MANAS Journal of Engineering

MJEN

Volume 6 (Issue 1) (2018) Pages 56-74

Gelecek elimizde...

Solutions of the Rational Difference Equation

$$x_{n+1} = \frac{x_{n-23}}{1 + x_{n-5}x_{n-11}x_{n-17}}$$

Burak Oğul[1], Dağıstan Şimşek*[1,2]

[1]Kyrgyz – Turkish Manas University, Faculty of Science, Department of Applied Mathematics and Informatics, Bishkek, Kyrgyzstan

[2]Selcuk University, Faculty of Engineering, Department of Industrial Engineering, Konya, Turkey

dagistan.simsek@manas.edu.kg; dsimsek@selcuk.edu.tr

Received: 02-11-2017; Accepted: 21-02-2018

Abstract: *In this paper the solutions of the following difference equation is examined,*

$$x_{n+1} = \frac{x_{n-23}}{1 + x_{n-5}x_{n-11}x_{n-17}}, \quad n = 0,1,2,...$$

where the initial conditions are positive real numbers.

Keywords: *Difference equation, period 24 solution*

$$x_{n+1} = \frac{x_{n-23}}{1 + x_{n-5}x_{n-11}x_{n-17}}$$

Rasyonel Fark Denkleminin Çözümleri

Özet: *Aşağıdaki Rasyonel fark denkleminin çözümlerini incelendi.*

$$x_{n+1} = \frac{x_{n-23}}{1 + x_{n-5}x_{n-11}x_{n-17}}, \quad n = 0,1,2,...$$

Burada başlangıç şartları reel sayılardır.

Anahtar Kelimeler: *Fark denklemleri, 24 periyotlu çözümler*

* Corresponding Author.

1. INTRODUCTION

Difference equations are always attracting very much interest, because these equations appear in the mathematical models of some problems in biology, ecology and physics, and numerical solutions of differential equations [29]. Recently there has been a lot of studies on the periodic nature of nonlinear difference equations. We refer readers to [1, 5, 18], for some recent results concerning among other problems and the periodicity of scalar nonlinear difference equations.

Cinar, studied the following problems with positive initial values:

$$x_{n+1} = \frac{x_{n-1}}{1 + a x_n x_{n-1}},$$

$$x_{n+1} = \frac{x_{n-1}}{-1 + a x_n x_{n-1}},$$

$$x_{n+1} = \frac{a x_{n-1}}{1 + b x_n x_{n-1}},$$

for $n = 0,1,2,...$ in $[2,3,4]$, respectively.

Simsek et. al., studied the following problems with positive initial values:

$$x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}},$$

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2}},$$

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-1} x_{n-3}},$$

$$x_{n+1} = \frac{x_{n-3}}{1 + x_n x_{n-1} x_{n-2}},$$

for $n = 0,1,2,...$ in $[20,21,22,25]$, respectively.

In this paper the folloving nonlinear difference equation

$$x_{n+1} = \frac{x_{n-23}}{1 + x_{n-5} x_{n-11} x_{n-17}}, \; n = 0,1,2,... \qquad (1)$$

where $x_{-23}, x_{-22}, ..., x_{-1}, x_0 \in (0,\infty)$ is investigated.

2. MAIN RESULT

Let $\overline{x}$ be the unique positive equilibrum of the (1), then clearly,

$$\overline{x} = \frac{\overline{x}}{1 + \overline{xxx}} \Rightarrow \overline{x} + \overline{x}^4 = \overline{x} \Rightarrow \overline{x}^4 = 0 \Rightarrow \overline{x} = 0$$

$\overline{x}=0$ can be obtained.

Theorem 1. Consider the difference equation (1). Then the following statements are true.

a) The sequences

$$(x_{24n-23}),\ (x_{24n-22}),(x_{24n-21}),(x_{24n-20}),(x_{24n-19}),(x_{24n-18}),(x_{24n-17}),(x_{24n-16}),(x_{24n-15}),$$
$$(x_{24n-14}),(x_{24n-13}),(x_{24n-12}),(x_{24n-11}),(x_{24n-10}),(x_{24n-9}),(x_{24n-8}),(x_{24n-7}),(x_{24n-6}),$$
$$(x_{24n-5}),(x_{24n-4}),(x_{24n-3}),(x_{24n-2}),(x_{24n-1}),(x_{24n})$$

are decreased and $a_1,a_2,...,a_{24}\geq 0$ is existed in such that:

$$\lim_{n\to\infty} x_{24n-23}=a_1,\ \lim_{n\to\infty} x_{24n-22}=a_2,\ \lim_{n\to\infty} x_{24n-21}=a_3,\ \lim_{n\to\infty} x_{24n-20}=a_4,\ \lim_{n\to\infty} x_{24n-19}=a_5,$$

$$\lim_{n\to\infty} x_{24n-18}=a_6,\lim_{n\to\infty} x_{24n-17}=a_7,\ \lim_{n\to\infty} x_{24n-16}=a_8,\ \lim_{n\to\infty} x_{24n-15}=a_9,\ \lim_{n\to\infty} x_{24n-14}=a_{10},$$

$$\lim_{n\to\infty} x_{24n-13}=a_{11},\ \lim_{n\to\infty} x_{24n-12}=a_{12},\lim_{n\to\infty} x_{24n-11}=a_{13},\ \lim_{n\to\infty} x_{24n-10}=a_{14},\ \lim_{n\to\infty} x_{24n-9}=a_{15},$$

$$\lim_{n\to\infty} x_{24n-8}=a_{16},\ \lim_{n\to\infty} x_{24n-7}=a_{17},\ \lim_{n\to\infty} x_{24n-6}=a_{18},\lim_{n\to\infty} x_{24n-5}=a_{19},\ \lim_{n\to\infty} x_{24n-4}=a_{20},$$

$$\lim_{n\to\infty} x_{24n-3}=a_{21},\ \lim_{n\to\infty} x_{24n-2}=a_{22},\ \lim_{n\to\infty} x_{24n-1}=a_{23},\ \lim_{n\to\infty} x_{24n}=a_{24}.$$

b) $(a_1,a_2,...,a_{24},a_1,a_2,...,a_{24},...)$ is a solution of (1) of period 24.

