

Uniform and Pointwise Polynomial Inequalities in Regions with Asymptotically Conformal Curve

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Abstract: *In this we continue studying the Nikolskii and Bernstein-Walsh type polynomial estimation in the Lebesgue spaces in the bounded and unbounded regions bounded by asymptotically conformal curve.*

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Asimptotik konform eğri ile sınırlı bölgelerde düzgün ve noktasal polinom eşitsizlikleri

Özet: *Bu çalışmada, asimptotik konform eğri ile sınırlı sonlu ve sonsuz bölgelerde Nikolskii ve Bernstein tipinde polinom eşitsizliklerini Lebesgue uzaylarında incelemeye devam ediyoruz.*

Anahtar kelimeler: *Polinomlar, Nikolskii eşitsizliği, Bernstein eşitsizliği, Konform dönüşüm, Asimptotik konform eğri.*

1. INTRODUCTION

Let \mathfrak{E} be a complex plane, and $\overline{\mathfrak{E}} := \mathfrak{E} \cup \{\infty\}$; $L \subset \mathfrak{E}$ be a closed rectifiable curve, $G := \text{int } L$ with $L := \partial G$ and $0 \in G$; $\Omega := \text{ext } L$. Let \wp_n denotes the class of arbitrary algebraic polynomials $P_n(z)$ of degree at most $n \in \mathbb{N}$.

Let $0 < p \leq \infty$. For a rectifiable Jordan curve L and a weight function $h(z)$ defined on a certain neighborhood of L we introduce:

$$\|P_n\|_p := \|P_n\|_{\mathfrak{L}_p(h,L)} := \left(\int_L h(z) |P_n(z)|^p |dz| \right)^{1/p}, \quad 0 < p < \infty;$$

$$\|P_n\|_\infty := \|P_n\|_{\mathfrak{L}_\infty(1,L)} := \max_{z \in L} |P_n(z)|, \quad p = \infty.$$

Clearly, $\|\cdot\|_{\mathfrak{L}_p}$ is the quasinorm (i.e. a norm for $1 \leq p \leq \infty$ and a p -norm for $0 < p < 1$). For ease of writing, we will use the word "norm" in both cases

Denoted by $w = \Phi(z)$, the univalent conformal mapping of Ω onto $\Delta := \{w : |w| > 1\}$ with normalization $\Phi(\infty) = \infty$, $\lim_{z \rightarrow \infty} \frac{\Phi(z)}{z} > 0$ and $\Psi := \Phi^{-1}$. For $t \geq 1$ we set

$$L_t := \{z : |\Phi(z)| = t\}, \quad L_1 \equiv L, \quad G_t := \text{int } L_t, \quad \Omega_t := \text{ext } L_t.$$

Throughout the article, we will use consider a so-called generalized Jacobi weight function $h(z)$ being defined as follows. Let $\{z_j\}_{j=1}^m$ be a fixed system of distinct points on curve L which is located in the positive direction. For some fixed R_0 , $1 < R_0 < \infty$, and $z \in G_{R_0}$, we define:

$$h(z) := h_0(z) \prod_{j=1}^m |z - z_j|^{\gamma_j}, \tag{1.1}$$

where $\gamma_j > -1$, for all $j = 1, 2, \dots, m$, and there exists a constant $c_0 := c_0(G_{R_0}) > 0$ such that for all $z \in G_{R_0}$

$$h_0(z) \geq c_0 > 0.$$

In this paper, our first goal is to continue studying the Nikolskii-type inequality:

$$\|P_n\|_{\mathfrak{L}_q(h,L)} \leq c \lambda_n(L, h, p, q) \|P_n\|_{\mathfrak{L}_p(h,L)}, \quad 0 < p < q \leq \infty, \tag{1.2}$$

where $c = c(G, p, q) > 0$ is the constant independent of n and P_n , and $\lambda_n(L, h, p, q) \rightarrow \infty$, $n \rightarrow \infty$, depending on the geometrical properties of curve L and weight function h and of p . In such a formulation of the problem for different p and q , $0 < p < q \leq \infty$, with respect to different norms, were the objects of the studying of many authors. First result of (1.2)-type, in case

$h(z) \equiv 1$ and $L = \{z : |z| = 1\}$ for $0 < p < \infty$ was found in [19]. The another results, similar to (0.2), for the sufficiently smooth curve, was obtained in [36] ($h(z) \equiv 1$), and in [33, Part 4 ($h(z) \neq 1$)]. The estimation of (1.2)-type for $0 < p < \infty$ and $h(z) \equiv 1$ when L is a rectifiable Jordan curve was investigated in [33], [34], [35], [22], [23], [25, pp 122-133], [31]. In [10, Theorem 6] obtained identical inequalities for more general curves and for another weighed function. There are more references regarding the inequality of (1.2)-type, we can find in Milovanovic et all. [24, Sect.5.3].

Further, analogous estimates of (1.2) for some regions and the weight function $h(z)$ were obtained: in [2] ($p > 1$) and in [26] ($p > 0, h \equiv h_0$) for regions bounded by rectifiable quasiconformal curve having some general properties; in [4] ($p > 1$) for piecewise Dini-smooth curve with interior and exterior cusps; in [3] ($p > 1$) for regions bounded by piecewise smooth curve with exterior cusps but without interior cusps; in [5] ($p > 0$) for regions bounded by piecewise rectifiable quasiconformal curve with cusps; in [6] ($p > 0$) for regions bounded by piecewise quasismooth (by Lavrentiev) curve with cusps.

Second our goal is a continue investigation estimation of the following type:

$$|P_n(z)| \leq c \eta_n(L, h, d(z, L), p) \|P_n\|_p |\Phi(z)|^{n+1}, \quad z \in \Omega, \quad p > 0, \quad (1.3)$$

where $c = c(L, p) > 0$ is a constant independent from n, z, P_n , and $\eta_n(L, h, d(z, L), p) \rightarrow \infty$ (in general!) as $n \rightarrow \infty$, depending on the geometrical properties of curve L , weight function h and parameter p .

