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Solutions of the Rational Difference Equations

$$x_{n+1} = \frac{x_{n-(2k+1)}}{1 + x_{n-k}}$$

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Abstract: *In this paper the solutions of the following difference equation is examined,*

$$x_{n+1} = \frac{x_{n-(2k+1)}}{1 + x_{n-k}}, \quad \text{n=0,1,2,...} \qquad (1)$$

where the initial conditions are positive real numbers.

Keywords: *Difference equation, period 2k+2 solution*

$$x_{n+1} = \frac{x_{n-(2k+1)}}{1 + x_{n-k}}$$

Rasyonel Fark Denkleminin Çözümleri

Özet: *Aşağıdaki Rasyonel fark denkleminin çözümlerini incelendi.*

$$x_{n+1} = \frac{x_{n-(2k+1)}}{1 + x_{n-k}}, \quad \text{n=0,1,2,...} \qquad (1)$$

Burada başlangıç şartları reel sayılardır.

Anahtar Kelimeler: *Fark denklemleri, 2k+2 periyotlu çözümler*

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1. INTRODUCTION

Recently there has been a lot of interest in studying the periodic nature of non-linear difference equations. For some recent results concerning among other problems, the periodic nature of scalar nonlinear difference equations see, [1-24].

Cinar, studied the following problems with positive initial values

$$x_{n+1} = \frac{x_{n-1}}{1 + ax_n x_{n-1}}$$

$$x_{n+1} = \frac{x_{n-1}}{-1 + ax_n x_{n-1}}$$

$$x_{n+1} = \frac{ax_{n-1}}{1 + bx_n x_{n-1}}$$

for $\mathrm{n} = 0,1,2,...$ in [2,3,4], respectively.

In [18] Stevic assumed that $\beta = 1$ and solved the following problem

$$x_{n+1} = \frac{x_{n-1}}{1 + x_n} \qquad \text{for} \quad \mathrm{n} = 0,1,2,...$$

Where $x_{-1}, x_0 \in (0,\infty)$. Also, this results was generalized to the equation of the following form:

$$x_{n+1} = \frac{x_{n-1}}{g(x_n)} \qquad \text{for} \quad \mathrm{n} = 0,1,2,...$$

Where $x_{-1}, x_0 \in (0,\infty)$.

Simsek et. al., studied the following problems with positive initial values

$$x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}}$$

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2}}$$

for $\mathrm{n} = 0,1,2,...$ in [19,20] respectively.

In this paper we investigated the following nonlinear difference equation

$$x_{n+1} = \frac{x_{n-(2k+1)}}{1 + x_{n-k}}, \quad \mathrm{n} = 0,1,2,... \qquad (1)$$

where $x_{-3}, x_{-2}, x_{-1}, x_0 \in (0,\infty)$.

2. MAİN RESULT

Let $\bar{x}$ be the unique positive equilibrum of Eq. (1), then clearly

$$\bar{x} = \frac{\bar{x}}{1+\bar{x}} \Rightarrow \bar{x} + \bar{x}^2 = \bar{x} \Rightarrow \bar{x}^2 = 0 \Rightarrow \bar{x} = 0$$

We can obtain $\bar{x} = 0$.

Theorem 1. Consider the difference equation (1). Then the following statements are true.

a) The sequences $(x_{(2k+2)n-(2k+1)})$, $(x_{(2k+2)n-(2k)})$, ... , $(x_{(2k+2)n})$ are decreasing and there exist $a_1, a_2, ..., a_{2k+2} \geq 0$ such that

$$\lim_{n\to\infty} x_{(2k+2)n-(2k+1)} = a_1,\ \lim_{n\to\infty} x_{(2k+2)n-(2k)} = a_2, ..., \lim_{n\to\infty} x_{(2k+2)n-(k)-1} = a_{k-1},$$

$$\lim_{n\to\infty} x_{(2k+2)n-(k)} = a_k,\ \lim_{n\to\infty} x_{(2k+2)n-(k)+1} = a_{k+1}, ..., \lim_{n\to\infty} x_{(2k+2)n} = a_{2k+2}.$$

b) $(a_1, a_2, ..., a_{2k+2}, a_1, a_2, ..., a_{2k+2}, ...)$ is a solution of equation (1) of period $2k+2$.

