MJEN

MANAS Journal of Engineering

Volume 5 (Issue 3) (2017) Pages 35-47

Gelecek elimizde...

Pointwise Bernstein-Walsh-Type Inequalities in Regions with Piecewise Dini-Smooth Boundary

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Received: 04-10-2017; Accepted: 30-11-2017

Abstract: *In this work, we investigate the order of growth of the modulus of an arbitrary algebraic polynomials in the weighted Bergman space, where the contour and the weight functions have some singularities. In particular, we obtain pointwise Berstein-Walsh -type estimation for algebraic polynomials in the unbounded regions with piecewise Dini-smooth boundary having exterior zero angles.*

Keywords: *Algebraic polynomials, conformal mapping, dini-smooth curve.*

Parçalı Dini-Düzgün Eğri ile Sınırlı Bölgelerde noktasal Bernstein-Walsh Tipi Eşitsizlikler

Özet: *Bu çalışmada, egri ve ağırlık fonksiyonlarının bazı tekilliklere sahip olduğu durumlarda, keyfi bir cebirsel polinomların modülünün büyüme hızını ağırlıklı Bergman uzayında inceliyoruz. Özellikle, parçalı Dini-düzgün dış sıfır açılara sahip olan sınırsız bölgelerde cebirsel polinomlar için noktasal Berstein-Walsh-tipi değerlendirmeler elde edilmiştir.*

Anahtar kelimeler: *Cebirsel polinomlar, Konform dönüşüm, Dini-düzgün eğri.*

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1. INTRODUCTION AND MAIN RESULTS

Let $\mathbb{C}$ be a complex plane, and $\overline{\mathbb{C}}:=\mathbb{C}\cup\{\infty\}$; $G\subset\mathbb{C}$ be the bounded Jordan region, with $0\in G$ and the boundary $L:=\partial G$ be a closed Jordan curve, $\Omega:=\overline{\mathbb{C}}\setminus\overline{G}=extL$. $\Delta:=\{w:|w|>1\}$ (with respect to $\overline{\mathbb{C}}$). Let function $w=\Phi(z)$ be the univalent conformal mapping of Ω onto the Δ normalized by $\Phi(\infty)=\infty, \lim_{z\to\infty}\frac{\Phi(z)}{z}>0,$ and $\Psi:=\Phi^{-1}$. For $R>1$ let us set $L_R:=\{z:|\Phi(z)|=R\}$, $G_R:=\operatorname{int}L_R, \Omega_R:=extL_R$. For $z\in\mathbb{C}$ and $M\subset\mathbb{C}$, we set: $d(z,M)=dist(z,M):=\inf\{|z-\zeta|:\zeta\in M\}$.

Let $\{z_j\}_{j=1}^m\in L$ be a fixed system of distinct points. Consider a so-called generalized Jacobi weight function $h(z)$ being defined as follows:

$$h(z):=\prod_{j=1}^{m}|z-z_j|^{\gamma_j}, \quad z\in G_{R_0}, \quad R_0>1, \tag{1.1}$$

where $\gamma_j>-2$ for every $j=1,2,...,m$.

Denote by $\wp_n$ the class of all complex algebraic polynomials $P_n(z)$ of degree at most $n\in\mathbb{N}$.

For any $p>0$ and for Jordan region G, let's define:

$$\|P_n\|_p:=\|P_n\|_{A_p(h,G)}:=\left(\iint_G h(z)|P_n(z)|^p\,d\sigma_z\right)^{1/p}<\infty, 0<p<\infty; \tag{1.2}$$

$$\|P_n\|_\infty:=\|P_n\|_{A_\infty(1,G)}:=\|P_n\|_{C(\overline{G})}, \ p=\infty,$$

where σ_z is the two-dimensional Lebesgue measure. Clearly, $\|.\|_p$ is the quasinorm (i.e. a norm for $1\le p\le\infty$ and a p-norm for $0<p<1$).

Well-known Bernstein-Walsh Lemma [17], (for more see also [28]) says that for any $R>1$

$$|P_n(z)|\le R^n\|P_n\|_{C(\overline{G})}, \quad z\in\Omega, \tag{1.3}$$

where

$$\|P_n\|_{C(\overline{G})}:=\max_{z\in\overline{G}}|P_n(z)|. \tag{1.4}$$

In [6, Theorem 1.1] was studied a similar problem in $A_p(1,G)$ for $p>0$ and for arbitrary Jordan region was obtained the following result: for any $p>0$, $P_n\in\wp_n$, $R_1=1+\frac{1}{n}$ and arbitrary R, $R>R_1$, the following is true:

$$\|P_n\|_{A_p(G_R)}\le c\cdot R^{n+\frac{2}{p}}\|P_n\|_{A_p(G_{R1})}, \tag{1.5}$$

where $c=\left(\frac{2}{e^{p}-1}\right)^{\frac{1}{p}}\left[1+O\left(\frac{1}{n}\right)\right],\ n\to\infty$.

