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Solutions of the Maximum of Difference Equations $x_{n+1}=\max\left\{\frac{1}{x_{n-1}},\frac{y_n}{x_{n-3}}\right\};y_{n+1}=\max\left\{\frac{1}{y_{n-1}},\frac{x_n}{y_{n-3}}\right\}$

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Abstract: *The behaviour of the solutions of the following system of difference equations is examined.*

$$x_{n+1}=\max\left\{\frac{1}{x_{n-1}},\frac{y_n}{x_{n-3}}\right\};y_{n+1}=\max\left\{\frac{1}{y_{n-1}},\frac{x_n}{y_{n-3}}\right\}, \qquad (1)$$

where the initial conditions are positive real numbers.

Keywords: *Difference Equation, Maximum Operations, Semicycle*

$x_{n+1}=\max\left\{\frac{1}{x_{n-1}},\frac{y_n}{x_{n-3}}\right\};y_{n+1}=\max\left\{\frac{1}{y_{n-1}},\frac{x_n}{y_{n-3}}\right\}$ Maksimumlu Fark Denkleminin Çözümleri

Özet: *Aşağıdaki Maksimumlu fark denklemin sisteminin çözümlerinin davranışları incelendi.*

$$x_{n+1}=\max\left\{\frac{1}{x_{n-1}},\frac{y_n}{x_{n-3}}\right\};y_{n+1}=\max\left\{\frac{1}{y_{n-1}},\frac{x_n}{y_{n-3}}\right\}, \qquad (1)$$

burada başlangıç şartları reel sayılardır.

Anahtar Kelimeler: *Fark Denklemleri,Maksimum Operatörü, Yarı Dönmeler*

INTRODUCTION

Investigations on the recent studies show that researches on the periodic nature of nonlinear difference equations have been an object of great interest. Although difference equations are relatively simple in form, it is, unfortunately, extremely difficult to understand thoroughly the periodic behavior of their solutions. The periodic nature of nonlinear difference equations of the max type has been investigated by many authors (see 1-30).

Definition 1.1. A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be eventually periodic with period p if there is $n_0 \in \{-k, \ldots\ldots, -1, 0, 1\}$ such that $x_{n+p} = x_n$ for all $n \geq n_0$. If $n_0 = -k$, then we say that the sequence $\{x_n\}_{n=-k}^{\infty}$ is periodic with p.

Definition 1.2. Let I be an interval of real numbers and let $f : I^{s+1} \rightarrow I$ be a continuously differentiable function, where s is a non-negative integer. Consider the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-s}) \text{ for } n = 0,1,2,\ldots \tag{2}$$

with the initial values $x_{-s}, \ldots, x_0 \in I$. A point $\bar{x}$ called an equilibrium point of Eq.(2) if $\bar{x} = f(\bar{x}, \ldots, \bar{x})$.

Definition 1.3. A positive semicycle of a solutions $\{x_n\}_{n=-s}^{\infty}$ of Eq.(2) consists of a string of terms $\{x_l, x_{l+1}, \ldots, x_m\}$, all greater than or equal to equilibrium $\bar{x}$ with $l \geq -s$ and $m \leq \infty$ such as that either $l = -s$ or $l >$ -s ve $x_{l-1} < \bar{x}$ and either $m = \infty$ or $m < \infty$ and $x_{m+1} < \bar{x}$.

Definition 1.4. A negative semicycle of a solutions $\{x_n\}_{n=-s}^{\infty}$ of Eq.(2) consists of a string of terms $\{x_l, x_{l+1}, \ldots, x_m\}$, all less than or equal to equilibrium $\bar{x}$ with $l \geq -s$ and $m \leq \infty$ such that either $l = -s$ or $l >$ -s and $x_{l-1} \geq \bar{x}$ and either $m = \infty$ or $m \leq \infty$ and $x_{m+1} \geq \bar{x}$.

MAIN RESULTS

Let $\bar{x}$ and $\bar{y}$ be the unique positive equilibrium of Eq.(1), then clearly

$$\bar{x} = \max\left\{\frac{1}{\bar{x}}, \frac{\bar{y}}{\bar{x}}\right\}; \bar{y} = \min\left\{\frac{1}{\bar{y}}, \frac{\bar{x}}{\bar{y}}\right\}$$

$$\bar{x} = \frac{1}{\bar{x}} \Rightarrow \bar{x}^2 = 1 \Rightarrow \bar{x} = 1 \text{and} \bar{x} = \frac{\bar{y}}{\bar{x}} \Rightarrow \bar{x}^2 = \bar{y},$$

$$\bar{y} = \frac{1}{\bar{y}} \Rightarrow \bar{y}^2 = 1 \Rightarrow \bar{y} = 1 \text{and} \bar{y} = \frac{\bar{x}}{\bar{y}} \Rightarrow \bar{y}^2 = \bar{x},$$

From which we can obtain $\bar{x} = 1$ and $\bar{y} = 1$.

Lemma 2.1. Assume that, $1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-1} < y_{-2} < y_{-3} < y_0$,
$1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-1} < y_{-3} < y_0 < y_{-2}$, $1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-2} < y_{-3} < y_0 < y_{-1}$,
$1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-3} < y_{-1} < y_0 < y_{-2}$, $1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-2} < y_{-1} < y_{-3} < y_0$,
$1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-3} < y_0 < y_{-1} < y_{-2}$, $1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_0 < y_{-2} < y_{-3} < y_{-1}$,
$1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_0 < y_{-3} < y_{-2} < y_{-1}$, $1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-1} < y_0 < y_{-3} < y_{-2}$,
$1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-3} < y_{-2} < y_{-1} < y_0$, $1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-3} < y_0 < y_{-2} < y_{-1}$,
$1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-2} < y_{-3} < y_{-1} < y_0$, $1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-2} < y_0 < y_{-3} < y_{-1}$,
$1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-1} < y_{-3} < y_{-2} < y_0$, $1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_0 < y_{-3} < y_{-1} < y_{-2}$,
$1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-3} < y_{-1} < y_{-2} < y_0$, $1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-3} < y_{-2} < y_0 < y_{-1}$

Then the following statements are true for the solutions of Eq.(1) :

- **a)** $(x_n . y_n)$ is the solution, solution x_n, for $n \geq 1$and solution y_n, for $n \geq 0$; every positive semicycle consists of six terms, every negative semicycle consists of two terms;
- **b)** $(x_n . y_n)$ is the solution, solution x_n, for $n \geq 1$and solution y_n, for $n \geq 0$; every positive semicycle of length six is followed by a negative semicycle of length two;
- **c)** $(x_n . y_n)$ is the solution, solution x_n, for $n \geq 1$and solution y_n, for $n \geq 0$; every negative semicycle of length two is followed by a positive semicycle of length six.

