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Formulas for Solutions of the Riccati's Equation

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Abstract: *In this paper we obtained the formula for the common solution of Riccati equations. Here Riccati equations was solved for common cases. Results obtained have been compared with the conventional ones and a comment has been made on them.*

Keywords: *Riccati's equation,particular solution, formula for the general solutatıon.*

Формулы о рещении уравнений Риккати

Аннотация: *На этом работе мы получили формулу для общего решения уравнений Риккати Для общего случае мы получили решений уравнения Риккати. Полученные результаты соответствует классическими результатами.*

Ключевые слова: *Уравнение Риккати ,частный решения, формулу для общего решения.*

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1. INTRODUCTION

We consider the equation

$$z'(t) = a(t)z^2 + b(t)z(t) + c(t),\ t \in I \tag{1}$$

where $I = (t_1, t_2), t_1 < t_2$ $a(t)$, $b(t)$ and $c(t)$ are known continuous functions, $a(t) \neq 0$ for all $t \in I$. Many works are dedicated to the determination of the common solutions of Riccati equations[1-6]. But in common case any formulas for the decision of Riccati's equatıons have not obtained. It is well known that any equation of the Riccati equation can always be reduced to the linear Differential equations of the second order. In [7] was obtained the formula for the general solutions of the linear Differential equation of the second order with the variable coefficients in the more common cases. In this theme the equations (1) is investigated in the more common cases.

2. FORMULA FOR SOLUTION OF THE EQUATION (1)

Depending on the correlation between $a(t)$, $b(t)$ and $c(t)$ formulas for the determination of the partıcular solutıon and the common solutions of this equation (1) were obtained.

Theorem. Let it be $a(t), b(t), c(t) \in C(I), a(t) \neq 0$ for all $t \in I$,

$$\begin{aligned} c(t) = {} & \frac{1}{a(t)}(\varphi'(t))^2 e^{\int \alpha(t)\varphi'(t)dt} \\ & \left\{ m'(t)(\varphi'(t))^{-1} - m^2(t)e^{\int \alpha(t)\varphi'(t)dt} + [-\frac{a'(t)}{a(t)} - b(t) + \frac{\varphi''(t)}{\varphi'(t)}](\varphi'(t))^{-1}m(t) - \alpha(t)m(t) \right\} + \\ & + \frac{(\varphi'(t))^2}{a(t)} \left\{ \alpha'(t)(\varphi'(t))^{-1} - \alpha^2(t) + \alpha(t)[-\frac{a'(t)}{a(t)} - b(t) + \frac{\varphi''(t)}{\varphi'(t)}](\varphi'(t))^{-1} \right\},\ t \in I, \end{aligned} \tag{2}$$

where $\varphi'(t)$ and $\varphi''(t)$ are the first and second derivatives of the functions $\varphi(t)$, $a'(t)$, $\alpha'(t)$ and $m'(t)$ respectively, the derivatives of the functions $a(t)$, $\alpha(t)$ and $m(t)$, $m(t) \neq 0$ and $\varphi'(t) \neq 0$ for all $t \in I$. Then the partıcular solution of the equation (1)is written as in the next form.

$$z_p(t) = \frac{\varphi'(t)}{a(t)}[\alpha(t) + m(t)e^{\int \alpha(t)\varphi'(t)dt}],\ t \in I \tag{3}$$

and the general solution of the Riccati equation (1) is given by

$$z(t) = z_p(t) + \frac{\exp\{2\varphi'(t)[\alpha(t) + m(t)e^{\int \alpha(t)\varphi'(t)dt}] + b(t)\}}{C - \int a(t)\exp\{2\varphi'(t)[\alpha(t) + m(t)e^{\int \alpha(t)\varphi'(t)dt}] + b(t)\}dt} \tag{4}$$

where $t \in I$, C is an arbitrary constant.
Proof. Differentiating (3), we obtain

$$z_p'(t) = [-\frac{a'(t)\varphi'(t)}{a^2(t)} + \frac{\varphi''(t)}{a(t)}][\alpha(t) + m(t)e^{\int \alpha(t)\varphi'(t)dt}] + \frac{\varphi'(t)}{a(t)}[\alpha'(t) + m'(t)e^{\int \alpha(t)\varphi'(t)dt} + m(t)\alpha(t)\varphi'(t)e^{\int \alpha(t)\varphi'(t)dt}]. \tag{5}$$