c)

$$\lim_{n\to\infty} x_{24n-23}.\lim_{n\to\infty} x_{24n-17}.\lim_{n\to\infty} x_{24n-11}.\lim_{n\to\infty} x_{24n-5}=0,$$

$$\lim_{n\to\infty} x_{24n-22}.\lim_{n\to\infty} x_{24n-16}.\lim_{n\to\infty} x_{24n-10}.\lim_{n\to\infty} x_{24n-4}=0,$$

$$\lim_{n\to\infty} x_{24n-21}.\lim_{n\to\infty} x_{24n-15}.\lim_{n\to\infty} x_{24n-9}.\lim_{n\to\infty} x_{24n-3}=0,$$

$$\lim_{n\to\infty} x_{24n-20}.\lim_{n\to\infty} x_{24n-14}.\lim_{n\to\infty} x_{24n-8}.\lim_{n\to\infty} x_{24n-2}=0,$$

$$\lim_{n\to\infty} x_{24n-19}.\lim_{n\to\infty} x_{24n-13}.\lim_{n\to\infty} x_{24n-7}.\lim_{n\to\infty} x_{24n-1}=0,$$

$$\lim_{n\to\infty} x_{24n-18}.\lim_{n\to\infty} x_{24n-12}.\lim_{n\to\infty} x_{24n-6}.\lim_{n\to\infty} x_{24n}=0$$

or

$$a_1.a_7.a_{13}.a_{19}=0,\ a_2.a_8.a_{14}.a_{20}=0,\ a_3.a_9.a_{15}.a_{21}=0,$$

$$a_4.a_{10}.a_{16}.a_{22}=0,\ a_5.a_{11}.a_{17}.a_{23}=0,\ a_6.a_{12}.a_{18}.a_{24}=0.$$

d) If there exist $n_0\in¥$ such that $x_{n+1}\leq x_{n-17}$ for all $n\geq n_0$, then

$$\lim_{n\to\infty} x_n=0.$$

e) The following formulas can be generated:

$$x_{24n+1}=x_{-23}\left(1-\frac{x_{-5}.x_{-11}.x_{-17}}{1+x_{-5}.x_{-11}.x_{-17}}\sum_{j=0}^{n}\prod_{i=1}^{6j}\frac{1}{1+x_{6i-5}.x_{6i-11}.x_{6i-17}}\right),$$

$$x_{24n+2}=x_{-22}\left(1-\frac{x_{-4}.x_{-10}.x_{-16}}{1+x_{-4}.x_{-10}.x_{-16}}\sum_{j=0}^{n}\prod_{i=1}^{6j}\frac{1}{1+x_{6i-4}.x_{6i-10}.x_{6i-16}}\right),$$

$$x_{24n+3}=x_{-21}\left(1-\frac{x_{-3}.x_{-9}.x_{-15}}{1+x_{-3}.x_{-9}.x_{-15}}\sum_{j=0}^{n}\prod_{i=1}^{6j}\frac{1}{1+x_{6i-3}.x_{6i-9}.x_{6i-15}}\right),$$

$$x_{24n+4}=x_{-20}\left(1-\frac{x_{-2}.x_{-8}.x_{-14}}{1+x_{-2}.x_{-8}.x_{-14}}\sum_{j=0}^{n}\prod_{i=1}^{6j}\frac{1}{1+x_{6i-2}.x_{6i-8}.x_{6i-14}}\right),$$

$$x_{24n+5}=x_{-19}\left(1-\frac{x_{-1}.x_{-7}.x_{-13}}{1+x_{-1}.x_{-7}.x_{-13}}\sum_{j=0}^{n}\prod_{i=1}^{6j}\frac{1}{1+x_{6i-1}.x_{6i-7}.x_{6i-13}}\right),$$

$$x_{24n+6}=x_{-18}\left(1-\frac{x_{0}.x_{-6}.x_{-12}}{1+x_{0}.x_{-6}.x_{-12}}\sum_{j=0}^{n}\prod_{i=1}^{6j}\frac{1}{1+x_{6i}.x_{6i-6}.x_{6i-12}}\right),$$

$$x_{24n+7}=x_{-17}\left(1-\frac{x_{-5}.x_{-11}.x_{-23}}{1+x_{-5}.x_{-11}.x_{-17}}\sum_{j=0}^{n}\prod_{i=1}^{6j+1}\frac{1}{1+x_{6i-5}.x_{6i-11}.x_{6i-17}}\right),$$

$$x_{24n+8}=x_{-16}\left(1-\frac{x_{-4}.x_{-10}.x_{-22}}{1+x_{-4}.x_{-10}.x_{-16}}\sum_{j=0}^{n}\prod_{i=1}^{6j+1}\frac{1}{1+x_{6i-4}.x_{6i-10}.x_{6i-16}}\right),$$

$$x_{24n+9}=x_{-15}\left(1-\frac{x_{-3}.x_{-9}.x_{-21}}{1+x_{-3}.x_{-9}.x_{-15}}\sum_{j=0}^{n}\prod_{i=1}^{6j+1}\frac{1}{1+x_{6i-3}.x_{6i-9}.x_{6i-15}}\right),$$

$$x_{24n+10}=x_{-14}\left(1-\frac{x_{-2}.x_{-8}.x_{-20}}{1+x_{-2}.x_{-8}.x_{-14}}\sum_{j=0}^{n}\prod_{i=1}^{6j+1}\frac{1}{1+x_{6i-2}.x_{6i-8}.x_{6i-14}}\right),$$

$$x_{24n+11}=x_{-13}\left(1-\frac{x_{-1}.x_{-7}.x_{-19}}{1+x_{-1}.x_{-7}.x_{-13}}\sum_{j=0}^{n}\prod_{i=1}^{6j+1}\frac{1}{1+x_{6i-1}.x_{6i-7}.x_{6i-13}}\right),$$

$$x_{24n+12}=x_{-12}\left(1-\frac{x_{0}.x_{-6}.x_{-18}}{1+x_{0}.x_{-6}.x_{-12}}\sum_{j=0}^{n}\prod_{i=1}^{6j+1}\frac{1}{1+x_{6i}.x_{6i-6}.x_{6i-12}}\right),$$

$$x_{24n+13}=x_{-11}\left(1-\frac{x_{-5}.x_{-17}.x_{-23}}{1+x_{-5}.x_{-11}.x_{-17}}\sum_{j=0}^{n}\prod_{i=1}^{6j+2}\frac{1}{1+x_{6i-5}.x_{6i-11}.x_{6i-17}}\right),$$

$$x_{24n+14}=x_{-10}\left(1-\frac{x_{-4}.x_{-16}.x_{-22}}{1+x_{-4}.x_{-10}.x_{-16}}\sum_{j=0}^{n}\prod_{i=1}^{6j+2}\frac{1}{1+x_{6i-4}.x_{6i-8}.x_{6i-14}}\right),$$

$$x_{24n+15}=x_{-9}\left(1-\frac{x_{-3}.x_{-15}.x_{-21}}{1+x_{-3}.x_{-9}.x_{-15}}\sum_{j=0}^{n}\prod_{i=1}^{6j+2}\frac{1}{1+x_{6i-3}.x_{6i-9}.x_{6i-15}}\right),$$

$$x_{24n+16}=x_{-8}\left(1-\frac{x_{-2}.x_{-14}.x_{-20}}{1+x_{-2}.x_{-8}.x_{-14}}\sum_{j=0}^{n}\prod_{i=1}^{6j+2}\frac{1}{1+x_{6i-2}.x_{6i-8}.x_{6i-14}}\right),$$