The results of the (1.3) type starts from the work of Bernstein [37]. Analogous results of (1.3)-type for some norms, weight function $h(z)$ and for different unbounded regions were obtained by Lebedev, Tamrazov, Dzjadyk, Shevchuk (see, for example, [14]), Stylianopoulos [32] and others. Corresponding results (1.3) for some regions and the weight function $h(z)$ defined as in (0.1) with $\gamma_j > -1$ were also obtained: in [4] for $p > 1$ and in [27] for $p > 0$, for regions bounded by piecewise Dini-smooth boundary with interior and exterior zero angles; in [5] for $p > 0$ and for regions bounded by piecewise quasiconformal boundary with interior and exterior zero angles; in [3] for $p > 1$ and for regions bounded by piecewise smooth boundary with exterior zero angles (without interior zero angles); in [6] for $p > 0$ and for regions bounded by piecewise quasismooth boundary with interior and exterior zero angles and in others.

In this work, we investigate similar problem for asymptotically conformal curve L , weight function h defined in (1.1) and for all $p > 0$ and $q = \infty$. Finally, combining obtained estimates for $|P_n(z)|$ on \bar{G} and Ω , we get the evaluation for $|P_n(z)|$ in whole complex plane, depending on the geometrical properties of the region G , weight function $h(z)$ and p .

2. DEFINITIONS AND MAIN RESULTS

Throughout this paper, c, c_0, c_1, c_2, \dots are positive and $\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots$ are sufficiently small positive constants (generally, different in different relations), which depends on G in general and, on parameters inessential for the argument; otherwise, such dependence will be explicitly stated.

Let the function φ maps G conformally and univalently onto $B := \{w : |w| < 1\}$ which is normalized by $\varphi(0) = 0, \varphi'(0) > 0$, and $\psi := \varphi^{-1}$.

Following [28, pp. 286-294], we say that a bounded Jordan curve L is called a κ -quasicircle (or κ -quasiconformal curve) $0 \leq \kappa < 1$, if any conformal mapping ψ can be extended to a K -quasiconformal, $K = \frac{1+\kappa}{1-\kappa}$, homeomorphism of the plane $\bar{\mathbb{E}}$ on the $\bar{\mathbb{E}}$. In that case the region G with $L := \partial G$ is called a κ -quasidisk. The curve L (region G) is called a quasicircle (quasidisk), if it is κ -quasicircle (κ -quasidisk) for some $0 \leq \kappa < 1$. A Jordan curve L is called a quasicircle or quasiconformal curve, if it is the image of the unit circle under a quasiconformal mapping of \mathbb{E} (see: [20, p.105], [28, p.286]). On the other hand, in them was given some geometric criteria of quasiconformality of the curves (see, [8, p.81], [29, p.107], [28, p.286], [21, p.341]). We give one of them. Let z_1, z_2 be an arbitrary points on L and $L(z_1, z_2)$ denotes the subarc of L of shorter diameter with endpoints z_1 and z_2 . The curve L is a quasicircle if and only if the quantity

$$\frac{|z_1 - z| + |z - z_2|}{|z_1 - z_2|} \tag{2.1}$$

is bounded for all $z_1, z_2 \in L$ and $z \in L(z_1, z_2)$. Lesley [21, p.341] said that the curve L as " c -quasiconformal", if there exists the positive constant c , independent from points z_1, z_2 and z such that

$$\frac{|z_1 - z| + |z - z_2|}{|z_1 - z_2|} \leq c. \tag{2.2}$$

The Jordan curve L is called asymptotically conformal if

$$\max_{z \in L(z_1, z_2)} \frac{|z_1 - z| + |z - z_2|}{|z_1 - z_2|} \rightarrow 1, \quad |z_1 - z_2| \rightarrow 0. \tag{2.3}$$

Some various properties of the asymptotically conformal curves has been studied, for instance, by Anderson, Becker and Lesley [9]. Dyn'kin [15], Pommerenke, Warschawski [30], Gutlyanskii, Ryazanov [16], [17], [18] and others. According to the geometric criteria of quasiconformality of the curves ([8, p.81], [29, p.107]), every asymptotically conformal curve is a quasicircle. Every smooth curve is asymptotically conformal but corners are not allowed. It is well known that quasicircles can be non-rectifiable (see, for example, [12], [20, p.104]). The same is true for asymptotically conformal curves.

In this work, we study a similar problem to (1.2) with respect to $\|P_n\|_p$, $p > 0$, for regions with asymptotically conformal boundary. Now, we start to formulate the new results.

Theorem 1. Let $p > 0$, $L = \partial G$ be a rectifiable asymptotically conformal curve and $h(z)$ is defined as in (1.1). Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$, $j = \overline{1, m}$ and arbitrarily small $\varepsilon > 0$ there exist $c_i = c_i(G, p, \gamma_j, \varepsilon) > 0$, $i = 1, 2$ such that

$$|P_n(z_j)| \leq c_1 n^{\frac{1+\gamma_j}{p}+\varepsilon} \|P_n\|_p, \tag{2.4}$$

and, consequently,

$$\|P_n\|_\infty \leq c_2 n^{\frac{1+\gamma}{p}+\varepsilon} \|P_n\|_p, \tag{2.5}$$

where $\gamma := \max\{0; \gamma_j, j = \overline{1, m}\}$.

For any fixed $\rho > 1$ we divide Ω_ρ as $\Omega_\rho := \bigcup_{j=1}^m \Omega_\rho^j$, where Ω_ρ^j defined below in (4.8).

Theorem 2. Let $p > 0$, $L = \partial G$ be a rectifiable asymptotically conformal curve and $h(z)$ is defined as in (1.1). Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$, $R_1 := 1 + \frac{\varepsilon_1}{n}$, $j = \overline{1, m}$ and arbitrary small $\varepsilon > 0$ there exists $c_3 = c_3(G, p, \gamma_j, \varepsilon) > 0$ such that

$$|P_n(z)| \leq c_3 \left(\frac{\sqrt{\mu_n}}{d(z, L_{R_1})} \right)^{\frac{2}{p}} \|P_n\|_p |\Phi(z)|^{n+1}, \quad z \in \Omega_{R_1}^j, \tag{2.6}$$

where for any $j = \overline{1, m}$, μ_n defined as:

$$\mu_n := \begin{cases} n^\varepsilon, & \text{if } \gamma_j < 1 - \varepsilon, \text{ for all } j = \overline{1, m} \\ n^\varepsilon \ln n, & \text{if } \exists j_0, \gamma_{j_0} = 1 - \varepsilon, \text{ and for } j \neq j_0 \gamma_j < 1 - \varepsilon; \\ n^{\gamma_j - 1 + \varepsilon}, & \text{if } \exists j_0, \gamma_{j_0} > 1 - \varepsilon \text{ and for } j \neq j_0 \gamma_j \leq 1 - \varepsilon. \end{cases} \tag{2.7}$$

According to the Bernstein Lemma [37], the estimation (2.5) also is true for the $z \in \overline{G_{R_1}}$, with another constant. Therefore, combining estimation (2.5) (for the $z \in \overline{G_{R_1}}$) with (2.6), we obtain an estimation on the growth of $|P_n(z)|$ in the whole complex plane:

Corollary 3. Under the assumptions of Theorems 1 and 2 following is true:

$$|P_n(z)| \leq c_4 \|P_n\|_p \begin{cases} n^{\frac{2+\gamma}{p}+\varepsilon}, & z \in \overline{G_{R_1}}, \\ \left(\frac{\sqrt{\mu_n}}{d(z, L_{R_1})} \right)^{\frac{2}{p}} |\Phi(z)|^{n+1}, & z \in \Omega_{R_1}^j, \end{cases} \tag{2.8}$$

where $c_4 = c_4(G, p) > 0$.