c) $\lim_{n\to\infty} x_{(2k+2)n-(2k+1)} . \lim_{n\to\infty} x_{(2k+2)n-(k)} = 0, ..., \lim_{n\to\infty} x_{(2k+2)n-(k)-1} \lim_{n\to\infty} x_{(2k+2)n} = 0$

or

$$a_1 a_k = 0, ..., a_{k-1} a_{2k+2} = 0.$$

d) If there exist $n_0 \in N$ such that $x_{n-k} \geq x_{n+1}$ for all $n \geq n_0$, then $\lim_{n\to\infty} x_n = 0$.

e) The following formulas hold:

$$x_{(2k+2)n+1} = x_{-(2k+1)} \left(1 - \frac{x_{-k}}{1+x_{-k}} \sum_{j=0}^{n} \prod_{i=1}^{2j} \frac{1}{1+x_{(k+1)i-k}} \right)$$

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$$x_{(2k+2)n+k+1} = x_{-(k+1)} \left(1 - \frac{x_0}{1+x_0} \sum_{j=0}^{n} \prod_{i=1}^{2j} \frac{1}{1+x_{(k+1)i}} \right)$$

$$x_{(2k+2)n+k+2} = x_{-k}\left(1-\frac{x_{-(2k+1)}}{1+x_{-k}}\sum_{j=0}^{n}\prod_{i=1}^{2j+1}\frac{1}{1+x_{(k+1)i-k}}\right)$$

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$$x_{(2k+2)n+2k+2} = x_0\left(1-\frac{x_{-(k+1)}}{1+x_0}\sum_{j=0}^{n}\prod_{i=1}^{2j+1}\frac{1}{1+x_{(k+1)i}}\right).$$

f) If $x_{(2k+2)n+1} \to a_1 \neq 0$ then $x_{(2k+2)n+k+2} \to 0$ as $n \to \infty$,..., .If $x_{(2k+2)n+k+1} \to a_{k+1} \neq 0$ then $x_{(2k+2)n+2k+2} \to 0$ as $n \to \infty$

Proof. a) Firstly, we consider the equation (1). From this equation we obtain

$$x_{n+1}(1+x_{n-k}) = x_{n-(2k+1)}.$$

If $x_{n-k} \in (0,+\infty)$, then $(1+x_{n-k}) \in (1,+\infty)$. Since $x_{n+1} < x_{n-(2k+1)}$, $n \in N$, we obtain that

$$\lim_{n\to\infty} x_{(2k+2)n-(2k+1)} = a_1,\ \lim_{n\to\infty} x_{(2k+2)n-(2k)} = a_2, ..., \lim_{n\to\infty} x_{(2k+2)n-(k)-1} = a_{k-1},$$

$$\lim_{n\to\infty} x_{(2k+2)n-(k)} = a_k,\ \lim_{n\to\infty} x_{(2k+2)n-(k)+1} = a_{k+1}, ..., \lim_{n\to\infty} x_{(2k+2)n} = a_{2k+2}.$$

b) $(a_1, a_2, ..., a_{2k+2}, a_1, a_2, ..., a_{2k+2}, ...)$ is a solution of equation (1) of period $2k+2$.

c) In view of the equation (1), we obtain

$$x_{(2k+2)n+1} = \frac{x_{(2k+2)n-(2k+1)}}{1+x_{(2k+2)n-k}}.$$

Taking limit as $n \to \infty$ on both sides of the above equality, we get

$$\lim_{n\to\infty} x_{(2k+2)n+1} = \lim_{n\to\infty} \frac{x_{(2k+2)n-(2k+1)}}{1+x_{(2k+2)n-k}}.$$

Then

$$\lim_{n\to\infty} x_{(2k+2)n+1} \lim_{n\to\infty} x_{(2k+2)n-k} = 0 \text{ or } a_1.a_k = 0.$$

Similarly,

$$\lim_{n\to\infty} x_{(2k+2)n-k-1} \lim_{n\to\infty} x_{(2k+2)n+2k+2} = 0 \text{ or } a_{k-1}.a_{2k+2} = 0.$$

d) If there exist $n_0 \in N$ such that $x_{n-k} \geq x_{n+1}$ for all $n \geq n_0$, then $a_1 \leq ... \leq a_k, ..., a_{k-1} \leq ... \leq a_{2k+2} \leq a_{k-1}$. Since $a_1.a_k = 0, ..., a_{k-1}.a_{2k+2} = 0$ we obtain the result.