$$x_{24n+17}=x_{-7}\left(1-\frac{x_{-1}.x_{-13}.x_{-19}}{1+x_{-1}.x_{-7}.x_{-13}}\sum_{j=0}^{n}\prod_{i=1}^{6j+2}\frac{1}{1+x_{6i-1}.x_{6i-7}.x_{6i-13}}\right),$$

$$x_{24n+18}=x_{-6}\left(1-\frac{x_{0}.x_{-12}.x_{-18}}{1+x_{0}.x_{-6}.x_{-12}}\sum_{j=0}^{n}\prod_{i=1}^{6j+2}\frac{1}{1+x_{6i}.x_{6i-6}.x_{6i-12}}\right),$$

$$x_{24n+19}=x_{-5}\left(1-\frac{x_{-11}.x_{-17}.x_{-23}}{1+x_{-5}.x_{-11}.x_{-17}}\sum_{j=0}^{n}\prod_{i=1}^{6j+3}\frac{1}{1+x_{6i-5}.x_{6i-11}.x_{6i-17}}\right),$$

$$x_{24n+20}=x_{-4}\left(1-\frac{x_{-10}.x_{-16}.x_{-22}}{1+x_{-4}.x_{-10}.x_{-16}}\sum_{j=0}^{n}\prod_{i=1}^{6j+3}\frac{1}{1+x_{6i-4}.x_{6i-10}.x_{6i-16}}\right),$$

$$x_{24n+21}=x_{-3}\left(1-\frac{x_{-9}.x_{-15}.x_{-21}}{1+x_{-3}.x_{-9}.x_{-15}}\sum_{j=0}^{n}\prod_{i=1}^{6j+3}\frac{1}{1+x_{6i-3}.x_{6i-9}.x_{6i-15}}\right),$$

$$x_{24n+22}=x_{-2}\left(1-\frac{x_{-8}.x_{-14}.x_{-20}}{1+x_{-2}.x_{-8}.x_{-14}}\sum_{j=0}^{n}\prod_{i=1}^{6j+3}\frac{1}{1+x_{6i-2}.x_{6i-8}.x_{6i-14}}\right),$$

$$x_{24n+23}=x_{-1}\left(1-\frac{x_{-7}.x_{-13}.x_{-19}}{1+x_{-1}.x_{-7}.x_{-13}}\sum_{j=0}^{n}\prod_{i=1}^{6j+3}\frac{1}{1+x_{6i-1}.x_{6i-7}.x_{6i-13}}\right),$$

$$x_{24n+24}=x_{0}\left(1-\frac{x_{-6}.x_{-12}.x_{-18}}{1+x_{0}.x_{-6}.x_{-12}}\sum_{j=0}^{n}\prod_{i=1}^{6j+3}\frac{1}{1+x_{6i}.x_{6i-6}.x_{6i-12}}\right).$$

f) If $x_{24n+1}\to a_1\neq 0$, $x_{24n+7}\to a_7\neq 0$, $x_{24n+13}\to a_{13}\neq 0$, then $x_{24n+19}\to 0$ as $n\to\infty$.

If $x_{24n+2}\to a_2\neq 0$, $x_{24n+8}\to a_8\neq 0$, $x_{24n+14}\to a_{14}\neq 0$, then $x_{24n+20}\to 0$ as $n\to\infty$.

If $x_{24n+3}\to a_3\neq 0$, $x_{24n+9}\to a_9\neq 0$, $x_{24n+15}\to a_{15}\neq 0$, then $x_{24n+21}\to 0$ as $n\to\infty$.

If $x_{24n+4} \to a_4 \neq 0, x_{24n+10} \to a_{10} \neq 0, x_{24n+16} \to a_{16} \neq 0$, then $x_{24n+22} \to 0$ as $n \to \infty$.

If $x_{24n+5} \to a_5 \neq 0, x_{24n+11} \to a_{11} \neq 0, x_{24n+17} \to a_{17} \neq 0$, then $x_{24n+23} \to 0$ as $n \to \infty$.

If $x_{24n+6} \to a_6 \neq 0, x_{24n+12} \to a_{12} \neq 0, x_{24n+18} \to a_{18} \neq 0$, then $x_{24n+24} \to 0$ as $n \to \infty$.

Proof. a) Firstly, we consider the (1). From this equation we obtain:

$$x_{n+1}.(1+x_{n-5}.x_{n-11}.x_{n-17}) = x_{n-23}.$$

If $x_{n-5}.x_{n-11}.x_{n-17} \in (0,+\infty)$, then $(1+x_{n-5}.x_{n-11}.x_{n-17}) \in (1,+\infty)$. Since $x_{n+1} < x_{n-23}$, $n \in ¥$, we obtain that results:

$$\lim_{n\to\infty} x_{24n-23} = a_1,\ \lim_{n\to\infty} x_{24n-22} = a_2,\ \lim_{n\to\infty} x_{24n-21} = a_3,\ \lim_{n\to\infty} x_{24n-20} = a_4,\ \lim_{n\to\infty} x_{24n-19} = a_5,$$

$$\lim_{n\to\infty} x_{24n-18} = a_6, \lim_{n\to\infty} x_{24n-17} = a_7,\ \lim_{n\to\infty} x_{24n-16} = a_8,\ \lim_{n\to\infty} x_{24n-15} = a_9,\ \lim_{n\to\infty} x_{24n-14} = a_{10},$$

$$\lim_{n\to\infty} x_{24n-13} = a_{11}, \lim_{n\to\infty} x_{24n-12} = a_{12},\ \lim_{n\to\infty} x_{24n-11} = a_{13},\ \lim_{n\to\infty} x_{24n-10} = a_{14},\ \lim_{n\to\infty} x_{24n-9} = a_{15},$$

$$\lim_{n\to\infty} x_{24n-8} = a_{16},\ \lim_{n\to\infty} x_{24n-7} = a_{17}, \lim_{n\to\infty} x_{24n-6} = a_{18}, \lim_{n\to\infty} x_{24n-5} = a_{19},\ \lim_{n\to\infty} x_{24n-4} = a_{20},$$

$$\lim_{n\to\infty} x_{24n-3} = a_{21},\ \lim_{n\to\infty} x_{24n-2} = a_{22},\ \lim_{n\to\infty} x_{24n-1} = a_{23}, \lim_{n\to\infty} x_{24n} = a_{24}.$$

b) $(a_1, a_2, ..., a_{24}, a_1, a_2, ..., a_{24}, ...)$ is a solution of the (1) of period 24.

c) In view of the (1), we obtain:

$$x_{24n+1} = \frac{x_{24n-23}}{1+x_{24n-5}.x_{24n-11}.x_{24n-17}}.$$

Taking limit as $n \to \infty$ on both sides of the above equality, we get:

$$\lim_{n\to\infty} x_{24n+1} = \lim_{n\to\infty} \frac{x_{24n-23}}{1+x_{24n-5}.x_{24n-11}.x_{24n-17}}.$$

