Equation By Adomian Decomposition Method

ADM is applied to the given equation in (4). Then, using (5)-(8) we obtain the following equation

$$\boldsymbol{u(x,t) = 1 + L_t^{-1}(u_{xx}(x,t) + u(x,t)) - L_t^{-1}(u^3(x,t)) + L_t^{-1}\left(x^3t^3 + 3x^2t^2 + 2xt + x\right)}. \tag{34}$$

Putting (9) and (10) into the equation (34), we get the following equalities;

$$\boldsymbol{u_0(x,t) = 1 + L_t^{-1}(x^3t^3 + 3x^2t^2 + 2xt + x)} \tag{35}$$

$$\boldsymbol{u_{k+1}(x,t) = L_t^{-1}\left(u_{k_{xx}}(x,t) - \alpha u_k(x,t)\right) - L_t^{-1}(A_k(x,t)), \qquad k \geq 0}.$$

The first equality gives us $\boldsymbol{u_0(x,t)}$ since $\boldsymbol{L_t^{-1} = \int_0^t(.)\ dt}$

$$\boldsymbol{u_0(x,t) = 1 + \frac{1}{4}x^3t^4 + x^2t^3 + xt^2 + xt}. \tag{36}$$

Moreover, $\boldsymbol{u_1(x,t)}$, $\boldsymbol{u_2(x,t)}$ and the rest components can be derived from the second equality.

$$\boldsymbol{u_1(x,t) = L_t^{-1}\left(u_{0_{xx}}(x,t) + u_0(x,t)\right) - L_t^{-1}(A_0(x,t))} \tag{37}$$

where $\boldsymbol{A_0(x,t) = u_0{}^3(x,t)}$. Putting $\boldsymbol{u_0(x,t)}$, $\boldsymbol{u_{0_{xx}}(x,t)}$ and $\boldsymbol{A_0(x,t)}$ into (32), we have

$$\boldsymbol{u_1(x,t) = -t - xt^2 - \frac{1}{4}x^3t^4 - x^2t^3 - \cdots} \tag{38}$$

Here, we obtain the noise terms $\boldsymbol{\frac{1}{4}x^3t^4}$**,** $\boldsymbol{x^2t^3}$ and $\boldsymbol{xt^2}$**.** Let us remove the noise terms from the component $\boldsymbol{u_0(x,t)}$ and confirm that the remaning terms in $\boldsymbol{u_0(x,t)}$ satisfiy the equation (4), then we find the exact solution

$$\boldsymbol{u(x,t) = 1 + xt}. \tag{39}$$

Solving the Fitzhugh-Nagumo-Huxley Equatıon by Modified Decomposition Method

Here, we use MDM to find the solution of the equation (4). It can be written as;

$$L_t u(x,t) = L_{xx}u(x,t) + u(x,t) - u^3(x,t) + x^3t^3 + 3x^2t^2 + 2xt + x, \qquad u(x,0) = f(x).$$

Now, using (19)-(25), we have the following the equalities;

$$a_0(x) = 1 \tag{40}$$

$$a_1(x) = \ 1[(a_0(x))_{xx} + a_0(x) + r_0(x) - A_0(x)], \tag{41}$$

$$a_2(x) = \ \frac{1}{2}[(a_1(x))_{xx} + a_1(x) + r_1(x) - A_1(x)], \tag{42}$$

Also, we have

$$r_0(x) = x, \qquad r_1(x) = 2x, \qquad r_2(x) = 3x^2 \tag{43}$$

$$A_0(x) = 1, \qquad A_1(x) = 3x, \qquad A_2(x) = 3x^2. \tag{44}$$

Then, we find the coefficients as follows;

$$a_1(x) = x, \qquad a_k(x) = 0, \qquad k \geq 2 \tag{45}$$

In this method, the solution is given in the form of (19). Putting the coefficients $a_0(x) = 1$, $a_1(x) = x$ and $a_k(x) = 0$, $k \geq 2$ in (19), we find the exact solution as

$$u(x,t) = 1 + xt. \tag{46}$$

Solving The Fitzhugh-Nagumo-Huxley Equatıon By Laplace Decomposition Method

Firtsly, the laplace transfrom is applied to both sides of the given equation (4). Then, we have

$$L[u(x,t)] = \frac{1}{s}(1 + \ L[u_{xx}(x,t)] + L[u(x,t)] - L[u^3(x,t)] + L[x^3t^3 + 3x^2t^2 + 2xt + x]). \tag{47}$$