Following [22, p.97], [25], the Jordan curve (or arc) L is called K-quasiconformal $(K\geq 1)$, if there is a K- quasiconformal mapping f of the region $D\supset L$ such that $f(L)$ is a circle (or line segment).

For the regions G with quasiconformal boundary and weight function $h(z)$, as defined in (1.1) with $\gamma_j>-2$ for any $p>0$ was found in [5] as follows:

$$\|P_n\|_{A_p(h,G_R)}\leq c_1\cdot R^{*^{n+\frac{1}{p}}}\|P_n\|_{A_p(h,G)}, \tag{1.6}$$

where $R^{*}:=1+c_2(R-1), c_2>0$ and $c_1:=c_1(G,p,c_2)>0$ constants, independent from n and R.

In [26], by replaced the norm $\|P_n\|_{C(\overline{G})}$ with norm $\|P_n\|_{A_2(G)}$ were gives a new version of the Bernstein-Walsh Lemma as following: for any rectifiable quasiconformal curve L there exists a constant $c=c(L)>0$ depending only on L such that

$$|P_n(z)|\leq c_3(L)\frac{\sqrt{n}}{d(z,L)}\|P_n\|_{A_2(G)}|\Phi(z)|^{n+1},\quad z\in\Omega,$$

holds for every $P_n\in\wp_n$, where $c_3(L)>0$ constant independent from n and z.

So, in general, we can express the problem as follows: find an estimate of the type

$$|P_n(z)|\leq c\cdot\alpha_n(L,h,d(z,\mathrm{L}),p)\|P_n\|_p|\Phi(z)|^{n+1},\quad z\in\Omega,\ p>0, \tag{1.7}$$

where $c=c(L,\mathrm{p})>0$ is a constant independent from n,z,P_n and $\alpha_n(L,h,d(z,\mathrm{L}),p)\to\infty$ (in general!) as $n\to\infty$, depending on the geometrical properties of curve L, weight function h and parameter p.

Analogous results of (1.7)-type for some norms, weight function $h(z)$ and for di¤erent unbounded regions were obtained by Lebedev, Tamrazov, Dzjadyk (see, for example, [18, pp.418-428.]), Abdullayev and et all [8], [9], [13], [7], [12] and others.

In this work, we investigate similar problem in unbounded region Ω with piece-wise Dini-smooth boundary having interior angles (also cusps) for weight function h defined in (1.1), through $\|\cdot\|_{A_p(h,G)}$- quasinorm and $p>0$.

Let us give some definitions and notations that will be used later in the text. In what follows, we always assume that $p>0$ and the constants $c,c_0,c_1,c_2,\ldots,$ are positive and constants $\varepsilon_0,\varepsilon_1,\varepsilon_2,\ldots,$ are sufficiently small positive (generally, are different in different relations), which depends on G in general and, on parameters inessential for the argument, otherwise, the dependence will be explicitly stated. Also note that, for any $k\geq 0$ and $m>k$, notation

$j=\overline{k,m}$ denotes $j=k,k+1,...,m.$

Let S be a rectifiable Jordan curve or arc and $z=z(s),\quad s\in[0.|S|],\quad |S|:=mes\,S,$ denote the natural representation of S .

Definition1.1. [24, p.48] (see also [16, p.32]) We say that a Jordan curve or arc S called Dini-smooth, if it has a parametrization $z=z(s),\ 0\le s\le|S|$, such that $z'(s)\ne 0,\quad 0\le s\le|S|$ and

$$|z'(s_2)-z'(s_1)|<g(s_2-s_1),\, s_1<s_2,$$

where g is an increasing function for which

$$\int_0^1\frac{g(x)}{x}dx<\infty\,.$$

Definition1.2. [8] We say that Jordan region $G\in PDS(\lambda_1,...,\lambda_m),\, 0<\lambda_j\le 2,\quad \mathrm{j}=\overline{1,m},$ if $L=\partial G$ consists of the union of finite Dini-smooth arcs $\{L_J\}_{j=1}^m$, such that L is locally Dini-smooth at $z_0\in L\setminus\{z_J\}_{j=1}^m$ and have exterior (with respect to $\overline{G}$) angels $\lambda_j\pi,\quad 0<\lambda_j\le 2,$ at the corner points $\{z_J\}_{j=1}^m\in L$, where two arcs meet.