Corollary 4. For any compact subset $F \subset \Omega$ and $P_n \in \mathcal{P}_n$, $n \in \mathbb{N}$, we have

$$|P_n(z)| \leq c_5 \left(\frac{\sqrt{\mu_n}}{d(z, L)} \right)^{\frac{2}{p}} \|P_n\|_p |\Phi(z)|^{n+1}, \quad z \in F, \quad (2.9)$$

where $c_5 = c_5(G, F) > 0$.

3. SHARPNESS OF ESTIMATES

The sharpness of the estimations (2.4)-(2.9) can be discussed by comparing them with the following result.

Remark 5. a) The inequalities (2.4), (2.5) are sharp. For the polynomials $P_n^*(z) = 1 + z + \dots + z^n$,

a) $h^*(z) \equiv 1$, b) $h^{**}(z) = |z-1|^\gamma$, $\gamma > 0$, and $L := \{z : |z|=1\}$, there exists a constant $c_9 = c_9(p) > 0$ and $c_{10} = c_{10}(h^{**}, p) > 0$ such that:

$$\begin{aligned} a) \quad & \|P_n^*\|_{\gamma_\infty} \geq c_9 n^{\frac{1}{p}} \|P_n^*\|_{\gamma_p(1, L)}, \quad p > 1; \\ b) \quad & \|P_n^*\|_{\gamma_\infty} \geq c_{10} n^{\frac{\gamma+1}{p}} \|P_n^*\|_{\gamma_p(h^{**}, L)}, \quad p > \gamma + 1. \end{aligned}$$

SOME AUXILIARY RESULTS

Throughout this paper, for $a > 0$ and $b > 0$, we use the expression $a \mathbf{p} b$ (order inequality), if $a \leq cb$. The expression $a \neq b$ means that $a \mathbf{p} b$ and $b \mathbf{p} a$ " simultaneously.

For any $k \geq 0$ and $m > k$, notation $i = \overline{k, m}$ means $i = k, k+1, \dots, m$.

We give some facts from the theory of quasiconformal mapping, which will be used throughout of all proof below.

Let L be a K -quasiconformal curve, then there exists a quasiconformal reflection $y(\cdot)$ across L such that $y(G) = \Omega$, $y(\Omega) = G$ and $y(\cdot)$ is fixed the points of L [8]. There exists a quasiconformal reflection of $y(\cdot)$ satisfying the following condition [8], [11, p.126]:

$$\begin{aligned} |y(\zeta) - z| \mathbf{p} |\zeta - z|, \quad z \in L, \quad \varepsilon < |\zeta| < \frac{1}{\varepsilon}, \\ |y_{\bar{\zeta}}| \mathbf{p} |y_\zeta| \mathbf{p} 1, \quad \varepsilon < |\zeta| < \frac{1}{\varepsilon}, \\ |y_{\bar{\zeta}}| \mathbf{p} |y(\zeta)|^2, \quad |\zeta| < \varepsilon, \quad |y_{\bar{\zeta}}| \mathbf{p} |\zeta|^{-2}, \quad |\zeta| > \frac{1}{\varepsilon}, \end{aligned} \quad (3.1)$$

and for the Jacobian $J_y = |y_z|^2 - |y_{\bar{z}}|^2$ of $y(\cdot)$ the relation $|y_{\bar{z}}|^2 \leq \frac{1}{1-k^2} J_y$ is hold, where $k = \frac{K^2-1}{K^2+1}$. Such quasiconformal reflection of $y(\cdot)$ is called regular quasiconformal reflection across L .

Let L be a quasicircle and $y(\cdot)$ be a regular quasiconformal reflection across L . For any $R > 1$, we put $L^* := y(L_R)$, $G^* := \text{int} L^*$, $\Omega^* := \text{ext} L^*$, and denote by Φ_R the conformal mapping of Ω^* onto Δ with the normalization $\Phi_R(\infty) = \infty$, $\lim_{z \rightarrow \infty} \frac{\Phi_R(z)}{z} > 0$, and let $\Psi_R := \Phi_R^{-1}$. Moreover, for any $t > 1$, we set $L_t^* := \{z : |\Phi_R(z)| = t\}$, $G_t^* := \text{int} L_t^*$, $\Omega_t^* := \text{ext} L_t^*$. According to [10], for all $z \in L^*$ and $t \in L$ such that $|z - t| = d(z, L)$, we have

$$\begin{aligned} d(z, L) \rho d(t, L_R) \rho d(z, L_R^*) \\ |\Phi_R(z)| \leq |\Phi_R(t)| \leq 1 + c(R-1). \end{aligned} \tag{3.2}$$

Lemma 1. [1] Let L be a quasicircle, $z_1 \in L, z_2, z_3 \in \Omega \cap \{z : |z - z_1| \leq d(z_1, L_{r_0})\}$; $w_j = \Phi(z_j), j = 1, 2, 3$. Then

- The statements $|z_1 - z_2| \rho |z_1 - z_3|$ and $|w_1 - w_2| \rho |w_1 - w_3|$ are equivalent. So are $|z_1 - z_2| \approx |z_1 - z_3|$ and $|w_1 - w_2| \approx |w_1 - w_3|$.
- If $|z_1 - z_2| \rho |z_1 - z_3|$, then

$$\left| \frac{w_1 - w_3}{w_1 - w_2} \right|^\varepsilon \rho \left| \frac{z_1 - z_3}{z_1 - z_2} \right| \rho \left| \frac{w_1 - w_3}{w_1 - w_2} \right|^c,$$

where $\varepsilon < 1, c > 1; L_{r_0} := \{\zeta : |\varphi(\zeta)| = r_0, 0 < r_0 < 1\}$ and $r_0 := r_0(G)$ is a constant, depending on G .