By using (29)-(30), we get the following equalities;

$$L[u_0(x,t)] = \frac{1}{s} + \frac{1}{s}L[x^3t^3 + 3x^2t^2 + 2xt + x] \tag{48}$$

$$L[u_{k+1}(x,t)] = \frac{1}{s}L[u_{k_{xx}}(x,t)] + \frac{1}{s}L[u_k(x,t)] - L[A_k(x,t)], \qquad k \geq 0 \tag{49}$$

Firstly, we need to find $u_0(x,t)$.

$$L[u_0(x,t)] = \frac{1}{s} + \frac{1}{s}\left(x^3\frac{3!}{s^4} + 3x^2\frac{2!}{s^3} + 2x\frac{1!}{s^2} + x\frac{1!}{s}\right] = \frac{1}{s} + x^3\frac{3!}{s^5} + 3x^2\frac{2!}{s^4} + 2x\frac{1!}{s^3} + x\frac{1!}{s^2}. \tag{50}$$

Applying L^{-1} to both sides of the equation (50), we obtain

$$u_0(x,t) = 1 + \frac{1}{4}x^3t^4 + x^2t^3 + xt^2 + xt. \tag{51}$$

To get $u_1(x,t)$, we need to solve the following equation

$$L[u_1(x,t)] = \frac{1}{s}L\left[u_{0_{xx}}(x,t)\right] + \frac{1}{s}L[u_0(x,t)] - L[A_0(x,t)]. \tag{52}$$

Putting $u_0(x,t)$, $u_{0_{xx}}(x,t)$ and $A_0(x,t)$ into (52), we get

$$u_1(x,t) = -\frac{1}{4}x^3t^4 - x^2t^3 - xt^2 + \frac{1}{4}t^4 - \cdots \tag{53}$$

So, we have the noise terms $\frac{1}{4}x^3t^4$, x^2t^3 and xt^2. Firstly, let us remove the noise terms from the component $u_0(x,t)$. Then, we verify that the remaining part of $u_0(x,t)$ satisfies the equation (4). So,

$u(x,t) = 1 + xt$ is the solution.

An Application of Laplace Decomposition Method

In this chapter, LDM is applied to two nonlinear inhomogeneous partial differential equations. Firstly, the general form of inhomogeneous Boussinesq equation is considered.

$$u_{tt} + \alpha u u_{xx} + \alpha {u_x}^2 + \beta u_{xxx} = h(x,t), \qquad \alpha\beta \neq 0. \tag{54}$$

Here, we study the case $\alpha = -1$, $\beta = -1$ and $h(x,t) = 2x^2 - 6\,x^2t^4$ with the initial conditions

$$u(x,0) = u_t(x,0) = 0.$$

By applying laplace transform to both sides of (54), we get

$$L[u(x,t)] = \frac{1}{s^2}\left(L\left[u(x,t)u_{xx}(x,t) + {u_x}^2(x,t)\right] + L[u_{xxx}(x,t)] + L[2x^2 - 6\,x^2t^4]\right). \tag{55}$$

Using (29) and (30) gives us

$$L[u_0(x,t)] = \frac{1}{s^2}L[2x^2 - 6\,x^2t^4] \tag{56}$$

$$L[u_{k+1}(x,t)] = \frac{1}{s^2}L\left[u_{k_{xxx}}(x,t)\right] - \frac{1}{s^2}L[A_k(x,t)], \qquad k \geq 0 \tag{57}$$

where $A_k(x,t)$ is Adomian polynomials for the nonlinear terms $F\big(u(x,t)\big) = uu_{xx} + {u_x}^2$.
From (65), we find