Without loss of generality, we assume that these points on the curve $\mathrm{L}=\partial G$ are located in the positive direction such that, G has exterior $\lambda_j\pi,\quad 0<\lambda_j<2,\ j=\overline{1,m},$ angle at the points $\{z_J\}_{j=1}^m$, $m_1\le m,$ and interior zero angle (i.e. $\lambda_j=2$ -interior cusps) at the points $\{z_J\}_{j=m_1+1}^m$.

It is clear from Definition1.2. the each region $G\in PDS(\lambda_1,...,\lambda_m),\quad 0<\lambda_j\le 2,\ j=\overline{1,m},$ may have exterior nonzero $\lambda_j\pi,\quad 0<\lambda_j<2,$ angles at the points $\{z_J\}_{j=1}^m\in L,$ and interior zero angles $(\lambda_j=2)$ at the points $\{z_J\}_{j=m_1+1}^m\in L.$ If $m_1=m=0,$ then the region G doesn't have such angles, and in this case we will write: $G\in DS$; if $m_1=m\ge 1,$ then G has only $\lambda_i\pi,\quad 0<\lambda_i<2,\quad \mathrm{i}=\overline{1,m_1},$ exterior nonzero angels, and in this case we will write: $G\in PDS(2)$.

Throughout this work, we will assume that the points $\{z_J\}_{j=1}^m\in L$, defined in (1.1) and Definition 1.2 are identical and $w_j:=\Phi(z_j)$.

For simplicity of exposition, without loss of generality, we will take $m_1=1,\, m=2$. Then, after this assumption, in the future we will have region $G\in PDS(\lambda_1,2),\, 0<\lambda_1<2,$ such that at the point $\mathrm{z}_1\in L$ region G have exterior nonzero $\lambda_1\pi,\quad 0<\lambda_1<2,$ and at the point $z_2\in L$ - interior zero angle $2\pi,\quad i.e.\,\lambda_2=2.$

Now we can state our new results.

Theorem 1.1. Let $G\in PDS(\lambda_1,2),$ for some $0<\lambda_1<2;\, h(z)$ be defined as in (1.1). Then, for any $P_n\in\wp_n,\ n\in\mathbb{N},$ and $\gamma_j>-2,\ j=1,2,$ we have:

$$\left|P_n(z)\right| \le c_1 \frac{\left|\Phi(z)\right|^{n+1}}{d^{2/p}\left(z,L_{1+1/n}\right)} A_{n;1} \left\|P_n\right\|_{p}, \quad z\in\Omega_{1+1/n}, \tag{1.8}$$

where $c_1 = c_1(G,\gamma_1,\gamma_2,\lambda_1,p) > 0$ is the constant, independent from z and n,

$$A_{n;1} := \begin{cases} n^{\frac{\gamma\cdot\lambda}{p}}, & \text{if} \quad \gamma\cdot\lambda > 1, \\ (n\ln n)^{\frac{1}{p}}, & \text{if} \quad \gamma\cdot\lambda = 1, \\ n^{\frac{1}{p}}, & \text{if} \quad \gamma\cdot\lambda < 1, \end{cases} ; \tag{1.9}$$

$\lambda := \begin{cases} \max\{1;\lambda_1\}, & if \ 0<\lambda_1<2, \\ 2, & if \ \lambda_2 = 2, \end{cases}$; $\lambda := \begin{cases} \lambda_1, & \text{if} \ 0<\lambda_1<2, \\ \lambda_2, & if \ \lambda_2 = 2, \end{cases}$ *and* $\lambda_i := \max\{0;\lambda_i\}, \quad i=1,2$.

We can take individual cases when the curve L in the both points have the same type of angle: exterior nonzero or interior zero angle. In this case, from Theorem 1.1, we obtain the following:

Corollary 1.1. Let $G\in PDS(\lambda_1,\lambda_2)$, for some $0<\lambda_j<2$, $\ j=1,2$; $h(z)$ be defined as in (1.1). Then, for any $P_n \in \wp_n$, $n\in\mathbb{N}$, and $\gamma_j > -2$, $j=1,2$, we have:

$$\left|P_n(z)\right| \le \left|c_2 \frac{\left|\Phi(z)\right|^{n+1}}{d^{2/p}\left(z,L_{1+1/n}\right)} A_{n;2} \left\|P_n\right\|_{p}, \quad z\in\Omega_{1+1/n}, \tag{1.10}$$

where $c_2 = c_2(G,\gamma_1,\gamma_2,\lambda_1,p) > 0$ is the constant, independent from z and n,

$$A_{n,2} := \begin{cases} n^{\frac{\gamma\cdot\lambda}{p}}, & \text{if} \quad \gamma\cdot\lambda > 1, \\ (n\ln n)^{\frac{1}{p}}, & \text{if} \quad \gamma\cdot\lambda = 1, \\ n^{\frac{1}{p}}, & \text{if} \quad \gamma\cdot\lambda < 1, \end{cases} ; \tag{1.11}$$

and $\gamma^* := \max\{0,\gamma_1,\gamma_2\}, \quad \lambda := \max\{1;\lambda_1,\lambda_2\}$.

Corollary 1.2. Let $G\in PDS(2,2)$, $h(z)$ be defined as in (1.1). Then, for any $P_n\in\wp_n$, $n\in\mathbb{N}$, and $\gamma_j > -2$, $j=1,2$, we have:

$$\left|P_n(z)\right| \le \left|c_3 \frac{\left|\Phi(z)\right|^{n+1}}{d^{2/p}\left(z,L_{1+1/n}\right)} A_{n;3} \left\|P_n\right\|_{p}, \quad z\in\Omega_{1+1/n}, \tag{1.12}$$

where $c_3 = c_3(G,\gamma_1,\gamma_2,\lambda_1,p) > 0$ is the constant, independent from z and n, and

$$A_{n,3} := \begin{cases} n^{\frac{2\gamma}{p}}, & \text{if} \quad \gamma > 1/2, \\ (n \ln n)^{\frac{1}{p}}, & \text{if} \quad \gamma = 1/2, \\ n^{\frac{1}{p}}, & \text{if} \quad \gamma < 1/2. \end{cases}; \tag{1.13}$$

The sharpness of the estimations (1.8)-(1.13) for some special cases can be discussed by comparing them with the following result:

Remark 1.1. For any $n \in \mathbb{N}$, there exists a polynomial $P_n^* \in \wp_n$, region $G^* \subset \mathbb{C}$, compact $F^* \Subset \Omega/\overline{G}^*$ and constant $c_5 = c_5(G^*, F^*) > 0$ such that

$$|P_n^*(z)| \geq c_5 \frac{\sqrt{n}}{d(z, L_{1+1/n})} \|P_n^*\|_{A_2(G^*)} |\Phi(z)|^{n+1}, \quad \text{for all } z \in F^*. \tag{1.14}$$

2. SOME AUXILIARY RESULTS

Throughout this work, for the nonnegative functions $a > 0$ and $b > 0$, we shall use the notations "$a \prec b$" (order inequality), if $a \leq cb$ and "$a \approx b$" are equivalent to $c_1 a \leq b \leq c_2 a$ for some constants c, c_1, c_2 (independent of a and b), respectively.

Lemma 2.1. [1] Let L be a K - quasiconformal curve, $z_1 \in L$, $z_2, z_3 \in \Omega \cap \{z : |z - z_1| \prec d(z_1, L_{ro})\}$; $w_j = \Phi(z_j)$, $j = 1, 2, 3$. Then

a) The statements $|z_1 - z_2| \prec |z_1 - z_3|$ *and* $|w_1 - w_2| \prec |w_1 - w_3|$ are equivalent.

So statements $|z_1 - z_2| \approx |z_1 - z_3|$ *and* $|w_1 - w_2| \approx |w_1 - w_3|$ also equivalent;

b) If $|z_1 - z_2| \prec |z_1 - z_3|$, then

$$\left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{K^2} \prec \left|\frac{z_1 - z_3}{z_1 - z_2}\right| \prec \left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{K^{-2}},$$

where $0 < r_0 < 1$, $R_0 := r_0^{-1}$ are constants, depending on G.

Corollary 2.1. For $z_3 \in L_{r_0}$ $(z_3 \in L_{r_0})$

$$|w_1 - w_2|^{K^2} \prec |z_1 - z_2| \prec |w_1 - w_2|^{K^{-2}}$$

Recall that for $0 < \delta_j < \delta_0 := \frac{1}{4} \min\{|z_i - z_j| : i, j = 1, 2, ..., m, i \neq j\}$, we put

$\Omega(z_j, \delta_j) := \Omega \cap \{z : |z - z_j| \leq \delta_j\}$; $\delta := \min_{1 \leq j \leq m} \delta_j$, $\Omega(\delta) := \bigcup_{j=1}^{m} \Omega(z_j, \delta)$, $\Omega := \Omega/\Omega(\delta)$. Additionally, let $\Delta_j := \Phi\left(\Omega\left(z_j,\delta\right)\right)$, $\Delta(\delta) := \bigcup_{j=1}^{m} \Phi\left(\Omega\left(z_j,\delta\right)\right)$, $\Delta(\delta) := \Delta / \Delta(\delta)$.