Proof. a)

$$x_1 = \max\left\{\frac{1}{x_{-1}}, \frac{y_0}{x_{-3}}\right\} = \frac{y_0}{x_{-3}} > \bar{x}$$

$$\text{y}_1 = \max\left\{\frac{1}{y_{-3}}, \frac{x_0}{y_{-3}}\right\} = \frac{x_0}{y_{-3}} < \bar{y}$$

$$x_2 = \max\left\{\frac{1}{x_0}, \frac{y_1}{x_{-2}}\right\} = \max\left\{\frac{1}{x_0}, \frac{x_0}{y_{-3}x_{-2}}\right\} = \frac{x_0}{y_{-3}x_{-2}} < \bar{x}$$

$$\text{y}_2 = \max\left\{\frac{1}{y_0}, \frac{x_1}{y-2}\right\} = \max\left\{\frac{1}{y_0}, \frac{y_0}{x_{-3}y_{-2}}\right\} = \frac{y_0}{x_{-3}y_{-2}} < \bar{y}$$

$$x_3=\max\left\{\frac{1}{x_1},\frac{y_2}{x_{-1}}\right\}=\max\left\{\frac{x_{-3}}{y_0},\frac{y_0}{x_{-3}x_{-1}y_{-2}}\right\}=\frac{x_{-3}}{y_0}>\bar{x}$$

$$y_3=\max\left\{\frac{1}{y_1},\frac{x_2}{y_{-1}}\right\}=\max\left\{\frac{y_{-3}}{x_0},\frac{x_0}{y_{-3}y_{-1}x_{-2}}\right\}=\frac{y_{-3}}{x_0}>\bar{y}$$

$$x_4=\max\left\{\frac{1}{x_2},\frac{y_3}{x_0}\right\}=\max\left\{\frac{y_{-3}x_{-2}}{x_0},\frac{y_{-3}}{x_0^{\,2}}\right\}=\frac{y_{-3}x_{-2}}{x_0}>\bar{x}$$

$$y_4=\max\left\{\frac{1}{y_2},\frac{x_3}{y_0}\right\}=\max\left\{\frac{x_{-3}y_{-2}}{y_0},\frac{x_{-3}}{y_0^{\,2}}\right\}=\frac{x_{-3}y_{-2}}{y_0}>\bar{y}$$

$$x_5=\max\left\{\frac{1}{x_3},\frac{y_4}{x_1}\right\}=\max\left\{\frac{y_0}{x_{-3}},\frac{x_{-3}^{\,2}y_{-2}}{y_0^{\,2}}\right\}=\frac{y_0}{x_{-3}}>\bar{x}$$

$$y_5=\max\left\{\frac{1}{y_3},\frac{x_4}{y_1}\right\}=\max\left\{\frac{x_0}{y_{-3}},\frac{y_{-3}^{\,2}x_{-2}}{x_0^{\,2}}\right\}=\frac{y_{-3}^{\,2}x_{-2}}{x_0^{\,2}}>\bar{y}$$

$$x_6=\max\left\{\frac{1}{x_4},\frac{y_5}{x_2}\right\}=\max\left\{\frac{x_0}{y_{-3}x_{-2}},\frac{y_{-3}^{\,3}x_{-2}^{\,2}}{x_0^{\,3}}\right\}=\frac{y_{-3}^{\,3}x_{-2}^{\,2}}{x_0^{\,3}}>\bar{x}$$

$$y_6=\max\left\{\frac{1}{y_4},\frac{x_5}{y_2}\right\}=\max\left\{\frac{y_0}{x_{-3}y_{-2}},y_{-2}\right\}=y_{-2}>\bar{y}$$

$$x_7=\max\left\{\frac{1}{x_5},\frac{y_6}{x_2}\right\}=\max\left\{\frac{x_{-3}}{y_0},\frac{y_0y_{-2}}{x_{-3}}\right\}=\frac{y_0y_{-2}}{x_{-3}}>\bar{x}$$

$$y_7=\max\left\{\frac{1}{y_5},\frac{x_6}{y_3}\right\}=\max\left\{\frac{x_0^{\,2}}{y_{-3}^{\,2}x_{-2}},\frac{y_{-3}^{\,2}x_{-2}^{\,2}}{x_0^{\,2}}\right\}=\frac{y_{-3}^{\,2}x_{-2}^{\,2}}{x_0^{\,2}}>\bar{y}$$

$$x_8=\max\left\{\frac{1}{x_6},\frac{y_7}{x_4}\right\}=\max\left\{\frac{x_0^{\,3}}{y_{-3}^{\,3}x_{-2}^{\,2}},\frac{y_{-3}x_{-2}}{x_0}\right\}=y_{-1}>\bar{x}$$

$$y_8=\max\left\{\frac{1}{y_6},\frac{x_7}{y_4}\right\}=\max\left\{\frac{1}{y_{-2}},\frac{y_0^{\,2}}{x_{-3}^{\,2}}\right\}=\frac{y_0^{\,2}}{x_{-3}^{\,2}}>\bar{y}$$

$$x_9=\max\left\{\frac{1}{x_7},\frac{y_8}{x_5}\right\}=\max\left\{\frac{x_{-3}}{y_0y_{-2}},\frac{y_0}{x_{-3}}\right\}=\frac{y_0}{x_{-3}}>\bar{x}$$

$$y_9=\max\left\{\frac{1}{y_7},\frac{x_8}{y_5}\right\}=\max\left\{\frac{x_0^{\,2}}{y_{-3}^{\,2}x_{-2}^{\,2}},\frac{x_0}{y_{-3}}\right\}=\frac{x_0}{y_{-3}}<\bar{y}$$

Consequently, we have obtained,

$$x_1>\bar{x},\ x_2<\bar{x},\ x_3<\bar{x},\ x_4>\bar{x},\ x_5>\bar{x},\ x_6>\bar{x},\ x_7>\bar{x},\ x_8>\bar{x},\ \ldots$$
$$y_1<\bar{y},\ y_2<\bar{y},\ y_3>\bar{y},\ y_4>\bar{y},\ y_5>\bar{y},\ y_6>\bar{y},\ y_7>\bar{y},\ y_8>\bar{y},\ \ldots\ .$$

Hence, the solution x_n for $n \geq 1$ and the solution y_n for $n \geq 0$; every positive semicycle consists of six terms, every negative semicycle consists of two terms. The proof is:

- **a)** The solution is x_n for $n \geq 1$ and solution y_n for $n \geq 0$; every positive semicycle consists of six terms, every negative semicycle consists of two terms;
- **b)** Using the proof a);

Therefore, the solution x_n for $n \geq 1$ and solution y_n for $n \geq 0$; every positive semicycle of length six is followed by a negative semicycle of length two.