Hence taking into account (3), (5) and (2) we have

$$z_p'(t) - a(t)z_p^2(t) - b(t)z_p(t) = [-\frac{a'(t)\varphi'(t)}{a^2(t)} + \frac{\varphi''(t)}{a(t)}][\alpha(t) + m(t)e^{\int \alpha(t)\varphi'(t)dt}] + \frac{\varphi'(t)}{a(t)}[\alpha'(t) + m'(t)e^{\int \alpha(t)\varphi'(t)dt} + m(t)\alpha(t)\varphi'(t)(t)e^{\int \alpha(t)\varphi'(t)dt}] - \frac{(\varphi'(t))^2}{a(t)}[\alpha^2(t) + 2\alpha(t)m(t)e^{\int \alpha(t)\varphi'(t)dt} + m^2(t)e^{2\int \alpha(t)\varphi'(t)dt}] - \frac{b(t)\varphi'(t)}{a(t)}[\alpha(t) + m(t)e^{\int \alpha(t)\varphi'(t)dt}] = c(t),\ t \in I.$$

Therefore, $z_p(t)$ will be a particular solution of the equation (1). It is known that if one can find a particular solution $z_p(t)$ to the equation (1), then the general solution can be written as

$$z(t) = z_p(t) + \frac{1}{u(t)}, \tag{6}$$

where $u(t)$ is the general solution of an associated linear differential equation

$$u'(t) = -[2a(t)z_p(t) + b(t)]u(t) - a(t),\ t \in I \tag{7}$$

Solving this equation (7) we have

$$u(t) = e^{-\gamma(t)}[C - \int e^{\gamma(t)}a(t)dt], \tag{8}$$

where

$$\gamma(t) = 2a(t)z_p(t) + b(t) = 2\varphi'(t)\{\alpha(t) + m(t)\exp[\int \alpha(t)\varphi'(t)dt]\} + b(t)\ ,t \in I\ , \tag{9}$$

Taking into account (8) and (9) from (6), we obtain(4). Theorem has been proved.

Corollary. Let the conditions of Theorem hold. Then the function given by the formula

$$z(t) = \bar{z}_p(t) + \frac{(z_0 - \bar{z}_p(t_0))\exp\{2\varphi'(t)[\alpha(t) + m(t)e^{\int_{t_0}^{t}\alpha(s)\varphi'(s)ds}] + b(t)\}}{\exp\{2\varphi'(t_0)[\alpha(t_0) + m(t_0)] + b(t_0)\} - (z_0 - \bar{z}_p(t_0))\int_{t_0}^{t} a(s)\exp\{2\varphi'(s)[\alpha(s) + m(s)e^{\int_{t_0}^{s}\alpha(s)\varphi'(s)ds}] + b(s)\}ds},$$

is the solution of equation (1) with initial condition $z(t_0) = z_0$, where

$$\bar{z}_p(t) = \frac{\varphi'(t)}{a(t)}[\alpha(t) + m(t)e^{\int_{t_0}^{t}\alpha(s)\varphi'(s)ds}],\quad t, t_0 \in I.$$

3. COMPARISON WITH KNOWN RESULTS

1. According to the given study [5](1.2.8,1) for the equation

$$z' = z^2 + f(t)z - a_0^2 - a_0 f, a_0 \in R, \tag{10}$$

$z_0(t) = a_0$ has been shown a particular solution. The equation (10) provides all requirements of the our theorem for

$$a(t) = 1, b(t) = f(t), c(t) = -a_0^2 - a_0 f(t), \alpha(t) = 0, \varphi(t) = t, m(t) = a_0, \quad \text{at } t \in I.$$

Then from (3)we have $z_p(t) = a_0$.