e) Subracting $x_{n-(2k+1)}$ from the left and right-hand sides of equation (1) we obtain

$$x_{n+1}-x_{n-(2k+1)}=\frac{1}{1+x_{n-k}}\left(x_{n-k}-x_{n-(3k+2)}\right)$$

and the following formula

$$n\geq k+1 \text{ for } \begin{cases} x_{(k+1)n-\left[(k+1)^2-1\right]}-x_{(k+1)n-\left[(k+2)^2-2\right]}=\left(x_1-x_{-(2k+1)}\right)\prod_{i=1}^{n-(k+1)}\frac{1}{1+x_{(k+1)i-k}} \\ \vdots \\ x_{(k+1)n-\left[(k+1)^2-(k+1)\right]}-x_{(k+1)n-\left[(k+2)^2-(k+2)\right]}=\left(x_{k+1}-x_{-(k+1)}\right)\prod_{i=1}^{n-(k+1)}\frac{1}{1+x_{(k+1)i}} \end{cases} \tag{2}$$

holds. Replacing n by $2j$ in (2) and summing from $j=0$ to $j=n$ we obtain

$$\begin{aligned} x_{(2k+2)n+1}-x_{-(2k+1)}&=\left(x_1-x_{-(2k+1)}\right)\sum_{j=0}^{n}\prod_{i=1}^{2j}\frac{1}{1+x_{(k+1)i-k}} \quad (n=0,1,2,...) \\ &\vdots \\ x_{(2k+2)n+k+1}-x_{-(k+1)}&=\left(x_{k+1}-x_{-(k+1)}\right)\sum_{j=0}^{n}\prod_{i=1}^{2j}\frac{1}{1+x_{(k+1)i-k}} \quad (n=0,1,2,...) \end{aligned} \tag{3}$$

Also, replacing n by $2j+1$ in (2) and summing from $j=0$ to $j=n$ we obtain

$$\begin{aligned} x_{(2k+2)n+k+2}-x_{-k}&=\left(x_1-x_{-(2k+1)}\right)\sum_{j=0}^{n}\prod_{i=1}^{2j+1}\frac{1}{1+x_{(k+1)i-k}} \quad (n=0,1,2,...) \\ &\vdots \\ x_{(2k+2)n+2k+2}-x_{0}&=\left(x_{k+1}-x_{-(k+1)}\right)\sum_{j=0}^{n}\prod_{i=1}^{2j+1}\frac{1}{1+x_{(k+1)i-k}} \quad (n=0,1,2,...) \end{aligned} \tag{4}$$

Now, we obtained of the above formulas,

$$\begin{aligned} x_{(2k+2)n+1}&=x_{-(2k+1)}\left(1-\frac{x_{-k}}{1+x_{-k}}\sum_{j=0}^{n}\prod_{i=1}^{2j}\frac{1}{1+x_{(k+1)i-k}}\right) \\ &. \\ &. \\ &. \\ x_{(2k+2)n+k+1}&=x_{-(k+1)}\left(1-\frac{x_{0}}{1+x_{0}}\sum_{j=0}^{n}\prod_{i=1}^{2j}\frac{1}{1+x_{(k+1)i}}\right) \end{aligned} \tag{5}$$

$$x_{(2k+2)n+k+2} = x_{-k}\left(1 - \frac{x_{-(2k+1)}}{1+x_{-k}} \sum_{j=0}^{n} \prod_{i=1}^{2j+1} \frac{1}{1+x_{(k+1)i-k}}\right)$$

$$\vdots \qquad (6)$$

$$x_{(2k+2)n+2k+2} = x_0\left(1 - \frac{x_{-(k+1)}}{1+x_0} \sum_{j=0}^{n} \prod_{i=1}^{2j+1} \frac{1}{1+x_{(k+1)i}}\right).$$

f) Suppose that $a_1 = a_{k+2} = 0$. By **e)** we have

$$\lim_{n\to\infty} x_{(2k+2)n+1} = \lim_{n\to\infty} x_{-(2k+1)}\left(1 - \frac{x_{-k}}{1+x_{-k}} \sum_{j=0}^{n} \prod_{i=1}^{2j} \frac{1}{1+x_{(k+1)i-k}}\right)$$

$$a_1 = x_{-(2k+1)}\left(1 - \frac{x_{-k}}{1+x_{-k}} \sum_{j=0}^{\infty} \prod_{i=1}^{2j} \frac{1}{1+x_{(k+1)i-k}}\right)$$

$$a_1 = 0 \Rightarrow \frac{1+x_{-k}}{x_{-k}} = \sum_{j=0}^{\infty} \prod_{i=1}^{2j} \frac{1}{1+x_{(k+1)i-k}} \qquad (7)$$