- **c)** Using the proof a).

Therefore, the solution x_n for $n \geq 1$ and solution y_n for $n \geq 0$; every negative semicycle of length two is followed by a positive semicycle of length six.

Lemma 2.2. Assume that**,**

$$1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-2} < y_{-1} < y_0 < y_{-3},\ 1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_0 < y_{-2} < y_{-1} < y_{-3},$$
$$1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-1} < y_0 < y_{-2} < y_{-3},\ 1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-1} < y_{-2} < y_0 < y_{-3},$$
$$1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_0 < y_{-1} < y_{-2} < y_{-3},\ 1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-2} < y_0 < y_{-1} < y_{-3}$$

Then the following statements are true for the solutions of Eq.(1) :

a) $(x_n . y_n)$ is the solution, solution x_n for $n \geq 1$ and solution y_n for $n \geq 0$; every positive semicycle consists of six terms, every negative semicycle consists of two terms;

b) $(x_n . y_n)$ is the solution, solution x_n for $n \geq 1$ and solution y_n for $n \geq 0$; every positive semicycle of length six is followed by a negative semicycle of length two;

c) $(x_n . y_n)$ is the solution, solution x_n for $n \geq 1$ and solution y_n for $n \geq 0$; every negative semicycle of length two is followed by a positive semicycle of length six.

Proof. Similarly, we can obtain the proof of Lemma 2.2 as in the proof of Lemma 2.1.

Lemma 2.3. Assume that**,**

$$1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_0 < x_{-1} < x_{-2} < x_{-3},\ 1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_0 < x_{-2} < x_{-1} < x_{-3},$$
$$1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-2} < x_0 < x_{-1} < x_{-3},\ 1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-2} < x_{-1} < x_0 < x_{-3},$$
$$1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-1} < x_0 < x_{-2} < x_{-3},\ 1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-1} < x_{-2} < x_0 < x_{-3}$$

Then the following statements are true for the solutions of Eq.(1) :

- **d)** $(x_n . y_n)$ is the solution, solution x_n for $n \geq 0$ and solution y_n for $n \geq 1$; every positive semicycle consists of six terms, every negative semicycle consists of two terms;

- **e)** $(x_n.y_n)$ is the solution, solution x_n for $n \ge 0$ and solution y_n for $n \ge 1$; every positive semicycle of length six is followed by a negative semicycle of length two;
- **f)** $(x_n.y_n)$ *is* the solution, solution x_n for $n \ge 0$ and solution y_n for $n \ge 1$; every negative semicycle of length two is followed by a positive semicycle of length six.

Proof. Similarly, we can obtain the proof of Lemma 2.3 as in the proof of Lemma 2.1.

Lemma 2.4. Assume that**,**

$$1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-3} < x_{-2} < x_{-1} < x_0,\ 1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-3} < x_{-1} < x_{-2} < x_0,$$
$$1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-3} < x_{-1} < x_0 < x_{-2},\ 1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_0 < x_{-3} < x_{-1} < x_{-2},$$
$$1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_0 < x_{-3} < x_{-2} < x_{-1},\ 1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-1} < x_0 < x_{-3} < x_{-2},$$
$$1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-1} < x_{-2} < x_{-3} < x_0,\ 1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-1} < x_{-3} < x_0 < x_{-2},$$
$$1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-2} < x_{-3} < x_0 < x_{-1},\ 1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-3} < x_{-2} < x_0 < x_{-1},$$
$$1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-3} < x_0 < x_{-2} < x_{-1},\ 1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-2} < x_0 < x_{-3} < x_{-1},$$
$$1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_0 < x_{-2} < x_{-3} < x_{-1},\ 1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-3} < x_0 < x_{-1} < x_{-2},$$
$$1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_0 < x_{-1} < x_{-3} < x_{-2},\ 1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-2} < x_{-3} < x_{-1} < x_0,$$
$$1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-2} < x_{-1} < x_{-3} < x_0,\ 1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-1} < x_{-3} < x_{-2} < x_0$$

Then, the following statements are true for the solutions of Eq.(1) :

- **a)** $(x_n.y_n)$ is the solution, solution x_n for $n \ge 0$ and solution y_n for $n \ge 1$; every positive semicycle consists of six terms and every negative semicycle consists of two terms;
- **b)** $(x_n.y_n)$ is the solution, solution x_n for $n \ge 0$ and solution y_n for $n \ge 1$; every positive semicycle of length six is followed by a negative semicycle of length two;
- **c)** $(x_n.y_n)$ *is* the solution, solution x_n for $n \ge 0$ and solution y_n for $n \ge 1$; every negative semicycle of length two is followed by a positive semicycle of length six.

Proof. Similarly, we can obtain the proof of Lemma 2.4 as in the proof of Lemma 2.1.

Theorem 2.1. Let $(x_n.y_n)$ *be the solution of Eq.(1) for*

$$1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-1} < y_{-2} < y_{-3} < y_0,\ 1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-1} < y_{-3} < y_0 < y_{-2},$$
$$1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-2} < y_{-3} < y_0 < y_{-1},\ 1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-3} < y_{-1} < y_0 < y_{-2},$$
$$1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-2} < y_{-1} < y_{-3} < y_0,\ 1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-3} < y_0 < y_{-1} < y_{-2},$$
$$1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_0 < y_{-2} < y_{-3} < y_{-1},\ 1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_0 < y_{-3} < y_{-2} < y_{-1},$$
$$1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-1} < y_0 < y_{-3} < y_{-2},\ 1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-3} < y_{-2} < y_{-1} < y_0,$$

$$1<x_{-3}<x_{-2}<x_{-1}<x_0<y_{-3}<y_0<y_{-2}<y_{-1}\ ,\ 1<x_{-3}<x_{-2}<x_{-1}<x_0<y_{-2}<y_{-3}<y_{-1}<y_0\ ,$$

$$1<x_{-3}<x_{-2}<x_{-1}<x_0<y_{-2}<y_0<y_{-3}<y_{-1}\ ,\ 1<x_{-3}<x_{-2}<x_{-1}<x_0<y_{-1}<y_{-3}<y_{-2}<y_0\ ,$$

$$1<x_{-3}<x_{-2}<x_{-1}<x_0<y_0<y_{-3}<y_{-1}<y_{-2}\ ,\ 1<x_{-3}<x_{-2}<x_{-1}<x_0<y_{-3}<y_{-1}<y_{-2}<y_0\ ,$$

$$1<x_{-3}<x_{-2}<x_{-1}<x_0<y_{-3}<y_{-2}<y_0<y_{-1}$$

Then every $(x_n.y_n)$ is periodic with period eight.