Then

$$a_1 = \frac{a_1}{1+a_7.a_{13}.a_{19}} \Rightarrow a_1 + a_1.a_7.a_{13}.a_{19} = a_1 \Rightarrow a_1.a_7.a_{13}.a_{19} = 0.$$

Similarly,

$$x_{24n+2} = \frac{x_{24n-22}}{1+x_{24n-4}.x_{24n-10}.x_{24n-16}}.$$

Taking limit as $n \to \infty$ on both sides of the above equality, we get:

$$\lim_{n\to\infty} x_{24n+2} = \lim_{n\to\infty} \frac{x_{24n-22}}{1+x_{24n-4}.x_{24n-10}.x_{24n-16}}.$$

Then

$$a_2 = \frac{a_2}{1 + a_8 . a_{14} . a_{20}} \Rightarrow a_2 + a_2 . a_8 . a_{14} . a_{20} = a_2 \Rightarrow a_2 . a_8 . a_{14} . a_{20} = 0.$$

Similarly,

$$x_{24n+3} = \frac{x_{24n-21}}{1 + x_{24n-3} . x_{24n-9} . x_{24n-15}}.$$

Taking limit as $n \to \infty$ on both sides of the above equality, we get:

$$\lim_{n\to\infty} x_{24n+3} = \lim_{n\to\infty} \frac{x_{24n-21}}{1 + x_{24n-3} . x_{24n-9} . x_{24n-15}}.$$

Then

$$a_3 = \frac{a_3}{1 + a_9 . a_{15} . a_{21}} \Rightarrow a_3 + a_3 . a_9 . a_{15} . a_{21} = a_3 \Rightarrow a_3 . a_9 . a_{15} . a_{21} = 0.$$

Similarly,

$$x_{24n+4} = \frac{x_{24n-20}}{1 + x_{24n-2} . x_{24n-8} . x_{24n-14}}.$$

Taking limit as $n \to \infty$ on both sides of the above equality, we get:

$$\lim_{n\to\infty} x_{24n+4} = \lim_{n\to\infty} \frac{x_{24n-20}}{1 + x_{24n-2} . x_{24n-8} . x_{24n-14}}.$$

Then

$$a_4 = \frac{a_4}{1 + a_{10} . a_{16} . a_{22}} \Rightarrow a_4 + a_4 . a_{10} . a_{16} . a_{22} = a_4 \Rightarrow a_4 . a_{10} . a_{16} . a_{22} = 0.$$

Similarly,

$$x_{24n+5} = \frac{x_{24n-19}}{1 + x_{24n-1} . x_{24n-7} . x_{24n-13}}.$$

Taking limit as $n \to \infty$ on both sides of the above equality, we get:

$$\lim_{n\to\infty} x_{24n+5} = \lim_{n\to\infty} \frac{x_{24n-19}}{1 + x_{24n-1} . x_{24n-7} . x_{24n-13}}.$$

Then

$$a_5 = \frac{a_5}{1 + a_{11} . a_{17} . a_{23}} \Rightarrow a_5 + a_5 . a_{11} . a_{17} . a_{23} = a_5 \Rightarrow a_5 . a_{11} . a_{17} . a_{23} = 0.$$

Similarly,

$$x_{24n+5} = \frac{x_{24n-19}}{1 + x_{24n-1} . x_{24n-7} . x_{24n-13}}.$$

Taking limit as $n\to\infty$ on both sides of the above equality, we get:

$$\lim_{n\to\infty} x_{24n+5} = \lim_{n\to\infty} \frac{x_{24n-19}}{1+x_{24n-1}.x_{24n-7}.x_{24n-13}}.$$

Then

$$a_5 = \frac{a_5}{1+a_{11}.a_{17}.a_{23}} \Rightarrow a_5 + a_5.a_{11}.a_{17}.a_{23} = a_5 \Rightarrow a_5.a_{11}.a_{17}.a_{23} = 0.$$

d) If there exist $n_0 \in ¥$ such that $x_{n+1} \le x_{n-17}$ for all $n \ge n_0$, then

$a_1 \le a_7 \le a_{13} \le a_{19} \le a_1$ since $a_1.a_7.a_{13}.a_{19} = 0$, $a_2 \le a_8 \le a_{14} \le a_{20} \le a_2$ since $a_2.a_8.a_{14}.a_{20} = 0$,

$a_3 \le a_9 \le a_{15} \le a_{21} \le a_3$ since $a_3.a_9.a_{15}.a_{21} = 0$, $a_4 \le a_{10} \le a_{16} \le a_{22} \le a_4$ since $a_4.a_{10}.a_{16}.a_{22} = 0$,

$a_5 \le a_{11} \le a_{17} \le a_{23} \le a_5$ since $a_5.a_{11}.a_{17}.a_{23} = 0$, $a_6 \le a_{12} \le a_{18} \le a_{24} \le a_6$ since $a_6.a_{12}.a_{18}.a_{24} = 0$

we can obtain the results.

e) Subracting x_{n-23} from the left and right-hand sides of the (1) we obtain:

$$x_{n+1} - x_{n-23} = \frac{1}{1+x_{n-5}.x_{n-11}.x_{n-17}}(x_{n-5} - x_{n-29})$$

and the following formula:

$$\text{for } n \ge 6 \begin{cases} x_{6n-35} - x_{6n-59} = (x_1 - x_{-23}) = \prod_{i=1}^{n-6} \frac{1}{1+x_{6i-5}.x_{6i-11}.x_{6i-17}}, \\ x_{6n-34} - x_{6n-58} = (x_2 - x_{-22}) = \prod_{i=1}^{n-6} \frac{1}{1+x_{6i-4}.x_{6i-10}.x_{6i-16}}, \\ x_{6n-33} - x_{6n-57} = (x_3 - x_{-21}) = \prod_{i=1}^{n-6} \frac{1}{1+x_{6i-3}.x_{6i-9}.x_{6i-15}}, \\ x_{6n-32} - x_{6n-56} = (x_4 - x_{-20}) = \prod_{i=1}^{n-6} \frac{1}{1+x_{6i-2}.x_{6i-8}.x_{6i-14}}, \\ x_{6n-31} - x_{6n-55} = (x_5 - x_{-19}) = \prod_{i=1}^{n-6} \frac{1}{1+x_{6i-1}.x_{6i-7}.x_{6i-13}}, \\ x_{6n-30} - x_{6n-54} = (x_6 - x_{-18}) = \prod_{i=1}^{n-6} \frac{1}{1+x_{6i}.x_{6i-6}.x_{6i-12}}, \end{cases} \quad (2)$$

holds. Replacing n by $6j$ in (2) and summing from $j=0$ to $j=n$, we obtain:

$$x_{24n+1}-x_{-23}=(x_1-x_{-23})\sum_{j=0}^{n}\prod_{i=1}^{6j}\frac{1}{1+x_{6i-5}.x_{6i-11}.x_{6i-17}}\quad (n=0,1,2,...),$$