Lemma 2. [21, p.342] Let L be an asymptotically conformal curve, Then, Φ and Ψ are $Lip\alpha$ for all $\alpha < 1$ in $\bar{\Omega}$ and $\bar{\Delta}$, correspondingly.

Lemma 3. Let L be an asymptotically conformal curve, Then,

$$|\Psi(w_1) - \Psi(w_2)| \ll |w_1 - w_2|^{1+\varepsilon},$$

for all $w_1, w_2 \in \bar{\Delta}$ and $\forall \varepsilon > 0$.

This fact follows from Lemma 2. We also give estimation for the Ψ' (see, for example, [11, Th.2.8]):

$$|\Psi'(\tau)| \rho \frac{d(\Psi(\tau), L)}{|\tau| - 1}. \tag{3.3}$$

Lemma 4. [7, Lemma 2.3] Let L be a quasicircle. For arbitrary $R > 1$, there exist numbers $\rho_1, \rho_2 : \rho_1 < \rho_2$, and $\rho_3, \rho_4 : \rho_3 < \rho_4$ such that the following conditions are satisfied:

$$\overline{G}_{\rho_1}^* \subseteq \overline{G}, \quad \overline{G} \subseteq \overline{G}_{\rho_2}^* \quad \text{and} \quad \overline{G}_{\rho_3}^* \subseteq \overline{G}_R, \quad \overline{G}_R \subseteq \overline{G}_{\rho_4}^*,$$

$$\rho_1 - 1 \leq \rho_2 - 1 \leq \rho_3 - 1 \leq \rho_4 - 1 \leq R - 1.$$

Let $\{z_j\}_{j=1}^m$ be a fixed system of the points on L and the weight function $h(z)$ is defined as in (1.1).

Lemma 5. [3] Let L be a rectifiable Jordan curve, $P_n \in \mathcal{P}_n$, $n \in \mathbb{N}$, $p > 0$ and $R > 1$. Then, the following inequality holds:

$$\|P_n\|_{L_p(h, L_R)} \leq R^{n+\frac{\gamma}{p}} \|P_n\|_p,$$

where $\gamma = \max\{\gamma_j : j = \overline{1, m}\}$.

4. PROOF OF THEOREMS

4.1. Proof of Theorem 1.

Let $R = 1 + \frac{1}{n}$. For the $\varepsilon_1 = \frac{1}{2}$, let us set: $R_1 := 1 + \frac{R-1}{2}$. Let $\{\zeta_j\}$, $1 \leq j \leq m \leq n$, zeros of $P_n(z)$ lying on Ω (if such zeros exist) and

$$B_m(z) := \prod_{j=1}^m B_j(z) = \prod_{j=1}^m \frac{\Phi(z) - \Phi(\zeta_j)}{1 - \overline{\Phi(\zeta_j)}\Phi(z)} \tag{4.1}$$

denote a Blaschke function with respect of zeros of $P_n(z)$. For any $p > 0$ and $z \in \Omega$, let us set:

$$F_{n,p}(z) := \left(\frac{P_n(z)}{B_m(z) \Phi^{n+1}(z)} \right)^{\frac{p}{2}}. \tag{4.2}$$

The function $F_{n,p}(z)$, $F_{n,p}(\infty) = 0$, is analytic in Ω , continuous on $\overline{\Omega}$ and does not have zeros in Ω . We take an arbitrary continuous branch of the $F_{n,p}(z)$ and for this branch we maintain the same designation. Cauchy integral representation for the region Ω is given as:

$$F_{n,p}(z) = -\frac{1}{2\pi i} \int_{L_{R_1}} F_{n,p}(\zeta) \frac{d\zeta}{\zeta - z}, \quad z \in \Omega_{R_1}.$$

Since $|B_m(\zeta)| = 1$, for $\zeta \in L$, then, for arbitrary ε , $0 < \varepsilon < \varepsilon_1$, there exists a circle $|w| = 1 + \frac{\varepsilon}{n}$, such that for any $j = \overline{1, m}$ the following is satisfied:

$$|B_j(\zeta)| > 1 - \varepsilon.$$

Then, $|B_m(\zeta)| > (1 - \varepsilon)^m$, for $\zeta \in L_{R_1}$ and $|B_m(z)| \leq 1$, for $z \in \Omega_{R_1}$. On the other hand, $|\Phi(\zeta)| = R_1 > 1$, for $\zeta \in L_{R_1}$. Therefore, for any $z \in \Omega_{R_1}$ we have:

$$|F_{n,p}(z)| \leq \frac{1}{2\pi} \int_{L_{R_1}} |F_{n,p}(\zeta)| \frac{|d\zeta|}{|\zeta - z|}.$$

So that:

$$\begin{aligned} |P_n(z)|^{\frac{p}{2}} &\leq \frac{|B_m(z) \Phi^{n+1}(z)|^{\frac{p}{2}}}{2\pi} \int_{L_{R_1}} \left| \frac{P_n(\zeta)}{B_m(\zeta) \Phi^{n+1}(\zeta)} \right|^{\frac{p}{2}} \frac{|d\zeta|}{|\zeta - z|} \\ & \rho |\Phi^{n+1}(z)|^{\frac{p}{2}} \int_{L_{R_1}} |P_n(\zeta)|^{\frac{p}{2}} \frac{|d\zeta|}{|\zeta - z|}. \end{aligned} \tag{4.3}$$

Multiplying the numerator and the denominator of the last integrand by $h^{1/2}(\zeta)$, replacing the variable $w = \Phi(z)$ and applying the Hölder inequality, from (4.3) we obtain:

$$\begin{aligned} \int_{L_{R_1}} |P_n(\zeta)|^{\frac{p}{2}} \frac{|d\zeta|}{|\zeta - z|} &= \int_{L_{R_1}} h^{1/2}(\zeta) |P_n(\zeta)|^{\frac{p}{2}} \frac{|d\zeta|}{h^{1/2}(\zeta) |\zeta - z|} \\ &\leq \left(\int_{|t|=R_1} h(\Psi(t)) |P_n(\Psi(t))|^p |\Psi'(t)| |dt| \right)^{\frac{1}{2}} \cdot \left(\int_{|t|=R_1} \frac{|\Psi'(t)| |dt|}{h(\Psi(t)) |\Psi(t) - \Psi(w)|^2} \right)^{\frac{1}{2}} \\ &= \|P_n\|_{L_p(h, L_{R_1})}^{\frac{p}{2}} \cdot \left(\int_{|t|=R_1} \frac{|\Psi'(t)| |dt|}{h(\Psi(t)) |\Psi(t) - \Psi(w)|^2} \right)^{\frac{1}{2}}. \end{aligned} \tag{4.4}$$