Similarly,

$$\lim_{n\to\infty} x_{(2k+2)n+k+2} = \lim_{n\to\infty} x_{-k}\left(1 - \frac{x_{-(2k+1)}}{1+x_{-k}} \sum_{j=0}^{n} \prod_{i=1}^{2j+1} \frac{1}{1+x_{(k+1)i-k}}\right)$$

$$a_{k+2} = x_{-k}\left(1 - \frac{x_{-(2k+1)}}{1+x_{-k}} \sum_{j=0}^{\infty} \prod_{i=1}^{2j+1} \frac{1}{1+x_{(k+1)i-k}}\right)$$

$$a_{k+2} = 0 \Rightarrow \frac{1+x_{-k}}{x_{-(2k+1)}} = \sum_{j=0}^{\infty} \prod_{i=1}^{2j+1} \frac{1}{1+x_{(k+1)i-k}} \qquad (8)$$

From the equations (7) and (8),

$$\frac{1+x_{-k}}{x_{-k}} = \sum_{j=0}^{\infty} \prod_{i=1}^{2j} \frac{1}{1+x_{(k+1)i-k}} > \frac{1+x_{-k}}{x_{-(2k+1)}} = \sum_{j=0}^{\infty} \prod_{i=1}^{2j+1} \frac{1}{1+x_{(k+1)i-k}} \qquad (9)$$

thus, $x_{-(2k+1)} > x_{-k}$.

Suppose that $a_{k+1} = a_{2k+2} = 0$. From the equation (10) in **e)** follows, Proof of the equation (9) is similar and will be omitted.

$$\frac{1+x_0}{x_{-0}}=\sum_{j=0}^{\infty}\prod_{i=1}^{2j}\frac{1}{1+x_{(k+1)i-k}}>\frac{1+x_{-0}}{x_{-(k+1)}}=\sum_{j=0}^{\infty}\prod_{i=1}^{2j+1}\frac{1}{1+x_{(k+1)i-k}} \tag{10}$$

thus, $x_{-(k+1)}>x_0$.

From here we obtain $x_{-(2k+1)}>x_{-2k}>...>x_{-1}>x_0$. We arrive at a contradiction which completes the proof of theorem.

3. EXAMPLES

Example 3.1: Consider the following equation $x_{n+1}=\frac{x_{n-3}}{1+x_{n-1}}$ which is special case of $k=1$.

If the initial conditions are selected as follows:

$$x[-3]=2;x[-2]=3;x[-1]=4;x[0]=5;$$

The following solutions are obtained:

x(n)={ 0.0327869, 1.81188, 3.08397, 4.22581, 0.0013321, 1.78096, 3.05336, 4.19542, 0.000055937, 1.77969, 3.05208, 4.19414, 2.35212x10^{-6}, 1.77963, 3.05203, 4.19409, 9.89108x10^{-8}, 1.77963, 3.05203, 4.19409, 4.15939x10^{-9}, 1.77963, 3.05203, 4.19409, 1.7491x10^{-10}, 1.77963, 3.05203, 4.19409, 7.35532x10^{-12}, 1.77963, 3.05203, 4.19409, 3.09306x10^{-13},1.77963,3.05203, 4.19409,…}

The graph of the solutions is given below.