2. In [5](1.2.8,3) for the equation

$$z' = z^2 + tf(t)z + f(t) \tag{11}$$

$z_0(t) = -\frac{1}{t}, t \in I$ has been shown a particular solution.The equation (11) provides all requirements of the theorem for

$$a(t) = 1, b(t) = tf(t), c(t) = f(t), \alpha(t) = 0, \varphi(t) = t, m(t) = -\frac{1}{t}, \text{at } t \in I.$$

Then from (3) we obtain $z_p(t) = -\frac{1}{t}, t \in I$.

3. In [5](1.2.8,12) for the equation

$$z' = -f'(t)z^2 + f(t)g(t)z - g(t), \tag{12}$$

$z_0 = \frac{1}{f(t)}$ has been shown a particular solution. The equation (12) provides all necessary conditions of the Theorem for

$$a(t) = -f'(t), b(t) = f(t)g(t), a(t) = 0, c(t) = -g(t),$$
$$m(t) = -\frac{f'(t)}{f(t)}, \varphi(t) = t, t \in I.$$

Then from (3)we have $z_p(t) = \frac{1}{f(t)}, t \in I$.

4. In [5](1.2.8,13) for the equation

$$z' = f'(t)z^2 - f(t)g(t)z - g(t)z + g'(t) \tag{13}$$

$z_0(t) = g(t)\ t \in I$, has been shown a particular solution. The equation (13) provides all necessary conditions of the theorem for

$$a(t)=f(t),b(t)=-f(t)g(t),c(t)=g'(t),\alpha(t)=0,\varphi(t)=t,$$
$$m(t)=f(t)g(t),t\in I.$$

Then from (3) we have $z_p(t)=g(t)$,$t\in I$.

5. In [5] (1.2.8,14) for the equation

$$z'(t)=g(t)(z(t)-f(t))^2+f'(t) \tag{14}$$

$z_0(t)=f(t)\ \ t\in I$, has been shown a particular solution. The equation (14)provides all necessary requirements of the theorem for

$$a(t)=g(t),b(t)=-2f(t)g(t),c(t)=f'(t)+g(t)f^2(t),\alpha(t)=0,\varphi(t)=t,t\in I$$

Then from (3) we have $z_p(t)=f(t)$,$t\in I$.

6. In [5](1.2.8,15) for the equation

$$z'=\frac{f'(t)}{g(t)}z^2-\frac{g'(t)}{f(t)}\quad , \tag{15}$$

$z_0(t)=-\dfrac{g(t)}{f(t)}\ \ t\in I$, has been shown a particular solution. The equation (15) provides all necessary conditions of the theorem for

$$a(t)=\frac{f'(t)}{g(t)},\ b(t)=0,c(t)=-\frac{g'(t)}{f(t)},\alpha(t)=0,m(t)=-\frac{f'(t)}{f(t)},t\in I.$$

Then from (3) we have $z_p(t)=-\dfrac{g(t)}{f(t)}\ \ t\in I$.

7. In [5] (1.2.8,16) for the equation

$$f^2(t)z'-f'(t)z^2+g(t)(z-f(t))=0 \tag{16}$$

$z_0(t)=f(t),t\in I$ has been shown a particular solution. The equation (16) provides all necessary conditions of the theorem for

$$a(t)=\frac{f'(t)}{f^2(t)},\ b(t)=-\frac{g(t)}{f^2(t)},c(t)=\frac{g(t)}{f(t)},\alpha(t)=0,$$
$$\varphi(t)=t,m(t)=\frac{f'(t)}{f(t)},t\in I$$

Then from (3) we obtain $z_p(t)=f(t),t\in I$.

8. In [5] (1.2.8,17) for the equation

$$z' = z^2 - \frac{f''(t)}{f(t)}, \tag{17}$$

$z_0(t) = -\frac{f'(t)}{f(t)}, t \in I$ has been shown a particular solution. The equation (17) provides all necessary conditions of the theorem for

$$a(t) = 1,\ b(t) = 0, c(t) = -\frac{f''(t)}{f(t)}, \alpha(t) = 0,$$

$$\varphi(t) = t, m(t) = -\frac{f'(t)}{f(t)}, t \in I$$

Then from (3) we obtain $z_p(t) = -\frac{f'(t)}{f(t)}, t \in I$.