$$x_{24n+2}-x_{-22}=(x_2-x_{-22})\sum_{j=0}^{n}\prod_{i=1}^{6j}\frac{1}{1+x_{6i-4}.x_{6i-10}.x_{6i-16}}\quad (n=0,1,2,...),$$

$$x_{24n+3}-x_{-21}=(x_3-x_{-21})\sum_{j=0}^{n}\prod_{i=1}^{6j}\frac{1}{1+x_{6i-3}.x_{6i-9}.x_{6i-15}}\quad (n=0,1,2,...),\tag{3}$$

$$x_{24n+4}-x_{-20}=(x_4-x_{-20})\sum_{j=0}^{n}\prod_{i=1}^{6j}\frac{1}{1+x_{6i-2}.x_{6i-8}.x_{6i-14}}\quad (n=0,1,2,...),$$

$$x_{24n+5}-x_{-19}=(x_5-x_{-19})\sum_{j=0}^{n}\prod_{i=1}^{6j}\frac{1}{1+x_{6i-1}.x_{6i-7}.x_{6i-13}}\quad (n=0,1,2,...),$$

$$x_{24n+6}-x_{-18}=(x_6-x_{-18})\sum_{j=0}^{n}\prod_{i=1}^{6j}\frac{1}{1+x_{6i}.x_{6i-6}.x_{6i-12}}\quad (n=0,1,2,...).$$

Also, replacing n by $6j+1$ in (2) and summing from $j=0$ to $j=n$, we obtain:

$$x_{24n+7}-x_{-17}=(x_7-x_{-17})\sum_{j=0}^{n}\prod_{i=1}^{6j+1}\frac{1}{1+x_{6i-5}.x_{6i-11}.x_{6i-17}}\quad (n=0,1,2,...),$$

$$x_{24n+8}-x_{-16}=(x_8-x_{-16})\sum_{j=0}^{n}\prod_{i=1}^{6j+1}\frac{1}{1+x_{6i-4}.x_{6i-10}.x_{6i-16}}\quad (n=0,1,2,...),$$

$$x_{24n+9}-x_{-15}=(x_9-x_{-15})\sum_{j=0}^{n}\prod_{i=1}^{6j+1}\frac{1}{1+x_{6i-3}.x_{6i-9}.x_{6i-15}}\quad (n=0,1,2,...),$$

$$x_{24n+10}-x_{-14}=(x_{10}-x_{-14})\sum_{j=0}^{n}\prod_{i=1}^{6j+1}\frac{1}{1+x_{6i-2}.x_{6i-8}.x_{6i-14}}\quad (n=0,1,2,...),\tag{4}$$

$$x_{24n+11}-x_{-13}=(x_{11}-x_{-13})\sum_{j=0}^{n}\prod_{i=1}^{6j+1}\frac{1}{1+x_{6i-1}.x_{6i-7}.x_{6i-13}}\quad (n=0,1,2,...),$$

$$x_{24n+12}-x_{-12}=(x_{12}-x_{-12})\sum_{j=0}^{n}\prod_{i=1}^{6j+1}\frac{1}{1+x_{6i}.x_{6i-6}.x_{6i-12}}\quad (n=0,1,2,...).$$

Also, replacing n by $6j+2$ in (2) and summing from $j=0$ to $j=n$, we obtain:

$$x_{24n+13} - x_{-11} = (x_{13} - x_{-11}) \sum_{j=0}^{n} \prod_{i=1}^{6j+2} \frac{1}{1 + x_{6i-5}.x_{6i-11}.x_{6i-17}} \quad (n = 0,1,2,...),$$

$$x_{24n+14} - x_{-10} = (x_{14} - x_{-10}) \sum_{j=0}^{n} \prod_{i=1}^{6j+2} \frac{1}{1 + x_{6i-4}.x_{6i-10}.x_{6i-16}} \quad (n = 0,1,2,...),$$

$$x_{24n+15} - x_{-9} = (x_{15} - x_{-9}) \sum_{j=0}^{n} \prod_{i=1}^{6j+2} \frac{1}{1 + x_{6i-3}.x_{6i-9}.x_{6i-15}} \quad (n = 0,1,2,...),$$

$$x_{24n+16} - x_{-8} = (x_{16} - x_{-8}) \sum_{j=0}^{n} \prod_{i=1}^{6j+2} \frac{1}{1 + x_{6i-2}.x_{6i-8}.x_{6i-14}} \quad (n = 0,1,2,...), \tag{5}$$

$$x_{24n+17} - x_{-7} = (x_{17} - x_{-7}) \sum_{j=0}^{n} \prod_{i=1}^{6j+2} \frac{1}{1 + x_{6i-1}.x_{6i-7}.x_{6i-13}} \quad (n = 0,1,2,...),$$

$$x_{24n+18} - x_{-6} = (x_{18} - x_{-6}) \sum_{j=0}^{n} \prod_{i=1}^{6j+2} \frac{1}{1 + x_{6i}.x_{6i-6}.x_{6i-12}} \quad (n = 0,1,2,...).$$

Also, replacing n by $6j+3$ in (2) and summing from $j=0$ to $j=n$, we obtain:

$$x_{24n+19} - x_{-5} = (x_{19} - x_{-5}) \sum_{j=0}^{n} \prod_{i=1}^{6j+3} \frac{1}{1 + x_{6i-5}.x_{6i-11}.x_{6i-17}} \quad (n = 0,1,2,...),$$

$$x_{24n+20} - x_{-4} = (x_{20} - x_{-4}) \sum_{j=0}^{n} \prod_{i=1}^{6j+3} \frac{1}{1 + x_{6i-4}.x_{6i-10}.x_{6i-16}} \quad (n = 0,1,2,...),$$

$$x_{24n+21} - x_{-3} = (x_{21} - x_{-3}) \sum_{j=0}^{n} \prod_{i=1}^{6j+3} \frac{1}{1 + x_{6i-3}.x_{6i-9}.x_{6i-15}} \quad (n = 0,1,2,...),$$

$$x_{24n+22} - x_{-2} = (x_{22} - x_{-2}) \sum_{j=0}^{n} \prod_{i=1}^{6j+3} \frac{1}{1 + x_{6i-2}.x_{6i-8}.x_{6i-14}} \quad (n = 0,1,2,...), \tag{6}$$

$$x_{24n+23} - x_{-1} = (x_{23} - x_{-1}) \sum_{j=0}^{n} \prod_{i=1}^{6j+3} \frac{1}{1 + x_{6i-1}.x_{6i-7}.x_{6i-13}} \quad (n = 0,1,2,...),$$

$$x_{24n+24} - x_{0} = (x_{24} - x_{0}) \sum_{j=0}^{n} \prod_{i=1}^{6j+3} \frac{1}{1 + x_{6i}.x_{6i-6}.x_{6i-12}} \quad (n = 0,1,2,...).$$