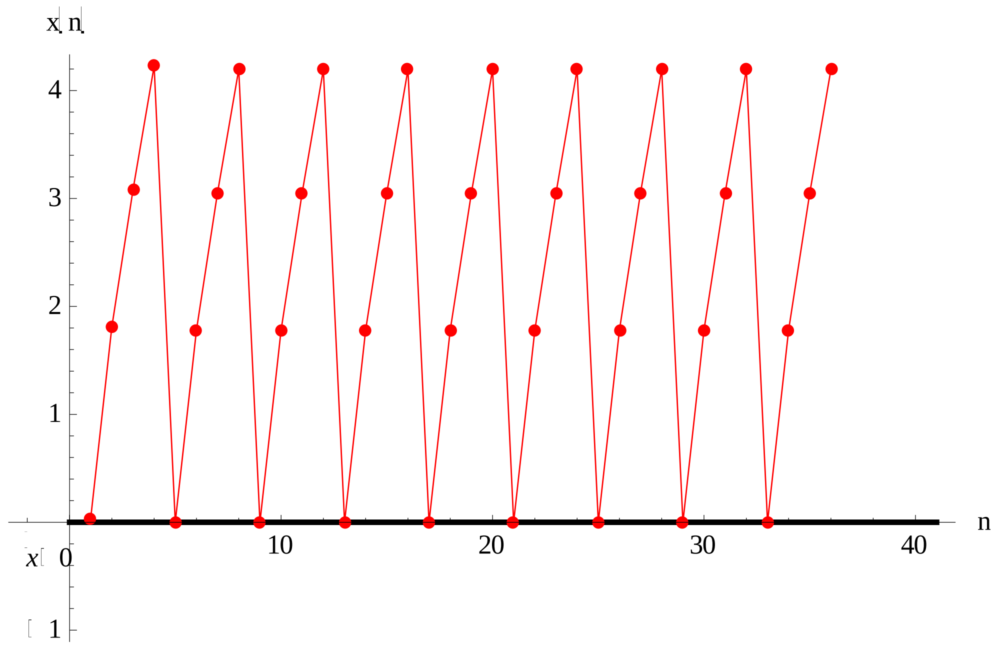

Figure 3.1. x(n) graph of the solutions.

Example 3.2: Consider the following equation $x_{n+1} = \dfrac{x_{n-3}}{1+x_{n-1}}$ which is special case of $k=1$.

If the initial conditions are selected as follows:

$x[-3]=2; x[-2]=0.1; x[-1]=0.01; x[0]=0.001;$

The following solutions are obtained:

x(n)={2, 0.099998, 0.009998, 0.000998004, 2, 0.099996, 0.00999601, 0.000996013, 1.99999, 0.099994, 0.00999401, 0.000994027, 1.99999, 0.099992, 0.00999203, 0.000992044, 1.99999, 0.09999, 0.00999005, 0.000990066, 1.99999, 0.0999881, 0.00998807, 0.000988093, 1.99999, 0.0999861, 0.0099861, 0.000986123, 1.99998, 0.0999841, 0.00998413, 0.000984159, 1.99998, 0.0999822, 0.00998216, 0.000982198, …}

The graph of the solutions is given below.

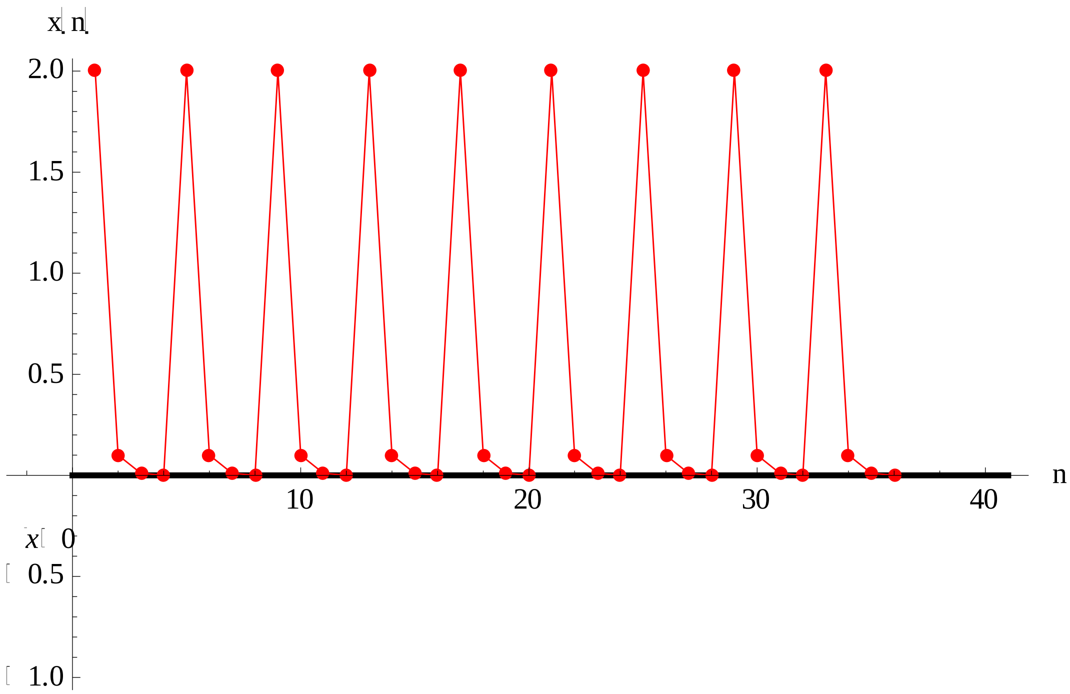

Figure 3.2. x(n) graph of the solutions.