9. In [5] (1.2.8,18) for the equation

$$z' = ae^{\lambda t} z^2 + ae^{\lambda t} f(t) z + \lambda f(t), \tag{18}$$

$z_0(t) = -\frac{\lambda}{a} e^{-\lambda t}, t \in I$ has been shown a particular solution. The equation (18) provides all necessary conditions of the theorem for

$$a(t) = ae^{\lambda t},\ b(t) = ae^{\lambda t} f(t), c(t) = \lambda f(t), \alpha(t) = 0,$$
$$\varphi(t) = t, m(t) = -\lambda, t \in I.$$

Then from (3)we have $z_p(t) = -\frac{\lambda}{a} e^{-\lambda t}, t \in I$.

10. In [5](1.2.8.19) for the equation

$$z' = f(t) z^2 - ae^{\lambda t} f(t) z + a\lambda e^{\lambda t}, \tag{19}$$

$z_0(t) = ae^{\lambda t}, t \in I$ has been shown a particular solution. The equation (19) provides all necessary conditions of the theorem for

$$a(t) = f(t),\ b(t) = -ae^{\lambda t} f(t), c(t) = a\lambda e^{\lambda t}, \alpha(t) = 0,$$
$$\varphi(t) = t, m(t) = ae^{\lambda t} f(t), t \in I$$

Then from (3)we obtain $z_p(t) = ae^{\lambda t}, t \in I$.

11. In [5](1.2.8.11) for the equation

$$z' = f(t) z^2 - at^n g(t) z + ant^{n-1} + a^2 t^{2n} (g(t) - f(t)) \tag{20}$$

$z_0(t) = at^n$ has been shown a particular solution. The equation (20) provides all necessary conditions of the theorem for

$$a(t)=f(t), b(t)=-at^n g(t), c(t)=ant^{n-1}+a^2t^{2n}(g(t)-f(t)), \alpha(t)=0, \varphi(t)=t,$$
$$m(t)=ae^{\lambda t}f(t), t\in I.$$

Then from (3)we have $z_p(t)=at^n$.

12. In [5](1.2.8,24) for the equation

$$z'=f(t)z^2+g(t)z+a\lambda e^{\lambda t}-ae^{\lambda t}g(t)-a^2e^{2\lambda t}f(t), \tag{21}$$

$z_0(t)=ae^{\lambda t}$ has been shown a particular solution. The equation (21) provides all necessary conditions of the theorem for

$$a(t)=f(t), b(t)=g(t), c(t)=a\lambda e^{\lambda t}-ae^{\lambda t}g(t)-a^2e^{2\lambda t}f(t), \alpha(t)=0, \varphi(t)=t,$$
$$m(t)=ae^{\lambda t}f(t), t\in I.$$

Then from (3) we have $z_p(t)=ae^{\lambda t}$.

13. In [5](1.2.8,35) for the equation

$$z'=f(t)z^2-(at\ln t)f(t)z+a\ln t+a, \tag{22}$$

$z_0(t)=at\ln t$ has been shown a particular solution. The equation (22) provides all necessary conditions of the theorem for

$$a(t)=f(t), b(t)=f(t)(a+t\ln t), c(t)=a\ln t+a, \alpha(t)=0, \varphi(t)=t,$$
$$m(t)=at(\ln t)f(t), t\in I.$$

Then from (3) we have $z_p(t)=at\ln t$.