According to Lemma 5, we get:

$$\|P_n\|_{L_p(h, L_{R_1})}^{\frac{p}{2}} \rho \|P_n\|_p^{\frac{p}{2}}. \tag{4.5}$$

Then, from (4.3), (4.4.) and (4.5), we have:

$$|P_n(z)|^{\frac{p}{2}} \rho |\Phi^{n+1}(z)|^{\frac{p}{2}} \|P_n\|_p^{\frac{p}{2}} \left(\int_{|t|=R_1} \frac{|\Psi'(t)| |dt|}{h(\Psi(t)) |\Psi(t) - \Psi(w)|^2} \right)^{\frac{1}{2}}. \tag{4.6}$$

Applying Lemma 1 and Lemma 3, from (3.3), for arbitrary small $\varepsilon > 0$, we obtain:

$$|\Psi'(t)| \rho (|t| - 1)^{-\varepsilon} \rho n^\varepsilon,$$

and so, from (4.6), we get:

$$\left|P_n(z)\right|^{\frac{p}{2}} \rho \left|\Phi^{n+1}(z)\right|^{\frac{p}{2}} \|P_n\|_p^{\frac{p}{2}} \cdot n^\varepsilon \cdot A_n(w), \tag{4.7}$$

for arbitrary small $\varepsilon > 0$ and

$$A_n(w) := \left(\int_{|t|=R_1} \frac{|dt|}{h(\Psi(t)) |\Psi(t) - \Psi(w)|^2} \right)^{\frac{1}{2}}.$$

To estimate the integral $A_n(w)$, for any fixed $\rho > 1$ we introduce:

$$w_j := \Phi(z_j), \quad \varphi_j := \arg w_j, \quad L^j := L \cap \overline{\Omega}^j, \quad L_\rho^j := L_\rho \cap \overline{\Omega}_\rho^j, \quad j = \overline{1, m}, \tag{4.8}$$

where $\Omega^j := \Psi(\Delta_j)$, $\Omega_\rho^j := \Psi(\Delta_j(\rho))$, $\Delta_j := \Delta_j(1)$, and

$$\begin{aligned} \Delta_1(\rho) &:= \left\{ t = re^{i\theta} : r > \rho, \quad \frac{\varphi_m + \varphi_1}{2} \leq \theta < \frac{\varphi_1 + \varphi_2}{2} \right\}, \\ \Delta_j(\rho) &:= \left\{ t = re^{i\theta} : r > \rho, \quad \frac{\varphi_{j-1} + \varphi_j}{2} \leq \theta < \frac{\varphi_j + \varphi_{j+1}}{2} \right\}, \quad j = \overline{2, m-1}, \\ \Delta_m(\rho) &:= \left\{ t = re^{i\theta} : r > \rho, \quad \frac{\varphi_{m-1} + \varphi_m}{2} \leq \theta < \frac{\varphi_m + \varphi_1}{2} \right\}. \end{aligned}$$

Then, we get

$$\begin{aligned} (A_n(w))^2 &= \int_{|t|=R_1} \frac{|dt|}{h(\Psi(t)) |\Psi(t) - \Psi(w)|^2} \tag{4.9} \\ \rho \sum_{j=1}^m \int_{\Phi(L_{R_1}^j)} \frac{|dt|}{\prod_{j=1}^m |\Psi(t) - \Psi(w_j)|^{\gamma_j} |\Psi(t) - \Psi(w)|^2} \\ \rho \sum_{j=1}^m \int_{\Phi(L_{R_1}^j)} \frac{|dt|}{|\Psi(t) - \Psi(w_j)|^{\gamma_j} |\Psi(t) - \Psi(w)|^2} &:= \sum_{j=1}^m A_{n,j}(w), \end{aligned}$$

since the points $\{z_j\}_{j=1}^m \in L$ are distinct. It remains to estimate the integrals $A_{n,j}(w)$ for each $j = \overline{1, m}$. Firstly, we assume that $z \in L_R$. For simplicity of our next calculations, we assume that $m = 1$. We put: $\Phi(L_{R_1}^1) = \bigcup_{i=1}^3 K_i(R_1)$, where

$$\begin{aligned}
 K_1(R_1) &:= \left\{ t \in \Phi(L_{R_1}^1) : |t - w_1| < \frac{c_1}{n} \right\} \\
 K_2(R_1) &:= \left\{ t \in \Phi(L_{R_1}^1) : \frac{c_1}{n} \leq |t - w_1| < c_2 \right\} \\
 K_3(R_1) &:= \left\{ t \in \Phi(L_{R_1}^1) : c_2 \leq |t - w_1| < c_3 < \text{diam } \overline{G} \right\}
 \end{aligned}$$

and $\Phi(L_R^1) = \bigcup_{i=1}^3 K_i(R)$, where

$$\begin{aligned}
 K_1(R) &:= \left\{ \tau \in \Phi(L_R^1) : |\tau - w_1| < \frac{2c_1}{n} \right\}, \\
 K_2(R) &:= \left\{ \tau \in \Phi(L_R^1) : \frac{2c_1}{n} \leq |\tau - w_1| < c_2 \right\}, \\
 K_3(R) &:= \left\{ \tau \in \Phi(L_R^1) : c_2 \leq |\tau - w_1| < c_3 < \text{diam } \overline{G} \right\}.
 \end{aligned}$$

Let $w \in \Phi(L_R^1)$ arbitrary fixed point. We will estimate the following integral for each cases with $w \in K_i(R)$ and $t \in K_i(R_1)$, $i = 1, 2, 3$.

$$\begin{aligned}
 A_{n,1}(w) &:= \int_{\Phi(L_{R_1}^1)} \frac{|dt|}{|\Psi(t) - \Psi(w_1)|^{\gamma_1} |\Psi(t) - \Psi(w)|^2} \\
 &= \sum_{i=1}^3 \int_{K_i(R_1)} \frac{|dt|}{|\Psi(t) - \Psi(w_1)|^{\gamma_1} |\Psi(t) - \Psi(w)|^2}, \\
 &=: \sum_{i=1}^3 A_{n,1}^i(w).
 \end{aligned} \tag{4.10}$$