$$x_n=\left(\frac{y_0}{x_{-3}},\frac{x_0}{y_{-3}x_{-2}},\frac{x_{-3}}{y_0},\frac{y_{-3}x_{-2}}{x_0},\frac{y_0}{x_{-3}},\frac{y_{-3}^{\,3}x_{-2}^{\,2}}{x_0^{\,3}},\frac{y_0y_{-2}}{x_{-3}},\frac{y_{-3}x_{-2}}{x_0},\ldots\right)$$

$$y_n=\left(\frac{x_0}{y_{-3}},\frac{y_0}{x_{-3}y_{-2}},\frac{y_{-3}}{x_0},\frac{x_{-3}y_{-2}}{y_0},\frac{y_{-3}^{\,2}x_{-2}}{x_0^{\,2}},y_{-2},\frac{y_{-3}^{\,2}x_{-2}^{\,2}}{x_0^{\,2}},\frac{y_0^{\,2}}{x_{-3}^{\,2}},\ldots\right)$$

Proof.

$$x_n=\max\left\{\frac{1}{x_{n-1}},\frac{y_n}{x_{n-3}}\right\};y_n=\max\left\{\frac{1}{y_{n-1}},\frac{x_n}{y_{n-3}}\right\}$$

$$x_1=\max\left\{\frac{1}{x_{-1}},\frac{y_0}{x_{-3}}\right\}=\frac{y_0}{x_{-3}}$$

$$y_1=\max\left\{\frac{1}{y_{-3}},\frac{x_0}{y_{-3}}\right\}=\frac{x_0}{y_{-3}}$$

$$x_2=\max\left\{\frac{1}{x_0},\frac{y_1}{x_{-2}}\right\}=\max\left\{\frac{1}{x_0},\frac{x_0}{y_{-3}x_{-2}}\right\}=\frac{x_0}{y_{-3}x_{-2}}$$

$$y_2=\max\left\{\frac{1}{y_0},\frac{x_1}{y-2}\right\}=\max\left\{\frac{1}{y_0},\frac{y_0}{x_{-3}y_{-2}}\right\}=\frac{y_0}{x_{-3}y_{-2}}$$

$$x_3=\max\left\{\frac{1}{x_1},\frac{y_2}{x_{-1}}\right\}=\max\left\{\frac{x_{-3}}{y_0},\frac{y_0}{x_{-3}x_{-1}y_{-2}}\right\}=\frac{x_{-3}}{y_0}$$

$$y_3=\max\left\{\frac{1}{y_1},\frac{x_2}{y_{-1}}\right\}=\max\left\{\frac{y_{-3}}{x_0},\frac{x_0}{y_{-3}y_{-1}x_{-2}}\right\}=\frac{y_{-3}}{x_0}$$

$$x_4=\max\left\{\frac{1}{x_2},\frac{y_3}{x_0}\right\}=\max\left\{\frac{y_{-3}x_{-2}}{x_0},\frac{y_{-3}}{x_0^{\,2}}\right\}=\frac{y_{-3}x_{-2}}{x_0}$$

$$y_4=\max\left\{\frac{1}{y_2},\frac{x_3}{y_0}\right\}=\max\left\{\frac{x_{-3}y_{-2}}{y_0},\frac{x_{-3}}{y_0^{\,2}}\right\}=\frac{x_{-3}y_{-2}}{y_0}$$

$$x_5=\max\left\{\frac{1}{x_3},\frac{y_4}{x_1}\right\}=\max\left\{\frac{y_0}{x_{-3}},\frac{x_{-3}^{\,2}y_{-2}}{y_0^{\,2}}\right\}=\frac{y_0}{x_{-3}}$$

$$y_5=\max\left\{\frac{1}{y_3},\frac{x_4}{y_1}\right\}=\max\left\{\frac{x_0}{y_{-3}},\frac{{y_{-3}}^2x_{-2}}{{x_0}^2}\right\}=\frac{{y_{-3}}^2x_{-2}}{{x_0}^2}$$

$$x_6=\max\left\{\frac{1}{x_4},\frac{y_5}{x_2}\right\}=\max\left\{\frac{x_0}{y_{-3}x_{-2}},\frac{{y_{-3}}^3{x_{-2}}^2}{{x_0}^3}\right\}=\frac{{y_{-3}}^3{x_{-2}}^2}{{x_0}^3}$$

$$y_6=\max\left\{\frac{1}{y_4},\frac{x_5}{y_2}\right\}=\max\left\{\frac{y_0}{x_{-3}y_{-2}},y_{-2}\right\}=y_{-2}$$

$$x_7=\max\left\{\frac{1}{x_5},\frac{y_6}{x_2}\right\}=\max\left\{\frac{x_{-3}}{y_0},\frac{y_0y_{-2}}{x_{-3}}\right\}=\frac{y_0y_{-2}}{x_{-3}}$$

$$y_7=\max\left\{\frac{1}{y_5},\frac{x_6}{y_3}\right\}=\max\left\{\frac{{x_0}^2}{{y_{-3}}^2x_{-2}},\frac{{y_{-3}}^2{x_{-2}}^2}{{x_0}^2}\right\}=\frac{{y_{-3}}^2{x_{-2}}^2}{{x_0}^2}$$

$$x_8=\max\left\{\frac{1}{x_6},\frac{y_7}{x_4}\right\}=\max\left\{\frac{{x_0}^3}{{y_{-3}}^3{x_{-2}}^2},\frac{y_{-3}x_{-2}}{x_0}\right\}=y_{-1}$$

$$y_8=\max\left\{\frac{1}{y_6},\frac{x_7}{y_4}\right\}=\max\left\{\frac{1}{y_{-2}},\frac{{y_0}^2}{{x_{-3}}^2}\right\}=\frac{{y_0}^2}{{x_{-3}}^2}$$

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If $n\geq 0$, then

$$x_n=\left(\frac{y_0}{x_{-3}},\frac{x_0}{y_{-3}x_{-2}},\frac{x_{-3}}{y_0},\frac{y_{-3}x_{-2}}{x_0},\frac{y_0}{x_{-3}},\frac{{y_{-3}}^3{x_{-2}}^2}{{x_0}^3},\frac{y_0y_{-2}}{x_{-3}},\frac{y_{-3}x_{-2}}{x_0},\ldots\right)$$

$$y_n=\left(\frac{x_0}{y_{-3}},\frac{y_0}{x_{-3}y_{-2}},\frac{y_{-3}}{x_0},\frac{x_{-3}y_{-2}}{y_0},\frac{{y_{-3}}^2x_{-2}}{{x_0}^2},y_{-2},\frac{{y_{-3}}^2{x_{-2}}^2}{{x_0}^2},\frac{{y_0}^2}{{x_{-3}}^2},\ldots\right).$$

Theorem 2.2.