Now, we obtained of the above formulas:

$$x_{24n+1}=x_{-23}\left(1-\frac{x_{-5}.x_{-11}.x_{-17}}{1+x_{-5}.x_{-11}.x_{-17}}\sum_{j=0}^{n}\prod_{i=1}^{6j}\frac{1}{1+x_{6i-5}.x_{6i-11}.x_{6i-17}}\right),$$

$$x_{24n+2}=x_{-22}\left(1-\frac{x_{-4}.x_{-10}.x_{-16}}{1+x_{-4}.x_{-10}.x_{-16}}\sum_{j=0}^{n}\prod_{i=1}^{6j}\frac{1}{1+x_{6i-4}.x_{6i-10}.x_{6i-16}}\right),$$

$$x_{24n+3}=x_{-21}\left(1-\frac{x_{-3}.x_{-9}.x_{-15}}{1+x_{-3}.x_{-9}.x_{-15}}\sum_{j=0}^{n}\prod_{i=1}^{6j}\frac{1}{1+x_{6i-3}.x_{6i-9}.x_{6i-15}}\right),$$

$$x_{24n+4}=x_{-20}\left(1-\frac{x_{-2}.x_{-8}.x_{-14}}{1+x_{-2}.x_{-8}.x_{-14}}\sum_{j=0}^{n}\prod_{i=1}^{6j}\frac{1}{1+x_{6i-2}.x_{6i-8}.x_{6i-14}}\right), \quad (7)$$

$$x_{24n+5}=x_{-19}\left(1-\frac{x_{-1}.x_{-7}.x_{-13}}{1+x_{-1}.x_{-7}.x_{-13}}\sum_{j=0}^{n}\prod_{i=1}^{6j}\frac{1}{1+x_{6i-1}.x_{6i-7}.x_{6i-13}}\right),$$

$$x_{24n+6}=x_{-18}\left(1-\frac{x_{0}.x_{-6}.x_{-12}}{1+x_{0}.x_{-6}.x_{-12}}\sum_{j=0}^{n}\prod_{i=1}^{6j}\frac{1}{1+x_{6i}.x_{6i-6}.x_{6i-12}}\right),$$

$$x_{24n+7}=x_{-17}\left(1-\frac{x_{-5}.x_{-11}.x_{-23}}{1+x_{-5}.x_{-11}.x_{-17}}\sum_{j=0}^{n}\prod_{i=1}^{6j+1}\frac{1}{1+x_{6i-5}.x_{6i-11}.x_{6i-17}}\right),$$

$$x_{24n+8}=x_{-16}\left(1-\frac{x_{-4}.x_{-10}.x_{-22}}{1+x_{-4}.x_{-10}.x_{-16}}\sum_{j=0}^{n}\prod_{i=1}^{6j+1}\frac{1}{1+x_{6i-4}.x_{6i-10}.x_{6i-16}}\right),$$

$$x_{24n+9}=x_{-15}\left(1-\frac{x_{-3}.x_{-9}.x_{-21}}{1+x_{-3}.x_{-9}.x_{-15}}\sum_{j=0}^{n}\prod_{i=1}^{6j+1}\frac{1}{1+x_{6i-3}.x_{6i-9}.x_{6i-15}}\right),$$

$$x_{24n+10}=x_{-14}\left(1-\frac{x_{-2}.x_{-8}.x_{-20}}{1+x_{-2}.x_{-8}.x_{-14}}\sum_{j=0}^{n}\prod_{i=1}^{6j+1}\frac{1}{1+x_{6i-2}.x_{6i-8}.x_{6i-14}}\right), \quad (8)$$

$$x_{24n+11}=x_{-13}\left(1-\frac{x_{-1}.x_{-7}.x_{-19}}{1+x_{-1}.x_{-7}.x_{-13}}\sum_{j=0}^{n}\prod_{i=1}^{6j+1}\frac{1}{1+x_{6i-1}.x_{6i-7}.x_{6i-13}}\right),$$

$$x_{24n+12}=x_{-12}\left(1-\frac{x_{0}.x_{-6}.x_{-18}}{1+x_{0}.x_{-6}.x_{-12}}\sum_{j=0}^{n}\prod_{i=1}^{6j+1}\frac{1}{1+x_{6i}.x_{6i-6}.x_{6i-12}}\right),$$

$$x_{24n+13}=x_{-11}\left(1-\frac{x_{-5}.x_{-17}.x_{-23}}{1+x_{-5}.x_{-11}.x_{-17}}\sum_{j=0}^{n}\prod_{i=1}^{6j+2}\frac{1}{1+x_{6i-5}.x_{6i-11}.x_{6i-17}}\right),$$

$$x_{24n+14}=x_{-10}\left(1-\frac{x_{-4}.x_{-16}.x_{-22}}{1+x_{-4}.x_{-10}.x_{-16}}\sum_{j=0}^{n}\prod_{i=1}^{6j+2}\frac{1}{1+x_{6i-4}.x_{6i-8}.x_{6i-14}}\right),$$

$$x_{24n+15}=x_{-9}\left(1-\frac{x_{-3}.x_{-15}.x_{-21}}{1+x_{-3}.x_{-9}.x_{-15}}\sum_{j=0}^{n}\prod_{i=1}^{6j+2}\frac{1}{1+x_{6i-3}.x_{6i-9}.x_{6i-15}}\right),$$

$$x_{24n+16}=x_{-8}\left(1-\frac{x_{-2}.x_{-14}.x_{-20}}{1+x_{-2}.x_{-8}.x_{-14}}\sum_{j=0}^{n}\prod_{i=1}^{6j+2}\frac{1}{1+x_{6i-2}.x_{6i-8}.x_{6i-14}}\right), \qquad (9)$$

$$x_{24n+17}=x_{-7}\left(1-\frac{x_{-1}.x_{-13}.x_{-19}}{1+x_{-1}.x_{-7}.x_{-13}}\sum_{j=0}^{n}\prod_{i=1}^{6j+2}\frac{1}{1+x_{6i-1}.x_{6i-7}.x_{6i-13}}\right),$$

$$x_{24n+18}=x_{-6}\left(1-\frac{x_{0}.x_{-12}.x_{-18}}{1+x_{0}.x_{-6}.x_{-12}}\sum_{j=0}^{n}\prod_{i=1}^{6j+2}\frac{1}{1+x_{6i}.x_{6i-6}.x_{6i-12}}\right),$$

$$x_{24n+19}=x_{-5}\left(1-\frac{x_{-11}.x_{-17}.x_{-23}}{1+x_{-5}.x_{-11}.x_{-17}}\sum_{j=0}^{n}\prod_{i=1}^{6j+3}\frac{1}{1+x_{6i-5}.x_{6i-11}.x_{6i-17}}\right),$$

$$x_{24n+20}=x_{-4}\left(1-\frac{x_{-10}.x_{-16}.x_{-22}}{1+x_{-4}.x_{-10}.x_{-16}}\sum_{j=0}^{n}\prod_{i=1}^{6j+3}\frac{1}{1+x_{6i-4}.x_{6i-10}.x_{6i-16}}\right),$$