Example 3.3: Consider the following equation $x_{n+1} = \dfrac{x_{n-5}}{1+x_{n-2}}$ which is special case of $k=2$.

If the initial conditions are selected as follows:

$x[-5]=2; x[-4]=3; x[-3]=4; x[-2]=5; x[-1]=6; x[0]=7;$

The following solutions are obtained:

x(n)={ 0.333333, 0.428571, 0.5,3.75, 4. 2, 4.66667, 0.0701754, 0.0824176, 0.0882353, 3.5041, 3.8802, 4.28829, 0.0155804, 0.0168881, 0.016685, 3.45034, 3.81576, 4.21791, 0.00350093, 0.00350685, 0.00319765, 3.4383, 3.80243, 4.20447, 0.0007888, 0.000730224, 0.000614404, 3.43559, 3.79965, 4.20189, 0.000177834, 0.000152141, 0.000118112,3 .43498,3.79907,…}

The graph of the solutions is given below.

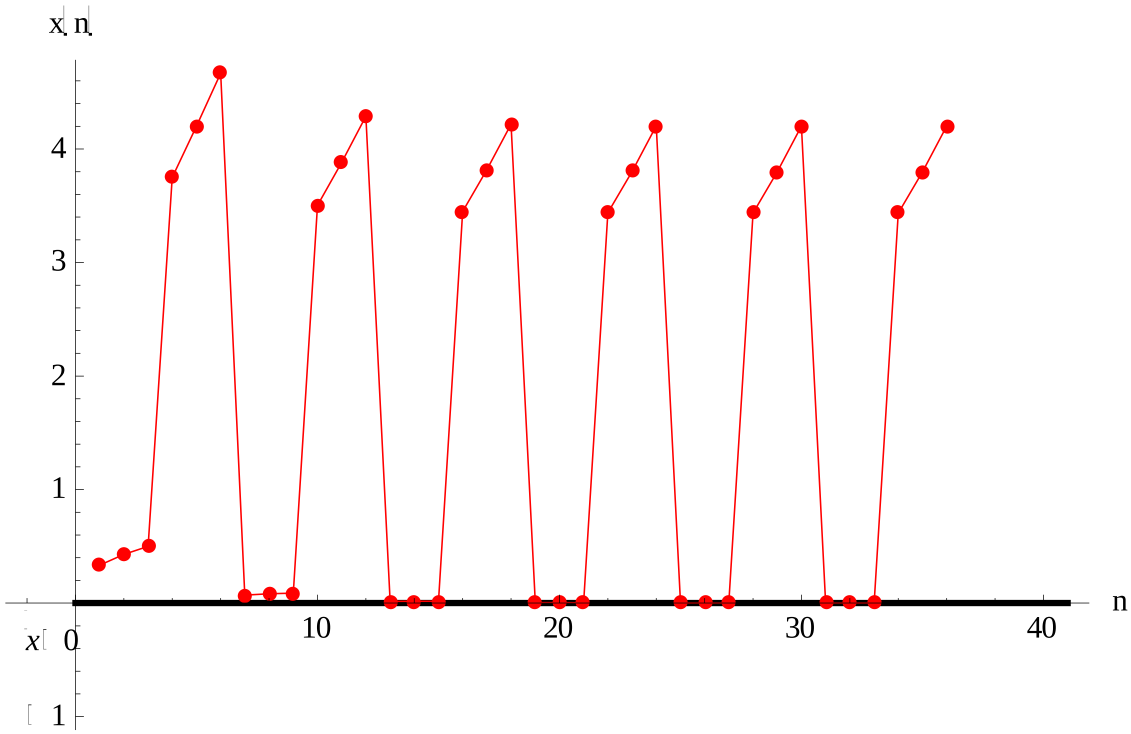

Figure 3.3. x(n) graph of the solutions

Example 3.4: Consider the following equation $x_{n+1} = \dfrac{x_{n-5}}{1+x_{n-2}}$ which is special case of $k=2$.