Let (x_n, y_n) be the solution of the system of difference equations (1) for

$1<x_{-3}<x_{-2}<x_{-1}<x_0<y_{-2}<y_{-1}<y_0<y_{-3}$, $1<x_{-3}<x_{-2}<x_{-1}<x_0<y_0<y_{-2}<y_{-1}<y_{-3}$, $1<x_{-3}<x_{-2}<x_{-1}<x_0<y_{-1}<y_0<y_{-2}<y_{-3}$, $1<x_{-3}<x_{-2}<x_{-1}<x_0<y_{-1}<y_{-2}<y_0<y_{-3}$, $1<x_{-3}<x_{-2}<x_{-1}<x_0<y_0<y_{-1}<y_{-2}<y_{-3}$, $1<x_{-3}<x_{-2}<x_{-1}<x_0<y_{-2}<y_0<y_{-1}<y_{-3}$

Then every (x_n, y_n) is periodic with period eight.

$$x_n = \left(\frac{y_0}{x_{-1}}, \frac{1}{x_0}, \frac{x_{-3}}{y_0}, x_0, \frac{y_0}{x_{-3}}, y_{-3}x_0, \frac{y_0 y_{-2}}{x_{-3}}, x_0, \ldots \right)$$

$$y_n = \left(\frac{x_0}{y_{-3}}, \frac{y_0}{x_{-3}y_{-2}}, \frac{y_{-3}}{x_0}, \frac{x_{-3}y_{-2}}{y_0}, y_{-3}, y_{-2}, x_0^2, \frac{y_0^2}{x_{-3}^2}, \ldots \right)$$

Proof. Similarly, we can obtain the proof of Theorem 2.2 as in the proof of Theorem 2.1. which completes the proofs of the given theorems.

Theorem 2.3.

Let (x_n, y_n) be the solution of the system of difference equations (1) for

$1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_0 < x_{-1} < x_{-2} < x_{-3}$, $1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_0 < x_{-2} < x_{-1} < x_{-3}$,
$1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-2} < x_0 < x_{-1} < x_{-3}$, $1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-2} < x_{-1} < x_0 < x_{-3}$,
$1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-1} < x_0 < x_{-2} < x_{-3}$, $1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-1} < x_{-2} < x_0 < x_{-3}$

Then every (x_n, y_n) is periodic with period eight.

$$x_n = \left(\frac{y_0}{x_{-3}}, \frac{x_0}{x_{-2}y_{-3}}, \frac{x_{-3}}{y_0}, \frac{y_{-3}x_{-2}}{x_0}, x_{-3}, x_{-2}, y_0^2, \frac{x_0^2}{y_{-3}^2}, \ldots \right)$$

$$y_n = \left(\frac{x_0}{y_{-3}}, \frac{1}{y_0}, \frac{y_{-3}}{x_0}, y_0, \frac{x_0}{y_{-3}}, x_{-3}y_0, \frac{x_0 x_{-2}}{y_{-3}}, y_0, \ldots \right)$$

Proof. Similarly, we can obtain the proof of Theorem 2.3 as the in the proof of Theorem 2.1. which completes the proofs of the given theorems.

Theorem 2.4.

Let (x_n, y_n) be the solution of the system of difference equations (1) for

$1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-3} < x_{-2} < x_{-1} < x_0$, $1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-3} < x_{-1} < x_{-2} < x_0$,
$1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-3} < x_{-1} < x_0 < x_{-2}$, $1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_0 < x_{-3} < x_{-1} < x_{-2}$,
$1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_0 < x_{-3} < x_{-2} < x_{-1}$, $1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-1} < x_0 < x_{-3} < x_{-2}$,

$1<y_{-3}<y_{-2}<y_{-1}<y_0<x_{-1}<x_{-2}<x_{-3}<x_0$, $1<y_{-3}<y_{-2}<y_{-1}<y_0<x_{-1}<x_{-3}<x_0<x_{-2}$,
$1<y_{-3}<y_{-2}<y_{-1}<y_0<x_{-2}<x_{-3}<x_0<x_{-1}$, $1<y_{-3}<y_{-2}<y_{-1}<y_0<x_{-3}<x_{-2}<x_0<x_{-1}$,
$1<y_{-3}<y_{-2}<y_{-1}<y_0<x_{-3}<x_0<x_{-2}<x_{-1}$, $1<y_{-3}<y_{-2}<y_{-1}<y_0<x_{-2}<x_0<x_{-3}<x_{-1}$,
$1<y_{-3}<y_{-2}<y_{-1}<y_0<x_0<x_{-2}<x_{-3}<x_{-1}$, $1<y_{-3}<y_{-2}<y_{-1}<y_0<x_{-3}<x_0<x_{-1}<x_{-2}$,
$1<y_{-3}<y_{-2}<y_{-1}<y_0<x_0<x_{-1}<x_{-3}<x_{-2}$, $1<y_{-3}<y_{-2}<y_{-1}<y_0<x_{-2}<x_{-3}<x_{-1}<x_0$,
$1<y_{-3}<y_{-2}<y_{-1}<y_0<x_{-2}<x_{-1}<x_{-3}<x_0$, $1<y_{-3}<y_{-2}<y_{-1}<y_0<x_{-1}<x_{-3}<x_{-2}<x_0$

Then every (x_n, y_n) is periodic with period eight.

$$x_n=\left(\frac{y_0}{x_{-3}},\frac{x_0}{x_{-2}y_{-3}},\frac{x_{-3}}{y_0},\frac{x_{-2}y_{-3}}{x_0},\frac{y_{-2}x_{-3}^2}{y_0^2},x_{-2},\frac{x_{-3}^2y_{-2}^2}{y_0^2},\frac{x_0^2}{y_{-3}^2},\ldots\right)$$

$$y_n=\left(\frac{x_0}{y_{-3}},\frac{y_0}{y_{-2}x_{-3}},\frac{y_{-3}}{x_0},\frac{y_{-2}x_{-3}}{y_0},\frac{x_0}{y_{-3}},\frac{y_{-2}^2x_{-3}^3}{y_0^3},\frac{x_0x_{-2}}{y_{-3}},\frac{x_{-3}y_{-2}}{y_0},\ldots\right)$$

Proof. Similarly, we can obtain the proof of Theorem 2.4 as in the proof of Theorem 2.1. which completes the proofs of the shown theorems.