$$x_{24n+21}=x_{-3}\left(1-\frac{x_{-9}.x_{-15}.x_{-21}}{1+x_{-3}.x_{-9}.x_{-15}}\sum_{j=0}^{n}\prod_{i=1}^{6j+3}\frac{1}{1+x_{6i-3}.x_{6i-9}.x_{6i-15}}\right), \qquad (10)$$

$$x_{24n+22}=x_{-2}\left(1-\frac{x_{-8}.x_{-14}.x_{-20}}{1+x_{-2}.x_{-8}.x_{-14}}\sum_{j=0}^{n}\prod_{i=1}^{6j+3}\frac{1}{1+x_{6i-2}.x_{6i-8}.x_{6i-14}}\right),$$

$$x_{24n+23}=x_{-1}\left(1-\frac{x_{-7}.x_{-13}.x_{-19}}{1+x_{-1}.x_{-7}.x_{-13}}\sum_{j=0}^{n}\prod_{i=1}^{6j+3}\frac{1}{1+x_{6i-1}.x_{6i-7}.x_{6i-13}}\right),$$

$$x_{24n+24}=x_{0}\left(1-\frac{x_{-6}.x_{-12}.x_{-18}}{1+x_{0}.x_{-6}.x_{-12}}\sum_{j=0}^{n}\prod_{i=1}^{6j+3}\frac{1}{1+x_{6i}.x_{6i-6}.x_{6i-12}}\right).$$

f) Suppose that $a_1=a_7=a_{13}=a_{19}=0$. By e), having:

$$\lim_{n\to\infty}x_{24n+1}=\lim_{n\to\infty}x_{-23}\left(1-\frac{x_{-5}.x_{-11}.x_{-17}}{1+x_{-5}.x_{-11}.x_{-17}}\sum_{j=0}^{n}\prod_{i=1}^{6j}\frac{1}{1+x_{6i-5}.x_{6i-11}.x_{6i-17}}\right),$$

$$a_1 = x_{-23}\left(1-\frac{x_{-5}.x_{-11}.x_{-17}}{1+x_{-5}.x_{-11}.x_{-17}}\sum_{j=0}^{\infty}\prod_{i=1}^{6j}\frac{1}{1+x_{6i-5}.x_{6i-11}.x_{6i-17}}\right),$$

$$a_1 = 0 \Rightarrow \frac{1+x_{-5}.x_{-11}.x_{-17}}{x_{-5}.x_{-11}.x_{-17}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j}\frac{1}{1+x_{6i-5}.x_{6i-11}.x_{6i-17}}. \quad (11)$$

Similarly,

$$\lim_{n\to\infty} x_{24n+7} = \lim_{n\to\infty} x_{-17}\left(1-\frac{x_{-5}.x_{-11}.x_{-23}}{1+x_{-5}.x_{-11}.x_{-17}}\sum_{j=0}^{n}\prod_{i=1}^{6j+1}\frac{1}{1+x_{6i-5}.x_{6i-11}.x_{6i-17}}\right),$$

$$a_7 = x_{-17}\left(1-\frac{x_{-5}.x_{-11}.x_{-23}}{1+x_{-5}.x_{-11}.x_{-17}}\sum_{j=0}^{\infty}\prod_{i=1}^{6j+1}\frac{1}{1+x_{6i-5}.x_{6i-11}.x_{6i-17}}\right),$$

$$a_7 = 0 \Rightarrow \frac{1+x_{-5}.x_{-11}.x_{-17}}{x_{-5}.x_{-11}.x_{-23}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j+1}\frac{1}{1+x_{6i-5}.x_{6i-11}.x_{6i-17}}. \quad (12)$$

Similarly,

$$\lim_{n\to\infty} x_{24n+13} = \lim_{n\to\infty} x_{-11}\left(1-\frac{x_{-5}.x_{-17}.x_{-23}}{1+x_{-5}.x_{-11}.x_{-17}}\sum_{j=0}^{n}\prod_{i=1}^{6j+2}\frac{1}{1+x_{6i-5}.x_{6i-11}.x_{6i-17}}\right),$$

$$a_{13} = x_{-11}\left(1-\frac{x_{-5}.x_{-17}.x_{-23}}{1+x_{-5}.x_{-11}.x_{-17}}\sum_{j=0}^{\infty}\prod_{i=1}^{6j+2}\frac{1}{1+x_{6i-5}.x_{6i-11}.x_{6i-17}}\right),$$

$$a_{13} = 0 \Rightarrow \frac{1+x_{-5}.x_{-11}.x_{-17}}{x_{-5}.x_{-17}.x_{-23}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j+2}\frac{1}{1+x_{6i-5}.x_{6i-11}.x_{6i-17}}. \quad (13)$$

Similarly,

$$\lim_{n\to\infty} x_{24n+19} = \lim_{n\to\infty} x_{-5}\left(1-\frac{x_{-11}.x_{-17}.x_{-23}}{1+x_{-5}.x_{-11}.x_{-17}}\sum_{j=0}^{n}\prod_{i=1}^{6j+3}\frac{1}{1+x_{6i-5}.x_{6i-11}.x_{6i-17}}\right),$$

$$a_{19} = x_{-5}\left(1-\frac{x_{-11}.x_{-17}.x_{-23}}{1+x_{-5}.x_{-11}.x_{-17}}\sum_{j=0}^{\infty}\prod_{i=1}^{6j+3}\frac{1}{1+x_{6i-5}.x_{6i-11}.x_{6i-17}}\right),$$

$$a_{19} = 0 \Rightarrow \frac{1+x_{-5}.x_{-11}.x_{-17}}{x_{-11}.x_{-17}.x_{-23}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j+3}\frac{1}{1+x_{6i-5}.x_{6i-11}.x_{6i-17}}. \quad (14)$$

From the (11) and (12);

$$\frac{1+x_{-5}.x_{-11}.x_{-17}}{x_{-5}.x_{-11}.x_{-17}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j}\frac{1}{1+x_{6i-5}.x_{6i-11}.x_{6i-17}} > \frac{1+x_{-5}.x_{-11}.x_{-17}}{x_{-5}.x_{-11}.x_{-23}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j+1}\frac{1}{1+x_{6i-5}.x_{6i-11}.x_{6i-17}} \quad (15)$$

thus, $x_{-23} > x_{-17}$.

From the (12) and (13);

$$\frac{1+x_{-5}.x_{-11}.x_{-17}}{x_{-5}.x_{-11}.x_{-23}}=\sum_{j=0}^{\infty}\prod_{i=1}^{6j+1}\frac{1}{1+x_{6i-5}.x_{6i-11}.x_{6i-17}}>\frac{1+x_{-5}.x_{-11}.x_{-17}}{x_{-5}.x_{-17}.x_{-23}}=\sum_{j=0}^{\infty}\prod_{i=1}^{6j+2}\frac{1}{1+x_{6i-5}.x_{6i-11}.x_{6i-17}}$$

thus, $x_{-17}>x_{-11}$.