14. In [6,case2] were investigated the equation (1), when

$$c(t)=\frac{d}{dt}\left[\frac{-b(t)\pm\sqrt{f_2(t)+b^2(t)}}{2a(t)}\right]-\frac{f_2(t)}{4a(t)}=$$
$$\frac{1}{a(t)}\left\{\frac{1}{2}\left[-b'(t)\pm\frac{f_2'(t)+2b(t)b'(t)}{4\sqrt{f_2(t)+b^2(t)}}\right]-\frac{a'(t)}{2a(t)}\left[-b(t)\pm\right.\right. \tag{23}$$
$$\left.\left.\sqrt{f_2(t)+b^2(t)}\right]-\frac{1}{4}f_2(t)\right\},\ t\in I$$

where $b'(t),\ f_2'(t),\ a'(t) \in C(I),\ a(t) \neq 0$ for all $t \in I,\ f_2(t) + b^2(t) > 0,\ t \in I$ .Then the general solutions of the Riccati equation (1) were given by

$$z(t) = \frac{e^{\pm\int\sqrt{f_2(t)+b^2(t)}dt}}{C_2 - \int a(t) e^{\pm\int\sqrt{f_2(t)+b^2(t)}dt} dt} + \frac{-b(t) \pm \sqrt{f_2(t) + b^2(t)}}{2a(t)},$$

where C_2 is arbitrary constant. In this case the equation (1) provides all necessary conditions of the theorem for

$$\alpha(t) = 0, \varphi(t) = t, m(t) = \frac{1}{2}[-b(t) \pm \sqrt{f_2(t) + b^2(t)}], t \in I, ,$$

where $c(t)$ is defined by formulas (23).Then from(3) we have

$$z_p(t) = \frac{1}{2a(t)}[-b(t) \pm \sqrt{f_2(t) + b^2(t)}],\ t \in I .$$

15. In [6,case7] were investigated the equation (1), when

$$c(t) = \frac{1}{a(t)}[\frac{1}{2} f_4'(t) - \frac{1}{2}\frac{a'(t)}{a(t)} f_4(t) - \frac{1}{4} f_4^2(t) - \frac{1}{2} b(t) f_4(t)] ,t \in I, \tag{24}$$

where $f_4{}'(t), a'(t), b(t) \in C(I), a(t) \neq 0$ for all $t \in I$.

Then the general solutions of the equation (1) were given by

$$z(t) = \frac{e^{\int[b(t)+f_4(t)]dt}}{C_1 - \int a(t) \exp\{\int[b(t) + f_4(t)]dt\}} + \frac{f_4(t)}{2a(t)}\ t \in I \tag{25}$$

where C_1 is arbitrary constant. In this case on the strength of (24), the equation (1) provides all necessary conditions of the theorem for

$$\alpha(t) = 0, \varphi(t) = t, m(t) = \frac{1}{2} f_4(t), t \in I .$$

Then from (3) we obtain $z_p(t) = \dfrac{f_4(t)}{2a(t)}$, $t \in I$.

16. In [6,case10]were investigated the equation (1), when

$$c(t) = \frac{b^2(t) - 4a^2(t) f_5^2(t)}{4a(t)} + \frac{a'(t)b(t)}{2a^2(t)} - \frac{b'(t)}{2a(t)} + f_5'(t),\ t \in I, \tag{26}$$

where $a'(t), b'(t), f_5'(t) \in C(I), a(t) \neq 0$ for all $t \in I$.

Then the general solutions of the equation (1) were given by

$$z(t) = \frac{\exp\{2\int a(t) f_5(t)dt\}}{C_4 - \int c(t)\exp\{2\int a(t) f_5(t)dt\}dt} - \frac{b(t)}{2a(t)} + f_5(t),\ t \in I,$$

where C_4 is arbitrary constant. In this case on the strength of (26) the equation (1) provides all necessary conditions of the theorem for

$$\alpha(t) = 0, \varphi(t) = t, m(t) = -\frac{1}{2}b(t) + a(t) f_5(t), t \in I.$$

Then from (3) we obtain

$$z_p(t) = -\frac{b(t)}{2a(t)} + f_5(t),\ t \in I.$$

4.CONCLUSIONS

In the present paper,we have obtained the integrability conditions of the Riccati Equation (1). The particular solution and the general solutions of the Riccati equations (1) is presented in the common cases and will be written in the forms (3) and (4).The possibility of the application of the obtained results is also considered.

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