EXAMPLES

EXAMPLE 3.1: If the initial conditions are selected as follows:

$x[-3]=2$; $x[-2]=3$; $x[-1]=4$; $x[0]=5$; y[−3]=8; y[−2]=7; y[−1]=6; y[0]=9;

then the following solutions are obtained:

x(n) = { 4.5, 0.208333, 0.222222, 4. 8, 4.5, 36.864, 31.5, 4.8, 4.5, 0.208333, 0.222222, 4.8, 4.5, 36.864, 31.5, 4.8, …}
y(n) = { 0.625, 0.642857, 1.6, 1.55556, 7.68, 7., 23.04, 20.25, 0.625, 0.642857, 1.6, 1.55556, 7.68, 7., 23.04, 20.25,…}

The graph of the solutions is given below.

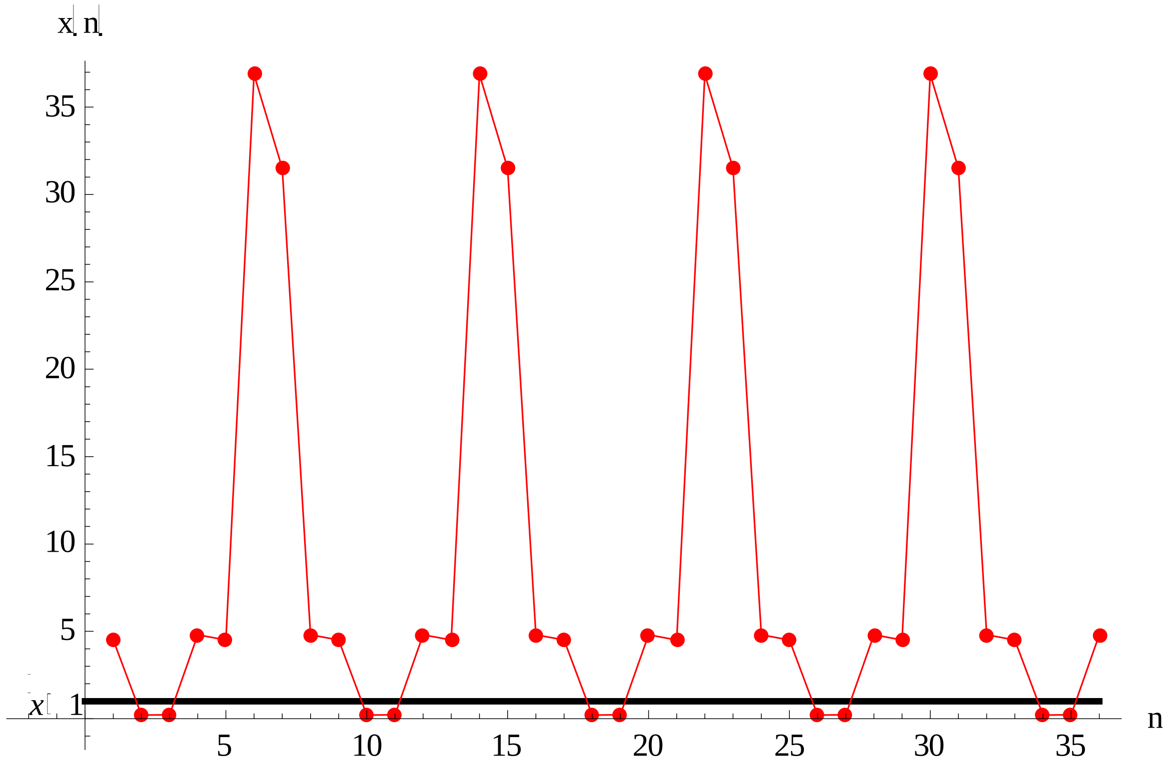

Figure 3.1. x(n) graph of the solutions

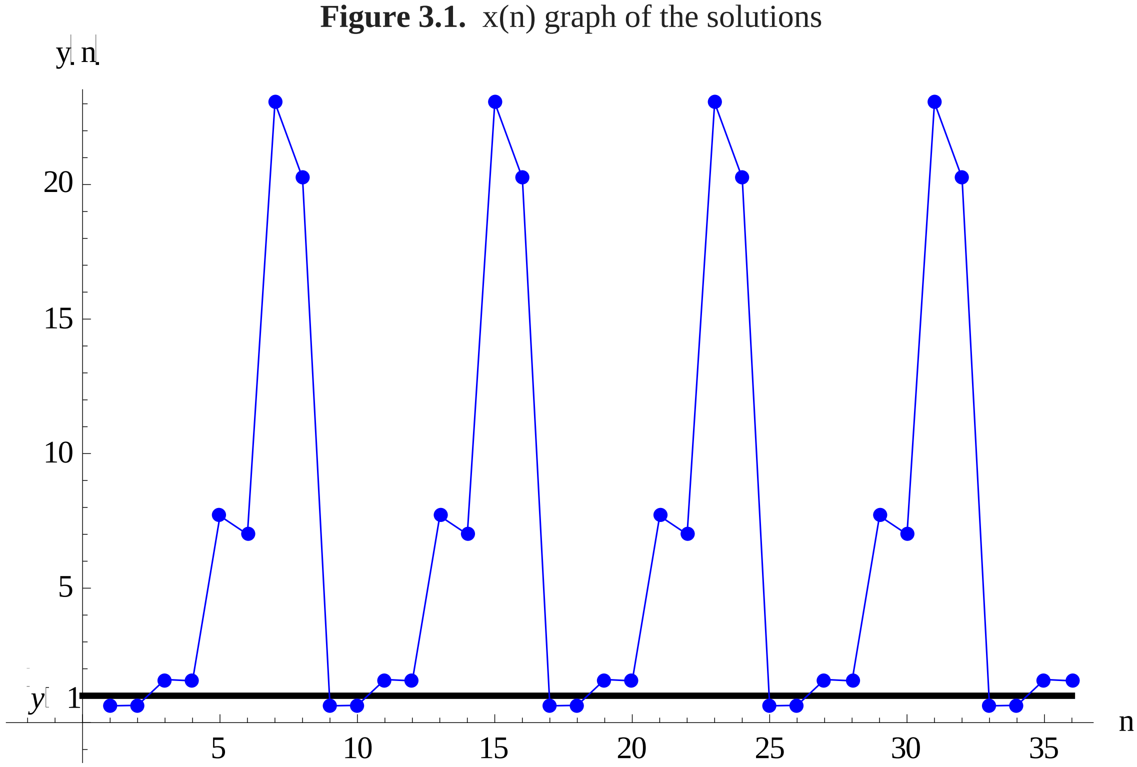

Figure 3.2. y(n) graph of the solutions

EXAMPLE 3.2: If the initial conditions are selected as follows:

$$x[-3]=9;\ x[-2]=8;\ x[-1]=7;\ x[0]=6;\ \ y[-3]=2;\ y[-2]=3;\ y[-1]=4;\ y[0]=5;$$

then the following solutions are obtained:
x(n) = { 0.555556, 0.375, 1.8, 2.66667, 9, 8, 25, 9, 0.555556, 0.375, 1.8, 2.66667, 9, 8, 25, 9, …}
y(n) = { 3, 0.2, 0.333333, 5, 3, 45, 24, 5, 3, 0.2, 0.333333, 5, 3, 45, 24, 5,…}

The graph of the solutions is given below.

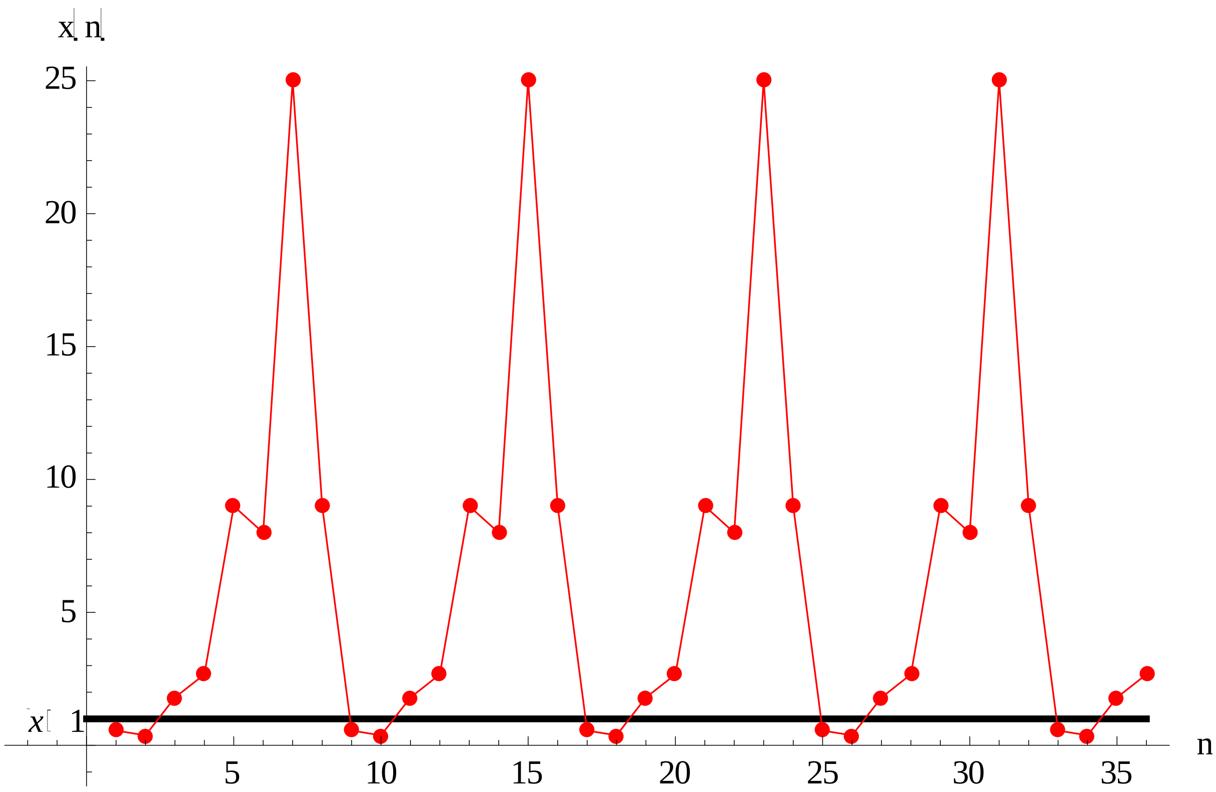

Figure 3.3. x(n) graph of the solutions

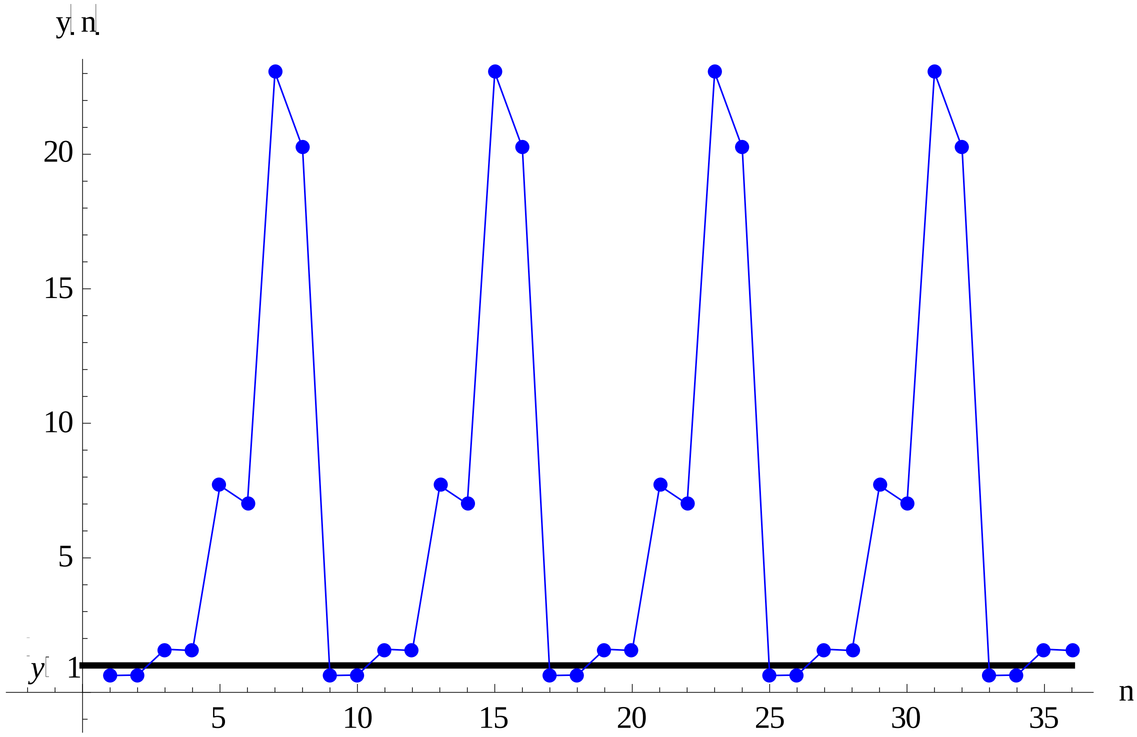

Figure 3.4. y(n) graph of the solutions

REFERENCES

[1] Amleh A.M., Boundedness periodicity and stability of some difference equations, University of Rhode Island, (1998), (PhD Thesis).