The following lemma is a consequence of the results given in [24, pp.41-58], [16, pp.32-36], and estimation for the $|\Psi'|$ (see, for example, [15, Th.2.8]):

$$|\Psi'(\tau)| \approx \frac{d\left(\Psi(\tau), L\right)}{|\tau|-1} \tag{2.1}$$

Lemma 2.2. Let a Jordan region $G \in PDS\left(\lambda_j;0\right)$, $0<\lambda_j \le 2$, $j=\overline{1,m_1}$. Then,

i) for any $w \in \Delta_j$, $\left|\Psi(w)-\Psi\left(w_j\right)\right| \approx \left|w-w_j\right|^{\lambda_j}$, $\left|\Psi'(w)\right| \approx \left|w-w_j\right|^{\lambda_j-1}$;

ii) for any $w \in \overline{\Delta}/\Delta_j$, $\left|\Psi(w)-\Psi\left(w_j\right)\right| \approx \left|w-w_j\right|$, $\left|\Psi'(w)\right| \approx 1.$

Let $\{z_J\}_{j=1}^{m}$ be a fixed system of the points L and the weight function $h(z)$ defined as in (1.1).

Lemma 2.3. [4] Let L be a K - quasiconformal curve; $h(z)$ is defined in (1.1).

Then, for arbitrary $P_n(z) \in \wp_n$, any $R>1$ and $n=1,2,...,$ we have

$$\|P_n\|_{A_p(h,G_R)} \prec \tilde{R}^{n+\frac{1}{p}} \|P_n\|_{A_p(h,G)}, \quad p>0, \tag{2.2}$$

Where $\tilde{R} = 1 + c(R-1)$ and c is independent from n and R.

Lemma 2.4. Let $G \in PDS\left(\lambda_1,...,\lambda_m\right)$, $0<\lambda_j \le 2$, $j=\overline{1,m_1}$ .Then, for arbitrary $P_n(z) \in \wp_n$, we have:

$$\|P_n\|_{A_p\left(h,G_{1+c/n}\right)} \prec \|P_n\|_{A_p(h,G)}, \tag{2.3}$$

2.1. Proof of Theorem 1.1.

Proof. Suppose that $G \in PDS(\lambda;2)$, for some $0<\lambda<2; h(z)$ be defined as in (1.1). Let $\left\{\xi_j\right\}$, $1 \le j \le m \le n$, be the zeros (if any) of $P_n(z)$ lying on Ω. Lets define the function Blashke with respect to the zeros $\left\{\xi_j\right\}$ of the polynomial $P_n(z)$:

$$\tilde{B}_j(z) := \frac{\Phi(z)-\Phi\left(\xi_j\right)}{1-\overline{\Phi\left(\xi_j\right)}\Phi(z)}, \quad z \in \Omega, \tag{2.4}$$

and let

$$B_m(z) := \prod_{j=1}^{m} \tilde{B}_j(z), \quad z \in \Omega. \tag{2.5}$$

According to the well known properties of the Blaschke function (2.4), we have:

$$B_m(\xi_j)=0, \quad |B_m(z)|\equiv 1, \quad z\in L; \quad |B_m(z)|<1, \quad z\in\Omega \tag{2.6}$$

Then, for each $\varepsilon_1,\ 0<\varepsilon_1<1,$ there exists circle $\left\{\omega:|\omega|=R_1:=1+\varepsilon_2,\ 0<\varepsilon_2<\frac{\varepsilon_1}{n}\right\}$ such that for any $j=1,2$ the following is holds:

$$\left|\tilde{B}_j(\zeta)\right|>1-\varepsilon_2, \qquad \zeta\in L_{R_1}$$

Then, the following estimate holds:

$$|B_m(\zeta)|>(1-\varepsilon_2)^m\succ 1, \qquad \zeta\in L_{R_1} \tag{2.7}$$

For any $p>0$ and $z\in\Omega$, we define:

$$Q_{n,p}(z):=\left[\frac{P_n(z)}{B_m(z)\Phi^{n+1}(z)}\right]^{p/2} \tag{2.8}$$

The function $Q_{n,p}(z)$ is analytic in Ω, continuous on $\bar{\Omega}$, $Q_{n,p}(\infty)=0$ and does not have zeros in Ω. We take on arbitrary continuous branch of the $Q_{n,p}(z)$ and for this branch, we maintain the same designation. According to Cauchy integral representation for the unbounded region Ω, we have:

$$Q_{n,p}(z)=-\frac{1}{2\pi i}\int_{L_{R_1}}Q_{n,p}(\zeta)\frac{d\zeta}{\zeta-z}, \qquad z\in\Omega_{R_1} \tag{2.9}$$