Case 1. Let $w \in K_1(R)$. We put $K_j^1(R_1) := \{t \in \Phi(L_{R_1}^1) : |t - w_1| < |t - w|\}$,

$K_j^2(R_1) := K_j(R) \setminus K_j^1(R_1)$, $j = 1, 2$. Then, according to Lemma 3 and Lemma 4, we get

$$A_{n,1}^1(w) := \sum_{i=1}^2 \int_{K_1^i(R_1)} \frac{|dt|}{|\Psi(t) - \Psi(w_1)|^{\gamma_1} |\Psi(t) - \Psi(w)|^2}$$

$$\rho \int_{K_1^1(R_1)} \frac{|dt|}{|t - w_1|^{2+\gamma_1+\varepsilon}} + \int_{K_1^2(R_1)} \frac{|dt|}{|t - w|^{2+\gamma_1+\varepsilon}}$$

$$\rho n^{1+\gamma_1+\varepsilon}, \quad \gamma_1 > 0, \quad \forall \varepsilon > 0.$$

$$A_{n,1}^1(w) = \sum_{i=1}^2 \int_{K_1^i(R_1)} \frac{|\Psi(t) - \Psi(w_1)|^{-\gamma_1} |dt|}{|\Psi(t) - \Psi(w)|^2}$$

$$\rho \int_{K_1^1(R_1)} \frac{|t - w_1|^{-\gamma_1-\varepsilon} |dt|}{|t - w_1|^{2+\varepsilon}} + \int_{K_1^2(R_1)} \frac{|t - w_1|^{-\gamma_1-\varepsilon} |dt|}{|t - w|^{2+\varepsilon}}$$

$$\rho \int_{K_1^1(R_1)} \frac{|dt|}{|t - w_1|^{2+\gamma_1+\varepsilon}} + \left(\frac{1}{n}\right)^{-\gamma_1-\varepsilon} \int_{K_1^2(R_1)} \frac{|dt|}{|t - w|^{2+\varepsilon}}$$

$$\rho n^{1+\gamma_1+\varepsilon}, \quad \gamma_1 \leq 0, \quad \forall \varepsilon > 0.$$

$$A_{n,1}^2(w) := \sum_{i=1}^2 \int_{K_2^i(R_1)} \frac{|dt|}{|\Psi(t) - \Psi(w_1)|^{\gamma_1} |\Psi(t) - \Psi(w)|^2}$$

$$\rho \int_{K_2^1(R_1)} \frac{|dt|}{|t - w_1|^{2+\gamma_1+\varepsilon}} + \int_{K_2^2(R_1)} \frac{|dt|}{|t - w|^{2+\gamma_1+\varepsilon}}$$

$$\rho n^{1+\gamma_1+\varepsilon}, \quad \gamma_1 > 0, \quad \forall \varepsilon > 0.$$

$$A_{n,1}^2(w) = \int_{K_2^1(R_1)} \frac{|\Psi(t) - \Psi(w_1)|^{-\gamma_1} |dt|}{|\Psi(t) - \Psi(w)|^2} + \int_{K_2^2(R_1)} \frac{|\Psi(t) - \Psi(w_1)|^{-\gamma_1} |dt|}{|\Psi(t) - \Psi(w)|^2}$$

$$\rho \int_{K_2^1(R_1)} \frac{|\Psi(t) - \Psi(w)|^{-\gamma_1} |dt|}{|\Psi(t) - \Psi(w)|^2} + \int_{K_2^2(R_1)} \frac{|\Psi(t) - \Psi(w)|^{-\gamma_1} |dt|}{|\Psi(t) - \Psi(w)|^2}$$

$$\rho \int_{K_2^1(R_1)} \frac{|dt|}{|t - w|^{2+\gamma_1+\varepsilon}} + \int_{K_2^2(R_1)} \frac{|dt|}{|t - w|^{2+\gamma_1+\varepsilon}}$$

$$\rho n^{1+\gamma_1+\varepsilon}, \quad \gamma_1 \leq 0, \quad \forall \varepsilon > 0.$$

Since $|t - w_1| \geq c_2$ and $|t - w| \geq |t - w_1| - |w - w_1| \geq c_2 - \frac{2c_1}{n} \geq 1$ for $t \in K_3(R_1)$ and $w \in K_1(R)$, then we obtain:

$$A_{n,1}^3(w) \leq \int_{K_3(R_1)} \frac{|dt|}{|t-w_1|^{\gamma_1+\varepsilon} |t-w|^{2+\varepsilon}} \leq \int_{K_3(R_1)} |dt| \leq |K_3(R_1)| \leq 1, \quad \gamma_1 > 0;$$

$$A_{n,1}^3(w) \leq (\text{diam} \bar{G})^{-\gamma_1+\varepsilon} \int_{K_3(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w)|^2}$$

$$\leq \int_{K_3(R_1)} |dt| \leq 1, \quad \gamma_1 \leq 0, \quad \forall \varepsilon > 0.$$

Case 2. Let $w \in K_2(R)$.

$$A_{n,1}^1(w) \leq \int_{K_1(R_1)} \frac{|dt|}{|t-w_1|^{\gamma_1+\varepsilon} |t-w|^{2+\varepsilon}}$$

$$\leq \frac{1}{(R_1-1)^{\gamma_1+\varepsilon} \left(\frac{c_2}{n}\right)^{2+\varepsilon}} \int_{K_1(R_1)} |dt| \leq n^{1+\gamma_1+\varepsilon}, \quad \gamma_1 > 0;$$

$$A_{n,1}^1(w) \leq \int_{K_1(R_1)} \frac{|t-w_1|^{-\gamma_1-\varepsilon} |dt|}{|t-w|^{2+\varepsilon}}$$

$$\leq \left(\frac{c_1}{n}\right)^{-\gamma_1-\varepsilon} \left(\frac{n}{2c_1}\right)^{2+\varepsilon} \int_{K_1(R_1)} |dt| \leq n^{1+\gamma_1+\varepsilon}, \quad \gamma_1 \leq 0, \quad \forall \varepsilon > 0.$$

$$A_{n,1}^2(w) := \sum_{i=1}^2 \int_{K_2^i(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w_1)|^{\gamma_1} |\Psi(t)-\Psi(w)|^2}$$