If the initial conditions are selected as follows:

$$x[-5] = 0.1; x[-4] = 0.01; x[-3] = 0.001; x[-2] = 2; x[-1] = 4; x[0] = 0.000001$$

The following solutions are obtained:

x(n)={ 0.0333333, 0.002, 0.000999999, 1.93548, 3.99202, $9.99001\text{x}10^{-7}$, 0.0113553, 0.00040064, 0.000999998, 1.91375, 3.99042, $9.98003\text{x}10^{-7}$,0.00389714, 0.0000802818, 0.000999997, 1.90632, 3.9901, $9.97006\text{x}10^{-7}$, 0.00134092, 0.0000160882, 0.000999996, 1.90377, 3.99003, $9.9601\text{x}10^{-7}$, 0.000461785, $3.22407\text{x}10^{-6}$, 0.000999995, 1.90289, 3.99002, $9.95015\text{x}10^{-7}$, 0.000159078, $6.46104\text{x}10^{-7}$, 0.000999994, 1.90259, 3.99002, $9.94021\text{x}10^{-7}$, …}

The graph of the solutions is given below.

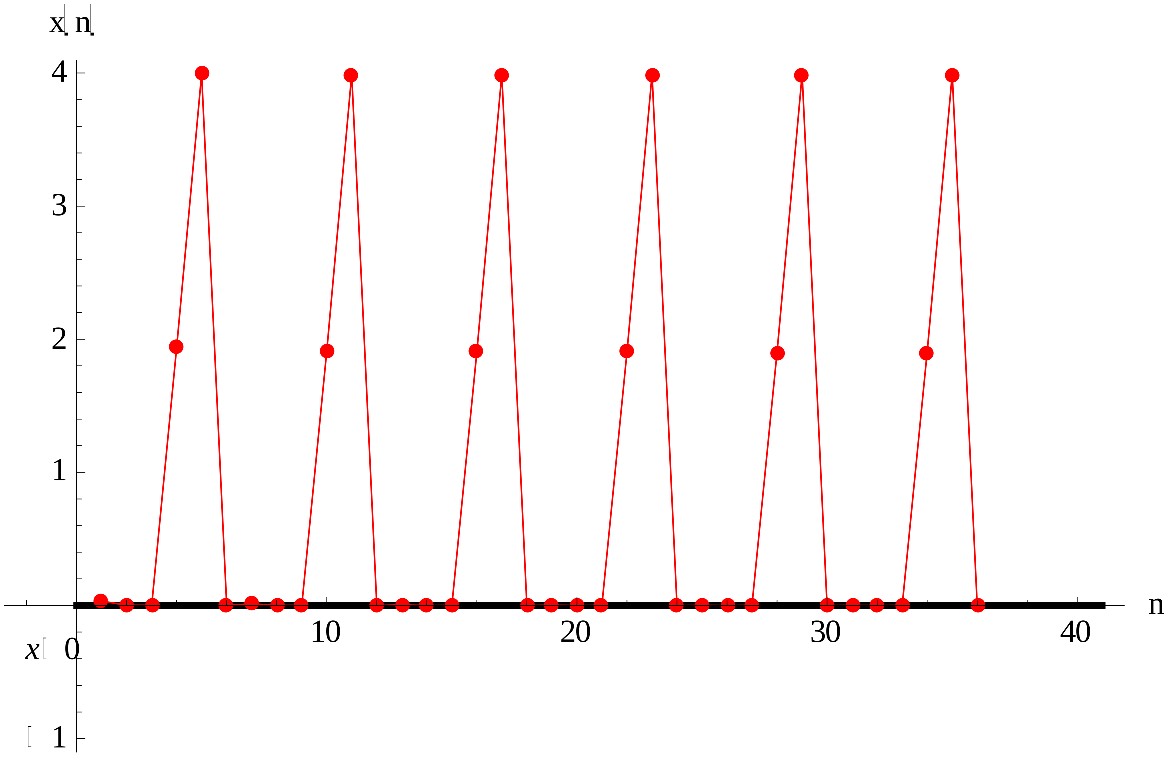

Figure 3.4. x(n) graph of the solutions.

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