From (2.4) – (2.8), we get:

$$|P_n(z)|^{p/2}=\frac{\left|B_m(z)\Phi^{n+1}(z)\right|^{p/2}}{2\pi d(z,L_{R_1})}\int_{L_{R_1}}\left|\frac{P_n(\zeta)}{B_m(\zeta)\Phi^{n+1}(\zeta)}\right|^{p/2}|d\zeta|\prec\frac{\left|\Phi^{n+1}(z)\right|^{p/2}}{d(z,L_{R_1})}\int_{L_{R_1}}|P_n(\zeta)|^{p/2}|d\zeta| \tag{2.10}$$

Multiplying the numerator and the denominator of the last integral by $h^{1/2}(\zeta)$, replacing the variable $w=\Phi(z)$ and applying the Holder inequality, we obtain: (2.11)

$$\left(\int_{L_{R_1}}\left|P_n(\zeta)\right|^{p/2}|d\zeta|\right)^2\leq\int_{|t|=R_1}h(\psi(t))\left|P_n(\psi(t))\right|^p\left|\psi'(t)\right|^2|dt|\cdot\left(\int_{|t|=R_1}\frac{|dt|}{\left|h(\psi(t))\right|}\right)\leq$$

$$\left(\int_{|t|=R_1}h(\psi(t))\left|P_n(\psi(t))\right|^p\left|\psi'(t)\right|^2|dt|\right)\cdot\left(\int_{|t|=R_1}\frac{|dt|}{\left|h(\psi(t))\right|}\right)=\left(\int_{|t|=R_1}\left|f_{n,p}(t)\right|^p|dt|\right)\left(\int_{|t|=R_1}\frac{|dt|}{\left|h(\psi(t))\right|}\right)=A_n\cdot D_n$$

where $f_{n,p}(t):=h^{\frac{1}{p}}(\psi(t))\left(\psi'(t)\right)^{\frac{2}{p}},\quad |t|=R_1.$

Analogously to $\left[8,(2.5)\right]$ and from Lemma 2.4, we get:

$$A_n\leq\|P_n\|_p^p. \tag{2.12}$$

To estimate the integral $B_n(w)$, denote by $w_J:=\Phi(z_j)$, $\varphi_j:=\arg w_j$, for any sixed $p>1$, we introduce:

$$\begin{aligned}\Delta_1(p)&:=\left\{t=re^{i\theta}:r>p,\quad \frac{\varphi_0+\varphi_1}{2}\leq\theta\leq\frac{\varphi_1+\varphi_2}{2}\right\},\\ \Delta_2(p)&:=\left\{t=re^{i\theta}:r>p,\quad \frac{\varphi_1+\varphi_2}{2}\leq\theta\leq\frac{\varphi_1+\varphi_0}{2}\right\};\end{aligned} \tag{2.13}$$

$$\Delta_j:=\Delta_j(1),\quad \Omega^j:=\psi(\Delta_j),\ \Omega_p^j=\Psi(\Delta_j(p));$$

$$L^j:=L\cap\overline{\Omega}^j,\quad L^1\cup L^1,\ L_p^j:=L_p\cap\overline{\Omega}_p^j,\quad L_p=L_p^1\cup L_p^2,\ j=1,2;$$

and

$$\Phi(L_{R_1})=\Phi\left(\bigcup_{j=1}^{2}L_{R_1}^j\right)=\bigcup_{j=1}^{2}\Phi\left(L_{R_1}^j\right)=\bigcup_{j=1}^{2}\bigcup_{i=1}^{3}K_i^j(R_1)\ ,$$

where

$$\begin{aligned}K_1^j(R_1)&:=\left\{t\in\Phi\left(L_{R_1}^J\right):\ |t-\omega_j|<\frac{c_1}{n}\right\},\\ K_2^j(R_1)&:=\left\{t\in\Phi\left(L_{R_1}^J\right):\ \frac{c_1}{n}\leq|t-\omega_j|<c_2\right\},\\ K_3^j(R_1)&:=\left\{t\in\Phi\left(L_{R_1}^J\right):\ c_2\leq|t-\omega_j|<diam\overline{G}\right\},\quad j=1,2.\end{aligned}$$