$$\leq \int_{K_2^1(R_1)} \frac{|dt|}{|t-w_1|^{2+\gamma_1+\varepsilon}} + \int_{K_2^2(R_1)} \frac{|dt|}{|t-w|^{2+\gamma_1+\varepsilon}} \leq n^{1+\gamma_1+\varepsilon}, \quad \gamma_1 > 0;$$

$$A_{n,1}^2(w) = \int_{K_2^1(R_1)} \frac{|\Psi(t)-\Psi(w_1)|^{-\gamma_1} |dt|}{|\Psi(t)-\Psi(w)|^2} + \int_{K_2^2(R_1)} \frac{|\Psi(t)-\Psi(w_1)|^{-\gamma_1} |dt|}{|\Psi(t)-\Psi(w)|^2}$$

$$\leq \int_{K_2^1(R_1)} \frac{|dt|}{|t-w|^{2+\gamma_1+\varepsilon}} + \int_{K_2^2(R_1)} \frac{|dt|}{|t-w|^{2+\gamma_1+\varepsilon}}$$

$$\leq n^{1+\gamma_1+\varepsilon}, \quad \gamma_1 \leq 0, \quad \forall \varepsilon > 0.$$

Since $|t-w_1| \geq c_2$ and $|t-w| \geq 1$, for $t \in K_3(R_1)$ and $w \in K_2(R)$, then we obtain:

$$\begin{aligned}
 A_{n,1}^3(w) &:= \int_{K_3(R_1)} \frac{|dt|}{|\Psi(t) - \Psi(w_1)|^{\gamma_1} |\Psi(t) - \Psi(w)|^2} \\
 &\rho \int_{K_3(R_1)} \frac{|dt|}{|t - w_1|^{\gamma_1 + \varepsilon} |t - w|^{2 + \varepsilon}} \\
 &\rho \left(\frac{1}{c_2} \right)^{\gamma_1 + \varepsilon} \int_{K_3(R_1)} \frac{|dt|}{|t - w|^{2 + \varepsilon}} \rho n^{1 + \varepsilon}, \quad \gamma_1 > 0, \quad \forall \varepsilon > 0; \\
 A_{n,1}^3(w) &= \int_{K_3(R_1)} \frac{|\Psi(t) - \Psi(w_1)|^{-\gamma_1} |dt|}{|\Psi(t) - \Psi(w)|^2} \\
 &\rho (\text{diam} \bar{G})^{-\gamma_1 - \varepsilon} \int_{K_3(R_1)} \frac{|dt|}{|\Psi(t) - \Psi(w)|^2} \\
 &\rho \int_{K_3(R_1)} \frac{|dt|}{|t - w|^{2 + \varepsilon}} \rho n^{1 + \varepsilon}, \quad \gamma_1 \leq 0, \quad \forall \varepsilon > 0.
 \end{aligned}$$

Case 3. Let $w \in K_3(R)$.

$$\begin{aligned}
 A_{n,1}^1(w) &\rho \int_{K_1(R_1)} \frac{|dt|}{|t - w_1|^{\gamma_1 + \varepsilon} |t - w|^{2 + \varepsilon}} \rho n^{\gamma_1 + \varepsilon}, \quad \gamma_1 > 0; \\
 A_{n,1}^1(w) &\rho \int_{K_1(R_1)} \frac{|dt|}{|t - w|^{2 + \gamma_1 + \varepsilon}} \\
 &\rho \left(c_2 - \frac{2c_1}{n} \right)^{-2 - \gamma_1 - \varepsilon} \int_{K_1(R_1)} |dt| \rho 1, \quad \gamma_1 \leq 0, \quad \forall \varepsilon > 0. \\
 A_{n,1}^2(w) &\rho \int_{K_2(R_1)} \frac{|dt|}{|\Psi(t) - \Psi(w_1)|^{2 + \gamma_1}} \\
 &\rho \int_{K_2(R_1)} \frac{|dt|}{|t - w_1|^{2 + \gamma_1 + \varepsilon}} \rho n^{1 + \gamma_1 + \varepsilon}, \quad \gamma_1 > 0; \\
 A_{n,1}^2(w) &\rho \left(c_2 - \frac{2c_1}{n} \right)^{-2 - \varepsilon} (c_2)^{-\gamma_1 - \varepsilon} \int_{K_2(R_1)} |dt| \rho 1, \quad \gamma_1 \leq 0, \quad \forall \varepsilon > 0.
 \end{aligned}$$

$$\begin{aligned}
 A_{n,1}^3(w) & \rho \int_{K_3(R_1)} \frac{|dt|}{|t-w_1|^{\gamma_1+\varepsilon} |t-w|^{2+\varepsilon}} \\
 & \rho \left(\frac{1}{c_2}\right)^{\gamma_1+\varepsilon} \int_{K_3(R_1)} \frac{|dt|}{|t-w|^{2+\varepsilon}} \rho n^{1+\varepsilon}, \quad \gamma_1 > 0; \\
 A_{n,1}^3(w) & \rho (\text{diam}\overline{G})^{-\gamma_1-\varepsilon} \int_{K_3(R_1)} \frac{|dt|}{|t-w|^{2+\varepsilon}} \rho 1, \quad \gamma_1 \leq 0, \quad \forall \varepsilon > 0.
 \end{aligned}$$

By combining the estimates obtained in the Cases 1-3 with (4.7), (4.9) and (4.10), for any $p > 0$ and all $z \in L_R$, we obtain:

$$|P_n(z)| \rho \|\Gamma_n(\varepsilon)\| \|P_n\|_p, \quad \forall \varepsilon > 0, \tag{4.11}$$

where

$$\Gamma_n(\varepsilon) := \begin{cases} \sum_{j=1}^m n^{\frac{\gamma_j+1}{p}+\varepsilon}, & \text{if } \gamma_j \neq 0 \text{ for at least one } j=1, m \\ n^{\frac{1}{p}+\varepsilon} & \text{if } \gamma_j = 0, \text{ for all } j=1, m \end{cases}$$