[2] Çinar C., Stevic S., Yalçınkaya I., On the positive solutions of reciprocal difference equation withminimum, Journal of Applied Mathematics and Computing, 17, (1-2), (2005), 307-314.

[3] Elaydi S., An introduction to difference equations, Spinger-Verlag, New York, (1996).

[4] Elsayed E.M., Stevic S., On the max-type equation x_{n+1}=max{A/x_{n},x_{n-2}}, Nonlinear Analysis, TMA 71, (2009), 910-922.

[5] Elsayed E.M., Iricanin B., Stevic S., On the max-type equation x_{n+1} = max{A_{n}/x_{n},x_{n-1}}, ARS Combin., (to appear).

[6] Feuer J., Periodic solutions of the Lyness max equation, Journal of Mathematical Analysis and Applications, 288, (2003), 147-160.

[7] Gelişken A., Çinar C., Karataş R., A note on the periodicity of the Lyness max equation, Advances in Difference Equations, Vol.2008, Article ID 651747, (2008), 5.

[8] Gelişken A., Çinar C., Yalçınkaya İ., On the periodicity of a difference equation with maximum, Discrete Dynamics in Nature and Society, (2008), 11.

[9] Gelişken A., Çinar, C., Kurbanlı A.S, On the asymptotic behavior and periodic nature of a difference equation with maximum, Computers & Mathematics with Applications, 59, (2010), 898-902.

[10] Iricanin B., Elsayed E. M., On a max-type equation x_{n+1}=max{A/x_{n},x_{n-3}}, Discrete Dynamics in Nature and Society, (2010).

[11] Kulonevic M. R. S., Ladas G., Dynamics of second order rational difference equations with open problems and conjecture, Boca Raton, London, (2002).

[12] Mishev D. P., Patula W. T., Voulov H. D., A reciprocal difference equation with maximum, Computers & Mathematics with Applications, 43, (2002), 1021-1026.

[13] Moybe L. A., Difference equations with public health applications, New York, USA, (2000).

[14] Огул Б., Шимшек Д., $x_{n+1} = \max\left\{\frac{1}{x_{n-4}}, \frac{y_{n-4}}{x_{n-4}}\right\}$; $y_{n+1} = \max\left\{\frac{1}{y_{n-4}}, \frac{x_{n-4}}{y_{n-4}}\right\}$ Система решениеразностного уравнения, Весник Кыргызского Государственного Технического Университета, 34, (2015).

[15] Papaschinopoulos G., Hatzifilippidis V., On a max difference equation, Journal of Mathematical Analysis and Applications, 258, (2001), 258-268.

[16] Papaschinopoulos G., Schinas J., Hatzifilippidis V., Global behaviour of the solutions of a max-equation and of a system of two max-equation, Journal of Computational Analysis and Applications, 5, 2, (2003), 237-247.

[17] Patula W. T., Voulov H. D., On a max type recursive relation with periodic coefficients, Journal of Difference Equations and Applications, 10, 3, (2004), 329-338.

[18] Stefanidou G., Papaschinopoulos G., The periodic nature of the positive solutions of a nonlinear fuzzy max--difference equation, Information Sciences, 176, (2006), 3694-3710.

[19] Stević S., On the recursive sequence x_{n+1}=max{c,x_{n}^{p}/x_{n-1}^{p}}, Applied Mathematics Letters, 21,8, (2008), 791-796.

[20] Simsek D., Cinar C., Yalçınkaya I., On the solutions of the difference equation x_{n+1}=max{1/x_{n-1},x_{n-1}}, International Journal of Contemporary Mathematical Sciences, Vol.1, No: 9--12, (2006), 481—487.

[21] Simsek D., Demir B., Kurbanlı A. S., x_{n+1} = max{(1/(x_{n})), ((y_{n})/(x_{n}))}, y_{n+1} = max{(1/(y_{n})), ((x_{n})/(y_{n}))} Denklem Sistemlerinin çözümleri üzerine, Ahmet Keleşoğlu Eğitim Fakültesi Dergisi, 28, (2009), 91-104.

[22] Simsek D., Demir B., Cinar C., On the solutions of the system of difference equations x_{n+1}=max{(A/(x_{n})),((y_{n})/(x_{n}))}, y_{n+1} = max {(A/(y_{n})), ((x_{n})/(y_{n}))}, Discrete Dynamics in Nature and Society, (2009), 11.

[23] Simsek D., Kurbanlı A.S., Erdoğan M.E., x(n+1) = max{1 \ x(n-1) ; y(n-1) \ x(n-1)} ; y(n+1) = max{1 \ y(n-1) ; x(n-1) \ y(n-1)} Fark denklem sisteminin çözümleri, XXIII. Ulusal Matematik Sempozyumu, (2010), 153.

[24] Şimşek D., Dogan A., Solutions of the system of maximum difference equations, Manas Journal of Engineering, 2(2): (2014), 9-22.

[25] Teixeria C.T., Existence stability boundedness and periodicity of some difference equations, University of Rhode Island, (2000), (PhD Thesis).

[26] Valicenti S., Periodicity and global attractivity of some difference equations, University of Rhode Island, (1999), (PhD Thesis).

[27] Voulov H. D., On the periodic character of some difference equations, Journal of Difference Equations and Applications, 8, (2002), 799-810.

[28] Voulov H. D., Periodic solutions to a difference equation with maximum, Proceedings of the American Mathematical Society, 131, (2002), 2155-2160.

[29] Yalçınkaya I., Iricanin B. D. and Çinar C., On a max-type difference equation, Discrete Dynamics in Nature and Society, (2007), 10.

[30] Yalçınkaya I., Çinar C. and Atalay M., On the solutions of systems of difference equations, Advances in Difference Equations, (2008), 9.