Then, we get

$$D_n=\int_{|t|=R_1}\frac{|dt|}{h(\psi(t))}\leq\sum_{j=1}^{2}\int_{\Phi(L_{R_1}^j)}\frac{|dt|}{\prod_{j=1}^{2}\left|\psi(t)-\psi(w_j)\right|^{\gamma_j}}\approx\sum_{j=1}^{2}\int_{\Phi(L_{R_1}^j)}\frac{|dt|}{\prod_{j=1}^{2}\left|\psi(t)-\psi(w_j)\right|^{\gamma_j}}:=\sum_{j=1}^{2}D_{n,j}, \tag{2.14}$$

since the points $\{z_j\}_{j=1}^{m} \in L$ are distinct. It remains to estimate the integrals

$$D_{n,j} := \int_{\Phi\left(L_{R_1}^j\right)} \frac{|dt|}{\left|\psi(t)-\psi(w_j)\right|^{\gamma_j}} \tag{2.15}$$

for each $j=1,2$. Now, we define

$$D_{n,1} = \sum_{i=1}^{3} \int_{K_i^1(R_1)} \frac{|dt|}{\left|\psi(t)-\psi(w_1)\right|^{\gamma_1}} =: \mathrm{D}_{n,1}^i \tag{2.16}$$

for $j=1$,

$$D_{n,2} = \sum_{i=1}^{3} \int_{K_i^2(R_1)} \frac{|dt|}{\left|\psi(t)-\psi(w_2)\right|^{\gamma_2}} =: \mathrm{D}_{n,2}^i , \tag{2.17}$$

for $j=2$, and let's estimate they separately.

Case 1.1. Applying Lemma 2.2, we get:

$$D_{n,1}^1 = \int_{K_1^1(R_1)} \frac{|dt|}{\left|\Psi(t)-\Psi(w_1)\right|^{\gamma_1}} \prec \int_{K_1^1(R_1)} \frac{|dt|}{\left|t-w_1\right|^{\gamma_1\lambda_1}} \prec \begin{cases} n^{\gamma_1\lambda_1-1}, & \text{if } \gamma_1\lambda_1>1, \\ \ln n, & \text{if } \gamma_1\lambda_1=1, \\ 1, & \text{if } -2<\gamma_1\lambda_1<1, \end{cases} \tag{2.18}$$

if $\gamma_1 \geq 0$, and

$$D_{n,1}^1 = \int_{K_1^1(R_1)} \left|\Psi(t)-\Psi(w_1)\right|^{(-\gamma_1)} |dt| \prec \int_{K_1^1(R_1)} \left|t-w_1\right|^{(-\gamma_1)\lambda_1} |dt| \prec \left(\frac{1}{n}\right)^{(-\gamma_1)\lambda_1} \cdot mesK_1^1(R_1) \prec 1, \tag{2.19}$$

if $\gamma_1 < 0$.

Case 1.2. Analogously, we obtain:

$$D_{n,1}^2 = \int_{K_2^1(R_1)} \frac{|dt|}{\left|\Psi(t)-\Psi(w_1)\right|^{\gamma_1}} \prec \int_{K_2^1(R_1)} \frac{|dt|}{\left|t-w_1\right|^{\gamma_1\lambda_1}} \prec \begin{cases} n^{\gamma_1\lambda_1-1}, & \text{if } \gamma_1\lambda_1>1, \\ \ln n, & \text{if } \gamma_1\lambda_1=1, \\ 1, & \text{if } -2<\gamma_1\lambda_1<1, \end{cases} \tag{2.20}$$

if $\gamma_1 \geq 0$, and

$$D_{n,1}^2 = \int_{K_1^2(R_1)} \left|t-w_1\right|^{(-\gamma_1)\lambda_1} |dt| \prec 1, \tag{2.21}$$

If $\gamma_1 < 0$.