The estimation (4.11) satisfied for all $z \in L_R$. We show that it is also carried out on L . For $R > 1$, let $w = \varphi_R(z)$ denotes the univalent conformal mapping of G_R onto B normalized by $\varphi_R(0) = 0, \varphi_R'(0) > 0$, and let $\{\xi_j\}, 1 \leq j \leq m \leq n$, zeros of $P_n(z)$, lying on G_R . Let

$$b_{m,R}(z) := \prod_{j=1}^m b_{j,R}(z) = \prod_{j=1}^m \frac{\varphi_R(z) - \varphi_R(\xi_j)}{1 - \overline{\varphi_R(\xi_j)}\varphi_R(z)} \tag{4.12}$$

denotes a Blaschke function with respect to zeros $\{\xi_j\}, 1 \leq j \leq m \leq n$, of $P_n(z)$. Clearly,

$$|b_{m,R}(z)| = 1, \quad z \in L_R; \quad |b_{m,R}(z)| < 1, \quad z \in G_R.$$

For any $z \in G_R$, the function

$$h_{n,p}(z) := \frac{P_n(z)}{b_{m,R}(z)} \tag{4.13}$$

is analytic in G_R , continuous on $\overline{G_R}$ and does not have zeros in G_R . Then, applying maximal modulus principle, we have:

$$\left| h_{n,p}(z) \right| = \left| \frac{P_n(z)}{b_{m,R}(z)} \right| \leq \max_{\zeta \in G_R} \left| \frac{P_n(\zeta)}{b_{m,R}(\zeta)} \right| \leq \max_{\zeta \in L_R} |P_n(\zeta)|$$

$$\rho \Gamma_n(\varepsilon) \|P_n\|_p, \forall \varepsilon > 0, \forall z \in L,$$

and, therefore, we find:

$$\max_{z \in L} |P_n(z)| \rho n^{\frac{\gamma+1}{p} + \varepsilon} \|P_n\|_p, \forall \varepsilon > 0, p > 0,$$

where $\gamma := \max\{0; \gamma_j, j = \overline{1, m}\}$, and the proof (2.5) is completed.

Now, we will begin to proof (2.4). Under the notations where we used in beginning of the proof of Theorem 1, we see that the function $h_{n,p}$ is analytic in G_R , continuous on $\overline{G_R}$ and does not have zeros in G_R . We take an arbitrary continuous branch of the $h_{n,p}(z)$ and for this branch we maintain the same designation. Using Cauchy integral representation for the region G_R , for $z = z_j$, we have:

$$h_{n,p}^{\frac{p}{2}}(z_j) = \frac{1}{2\pi i} \int_{L_R} h_{n,p}^{\frac{p}{2}}(\zeta) \frac{d\zeta}{\zeta - z_j}, \quad z \in G_R, \tag{4.14}$$

Since $|b_{m,R}(\zeta)| = 1$, for $\zeta \in L_R$, and $|b_{m,R}(z)| \leq 1$, for $z \in G_R$. Therefore, from (4.14), we get:

$$\left| P_n(z_j) \right|^{\frac{p}{2}} \rho \int_{L_R} |P_n(\zeta)|^{\frac{p}{2}} \frac{|d\zeta|}{|\zeta - z_j|}. \tag{4.15}$$

Therefore, multiplying the numerator and the denominator of the integrand by $h^{1/2}(\zeta)$, applying the Hölder inequality and Lemma 5, we obtain:

$$\left| P_n(z_j) \right| \rho \|P_n\|_p \times I_{n,1}, \tag{4.16}$$

where

$$I_{n,1} := \left(\int_{L_R} \frac{|d\zeta|}{\prod_{j=1}^m |\zeta - z_j|^{2+\gamma_j}} \right)^{1/p}.$$

Since the points $\{z_j\}_{j=1}^m \in L$ are distinct, by using designations from (4.8), we get:

$$(I_{n,1})^p = \sum_{i=1}^m \int_{L_R} \frac{|d\zeta|}{\prod_{j=1}^m |\zeta - z_j|^{2+\gamma_j}} \rho \sum_{i=1}^m \int_{L_R} \frac{|d\zeta|}{|\zeta - z_i|^{2+\gamma_i}} =: \sum_{i=1}^m I_{n,1}^i, \tag{4.17}$$

Therefore, it remains to estimate the integrals $I_{n,1}^i$ for each $i = \overline{1, m}$. In this case, we also assume that $m = 1$. By applying (3.3), according to [21], $\Psi \in Lip(1 - \varepsilon)$, for arbitrary small $\varepsilon > 0$, we obtain:

$$I_{n,1}^1 := \int_{L_R^1} \frac{|d\zeta|}{|\zeta - z_1|^{2+\gamma_1}} = \int_{\Phi(L_R^1)} \frac{d(\Psi(\tau), L)|d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{2+\gamma_1} (|\tau| - 1)} \tag{4.18}$$

$$\rho n^\varepsilon \int_{\Phi(L_R^1)} \frac{|d\tau|}{|\tau - w_1|^{\gamma_1+2+\varepsilon}} \rho n^{\gamma_1+1+\varepsilon}, \quad \forall \varepsilon > 0.$$

By combining the relations (4.16)-(4.18), we obtain:

$$|P_n(z_1)| \rho n^{\frac{\gamma_1+1+\varepsilon}{p}} \|P_n\|_p, \quad \forall \varepsilon > 0,$$

and, according to our assumption $m = 1$, we complete the proof (2.4)

4.2. Proof of Theorem 2

Let $z \in \Omega_{R_1}$ be an arbitrary fixed point. Then, $z \in \Omega_{R_1}^j$ for some $j = \overline{1, m}$. From (4.3) we have:

$$|P_n(z)|^{\frac{p}{2}} \rho \frac{|\Phi^{n+1}(z)|^{\frac{p}{2}}}{d(z, L_{R_1})} \int_{L_{R_1}} |P_n(\zeta)|^{\frac{p}{2}} |d\zeta|. \tag{4.19}$$

Analogously to the estimations (4.4)-(4.11), for each $j = \overline{1, m}$ we obtain:

$$\left(\int_{L_{R_1}} |P_n(\zeta)|^{\frac{p}{2}} |d\zeta| \right)^2 \rho \|P_n\|_p^p \int_{|t|=R_1} \frac{|\Psi'(t)| |dt|}{|\Psi(t) - \Psi(w_j)|^{\gamma_j}}$$

$$\rho \mu_n \cdot \|P_n\|_p^p.$$

Therefore, from (4.19) we get:

$$|P_n(z)| \rho \left(\frac{\sqrt{\mu_n}}{d(z, L_{R_1})} \right)^{\frac{2}{p}} |\Phi(z)|^{n+1} \|P_n\|_p, \quad z \in \Omega_{R_1}^j,$$

and we obtain the proof of (2.6).

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