(58)

$$u_0(x,t) = x^2t^2 - \frac{x^2t^6}{5}.$$

We need to find $u_1(x,t)$ to determine the noise terms. In (66), letting $k = 0$ provide us with

$$L[u_1(x,t)] = \frac{1}{s^2}L\left[u_{0_{xxx}}(x,t)\right] - \frac{1}{s^2}L[A_0(x,t)] \tag{59}$$

where $A_0(x,t) = u_0u_{0_{xx}} + (u_{0_x})^2$. So, we have

$$u_1(x,t) = \frac{x^2t^6}{5} - \frac{2x^2t^{10}}{75} + \frac{3x^2t^{14}}{2275}. \tag{60}$$

From (58) and (60), we obtain the noise term as $\frac{x^2t^6}{5}$. Now, cancelling the noise term from the first component $u_0(x,t)$ and verifying that the rest part of $u_0(x,t)$ satifies the equation and its initial conditions, we have the solution as $u(x,t) = x^2t^2$.

Secondly, let us consider the following equation

$$u_{tt} - u_{xx} + uu_x + u^2 = x^4t^2 + 2x^3t^2 - 2t \tag{61}$$

with the initial conditions $u(x,0) = 0$ and $u_t(x,0) = x^2$.

Applying the laplace transform to both sides of the equation (61), we have

$$L[u(x,t)] = \frac{1}{s^2}\left(x^2 + L[u_{xx}(x,t)] - L[u_x(x,t)u(x,t) + u^2(x,t)] + L[x^4t^2 + 2x^3t^2 - 2t]\right). \tag{62}$$

Using (29)-(30), we have the following equalities;

$$L[u_0(x,t)] = \frac{x^2}{s^2} + \frac{1}{s^2}L[x^4t^2 + 2x^3t^2 - 2t] \tag{63}$$

$$L[u_{k+1}(x,t)] = \frac{1}{s^2}L\left[u_{k_{xx}}(x,t)\right] - \frac{1}{s^2}L[A_k(x,t)]\,, \qquad k \geq 0 \tag{64}$$

where $A_k(x,t)$ is Adomian polynomials for the nonlinear terms $F(u(x,t)) = uu_x + u^2$. Applying L^{-1} to both sides of the equation (63), we obtain

$$u_0(x,t) = x^2\mathrm{t} + \frac{1}{12}\,x^4t^4 + \frac{1}{6}\,x^3t^4 - \frac{1}{3}\,t^3. \tag{65}$$

Letting $k = 0$ in (64), we get

$$L[u_1(x,t)] = \frac{1}{s^2}L\left[u_{0_{xx}}(x,t)\right] - \frac{1}{s^2}L[A_0(x,t)]. \tag{66}$$

To find $u_1(x,t)$, we need to compute $u_{0_{xx}}(x,t)$ and $A_0(x,t)$ where $A_0(x,t) = u_0u_{0_x} + {u_0}^2$. Now, putting $u_{0_{xx}}(x,t)$, $A_0(x,t)$ into (66) and applying L^{-1} provides us with

(67)

$$u_1(x,t) = \frac{1}{3}\,t^3 - \frac{1}{12}\,x^4t^4 - \frac{1}{6}\,x^3t^4 + \frac{1}{18}xt^6 + \frac{1}{18}x^2t^6 + \cdots$$

From (65) and (67), the noise terms, $\frac{1}{12}x^4t^4$, $\frac{1}{6}x^3t^4$, $-\frac{1}{3}t^3$, are observed. The noise terms of the component $u_0(x,t)$ are removed. Then, we check and see that the rest of $u_0(x,t)$ satisfies the given equation (61) with its inital conditions. So, the exact solution of the given equation is $u(x,t) = x^2t$.

CONCLUSION

In this study we consider the inhomogeneous Fitzhugh-Nagumo-Huxley equation with initial condition. To solve it, ADM, MDM and LDM are applied. Then, it is seen that each method provides us with the same exact solution. In addition to their effectiveness and usefulness in solving linear partial differential equations, we show that these decomposition methods are powerful tools in solving nonlinear ones with initial and boundary conditions. Compared to other methods for solving nonlinear ordinary or partial differential equations, there is no need for linearization of nonlinear terms thanks to the Adomian polynomials. Moreover, we can easily and rapidly attain the solution by means of the noise terms as shown in solving the Fitzhugh-Nagumo-Huxley equation, the Boussinesq equation and an example.

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