Case 1.3. For all cases of γ_1, we have:

$$D_{n,1}^{3}=\int\limits_{K_3^1(R_1)}\frac{|dt|}{\left|\Psi(t)-\Psi(w_1)\right|^{\gamma_1}}\prec c_2^{-\gamma_1}\cdot mesK_3^1(R_1)\prec 1. \tag{2.22}$$

Case 2.1. According to the estimation [27, p.181] for arbitrary continuum with simple connected complementary

$$\left|\Psi(t)-\Psi(w_2)\right|\succ\left|t-w_2\right|^2,$$

we get:

$$D_{n,2}^{1}+D_{n,2}^{2}=\sum_{i=1}^{2}\int\limits_{K_i^2(R_1)}\frac{|dt|}{\left|\Psi(t)-\Psi(w_2)\right|^{\gamma_2}}\prec\int\limits_{K_1^2(R_1)\cup K_2^2(R_1)}\frac{|dt|}{\left|t-w_2\right|^{2\gamma_2}}\prec\begin{cases}n^{2\gamma_2-1}, & \text{if } 2\gamma_2>1,\\ \ln n, & \text{if } 2\gamma_2=1,\\ 1, & \text{if } -2<2\gamma_2<1,\end{cases} \tag{2.23}$$

if $\gamma_2\geq 0$, and

$$D_{n,2}^{1}+D_{n,2}^{2}=\int\limits_{K_1^2(R_1)\cup K_2^2(R_1)}\left|\Psi(t)-\Psi(w_2)\right|^{(-\gamma_2)}|dt|\prec 1, \tag{2.24}$$

if $\gamma_2<0$.

Case 2.2. For all cases of γ_2, we have:

$$D_{n,2}^{3}=\int\limits_{K_3^2(R_1)}\frac{|dt|}{\left|\Psi(t)-\Psi(w_2)\right|^{\gamma_2}}\prec 1. \tag{2.25}$$

Therefore, comparing (2.14)-(2.25), we obtain:

$$D_n\prec\begin{cases}n^{\gamma_1\lambda_1-1}, & \text{if } \gamma_1>\frac{1}{\lambda_1},\\ \ln n, & \text{if } \gamma_{11}=\frac{1}{\lambda_1},\\ 1, & \text{if } -2<\gamma_1<\frac{1}{\lambda_1},\end{cases}+\begin{cases}n^{2\gamma_1-1}, & \text{if } \gamma_2>\frac{1}{2},\\ \ln n, & \text{if } \gamma_2=\frac{1}{2},\\ 1, & \text{if } -2<\Upsilon_2<\frac{1}{2},\end{cases}\prec\begin{cases}n^{\gamma_1\hat{\lambda}-1}, & if\ \gamma\hat{\lambda}>1,\\ \ln n, & if\ \gamma\hat{\lambda}=1,\\ 1, & if\ -2<\gamma\hat{\lambda}<1,\end{cases} \tag{2.26}$$

where

$$\hat{\lambda}:=\begin{cases}\max\{1;\lambda\}, & if\ 0<\lambda<2,\\ 2, & \text{if } \lambda=2.\end{cases};\ \gamma:=\begin{cases}\gamma_1, & \text{if } 0<\lambda<2,\\ \gamma_2, & \text{if } \lambda=2.\end{cases},\ \gamma_i:=\max\{0;\gamma_i\},\ i=1,2.$$

Combining relations (2.10), (2.11), (2.12) and (2.26), we complete the proof.

2.1.1. Proof of Remark 1.1.

Proof. Let the region G bounded by Dini-smooth curve $L=\partial G$. According to the "three-point" criterion [14, p.100], every piecewise Dini-smooth curve (without any cusps) is quasiconformal. Let $\{K_n(z)\}$, $\deg K_n=n$, denote of the system of Bergman polynomials for region G. According to [2], [3] for arbitrary quasidisk, we have

$$K_n(z)=\alpha_n\rho^{n+1}\Phi^n(z)\Phi'(z)A_n(z),\ z\in F\Subset\Omega,$$

where

$$\sqrt{\frac{n+1}{\pi}}\leq\alpha_n\rho^{n+1}\leq c_1\sqrt{\frac{n+1}{\pi}},$$

for some $c_1=c_1(G)>1$ and

$$c_2\leq\left|A_n(z)\right|\leq 1+\frac{c_3}{\sqrt{\left|\Phi(z)\right|-1}},$$

for some $c_i=c_i(G)>0,\quad i=2,3.$ Therefore, since $\left\|K_n\right\|_{A_2(G)}=1,$ according to (2.1), we have

$$\left|K_n(z)\right|\geq c_2\sqrt{\frac{n+1}{\pi}}\left|\Phi(z)\right|^n\frac{\left|\Phi(z)\right|-1}{d(z,L)}\geq c_3\frac{\sqrt{n}}{d(z,L)}\left|\Phi(z)\right|^{n+1}\left(1-\frac{1}{\left|\Phi(z)\right|}\right)\geq c_4\frac{\sqrt{n}}{d(z,L)}\left|\Phi(z)\right|^{n+1},$$

and we complete the proof.

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