From the (13) and (14);

$$\frac{1+x_{-5}.x_{-11}.x_{-17}}{x_{-5}.x_{-17}.x_{-23}}=\sum_{j=0}^{\infty}\prod_{i=1}^{6j+2}\frac{1}{1+x_{6i-5}.x_{6i-11}.x_{6i-17}}>\frac{1+x_{-5}.x_{-11}.x_{-17}}{x_{-11}.x_{-17}.x_{-23}}=\sum_{j=0}^{\infty}\prod_{i=1}^{6j+3}\frac{1}{1+x_{6i-5}.x_{6i-11}.x_{6i-17}}$$

thus, $x_{-11}>x_{-5}$.

We obtain $x_{-23}>x_{-17}>x_{-11}>x_{-5}$.

It is supposed that $a_2=a_8=a_{14}=a_{20}=0$. From the (16), it is followed as the proof of the (15) is similar and is omitted:

$$\begin{aligned}
&\frac{1+x_{-4}.x_{-10}.x_{-16}}{x_{-4}.x_{-10}.x_{-16}}=\sum_{j=0}^{\infty}\prod_{i=1}^{6j}\frac{1}{1+x_{6i-4}.x_{6i-10}.x_{6i-16}}>\frac{1+x_{-4}.x_{-10}.x_{-16}}{x_{-4}.x_{-10}.x_{-22}}=\sum_{j=0}^{\infty}\prod_{i=1}^{6j+1}\frac{1}{1+x_{6i-4}.x_{6i-10}.x_{6i-16}},\\
&\frac{1+x_{-4}.x_{-10}.x_{-16}}{x_{-4}.x_{-10}.x_{-22}}=\sum_{j=0}^{\infty}\prod_{i=1}^{6j+1}\frac{1}{1+x_{6i-4}.x_{6i-10}.x_{6i-16}}>\frac{1+x_{-4}.x_{-10}.x_{-16}}{x_{-4}.x_{-16}.x_{-22}}=\sum_{j=0}^{\infty}\prod_{i=1}^{6j+2}\frac{1}{1+x_{6i-4}.x_{6i-10}.x_{6i-16}}\\
&\frac{1+x_{-4}.x_{-10}.x_{-16}}{x_{-4}.x_{-16}.x_{-22}}=\sum_{j=0}^{\infty}\prod_{i=1}^{6j+2}\frac{1}{1+x_{6i-4}.x_{6i-10}.x_{6i-16}}>\frac{1+x_{-4}.x_{-10}.x_{-16}}{x_{-10}.x_{-16}.x_{-22}}=\sum_{j=0}^{\infty}\prod_{i=1}^{6j+3}\frac{1}{1+x_{6i-4}.x_{6i-10}.x_{6i-16}}
\end{aligned}\tag{16}$$

thus, $x_{-22}>x_{-16}>x_{-10}>x_{-4}$.

It is supposed that $a_3=a_9=a_{15}=a_{21}=0$. From the (17), it is followed as the proof of the (16) is similar and is omitted:

$$\begin{aligned}
&\frac{1+x_{-3}.x_{-9}.x_{-15}}{x_{-3}.x_{-9}.x_{-15}}=\sum_{j=0}^{\infty}\prod_{i=1}^{6j}\frac{1}{1+x_{6i-3}.x_{6i-9}.x_{6i-15}}>\frac{1+x_{-3}.x_{-9}.x_{-15}}{x_{-3}.x_{-9}.x_{-21}}=\sum_{j=0}^{\infty}\prod_{i=1}^{6j+1}\frac{1}{1+x_{6i-3}.x_{6i-9}.x_{6i-15}},\\
&\frac{1+x_{-3}.x_{-9}.x_{-15}}{x_{-3}.x_{-9}.x_{-21}}=\sum_{j=0}^{\infty}\prod_{i=1}^{6j+1}\frac{1}{1+x_{6i-3}.x_{6i-9}.x_{6i-15}}>\frac{1+x_{-3}.x_{-9}.x_{-15}}{x_{-3}.x_{-15}.x_{-21}}=\sum_{j=0}^{\infty}\prod_{i=1}^{6j+2}\frac{1}{1+x_{6i-3}.x_{6i-9}.x_{6i-15}},\\
&\frac{1+x_{-3}.x_{-9}.x_{-15}}{x_{-3}.x_{-15}.x_{-21}}=\sum_{j=0}^{\infty}\prod_{i=1}^{6j+2}\frac{1}{1+x_{6i-3}.x_{6i-9}.x_{6i-15}}>\frac{1+x_{-3}.x_{-9}.x_{-15}}{x_{-9}.x_{-15}.x_{-21}}=\sum_{j=0}^{\infty}\prod_{i=1}^{6j+3}\frac{1}{1+x_{6i-3}.x_{6i-9}.x_{6i-15}},
\end{aligned}\tag{17}$$

thus, $x_{-21}>x_{-15}>x_{-9}>x_{-3}$.

It is supposed that $a_4=a_{10}=a_{16}=a_{22}=0$. From the (18), it is followed as the proof of the (17) is similar and is omitted:

$$\frac{1+x_{-2}.x_{-8}.x_{-14}}{x_{-2}.x_{-8}.x_{-14}}=\sum_{j=0}^{\infty}\prod_{i=1}^{6j}\frac{1}{1+x_{6i-2}.x_{6i-8}.x_{6i-14}}>\frac{1+x_{-2}.x_{-8}.x_{-14}}{x_{-2}.x_{-8}.x_{-20}}=\sum_{j=0}^{\infty}\prod_{i=1}^{6j+1}\frac{1}{1+x_{6i-2}.x_{6i-8}.x_{6i-14}},\tag{18}$$

$$\frac{1+x_{-2}.x_{-8}.x_{-14}}{x_{-2}.x_{-8}.x_{-20}}=\sum_{j=0}^{\infty}\prod_{i=1}^{6j+1}\frac{1}{1+x_{6i-2}.x_{6i-8}.x_{6i-14}}>\frac{1+x_{-2}.x_{-8}.x_{-14}}{x_{-2}.x_{-14}.x_{-20}}=\sum_{j=0}^{\infty}\prod_{i=1}^{6j+2}\frac{1}{1+x_{6i-2}.x_{6i-8}.x_{6i-14}},$$
$$\frac{1+x_{-2}.x_{-8}.x_{-14}}{x_{-2}.x_{-14}.x_{-20}}=\sum_{j=0}^{\infty}\prod_{i=1}^{6j+2}\frac{1}{1+x_{6i-2}.x_{6i-8}.x_{6i-14}}>\frac{1+x_{-2}.x_{-8}.x_{-14}}{x_{-8}.x_{-14}.x_{-20}}=\sum_{j=0}^{\infty}\prod_{i=1}^{6j+3}\frac{1}{1+x_{6i-2}.x_{6i-8}.x_{6i-14}},$$

thus, $x_{-20}>x_{-14}>x_{-8}>x_{-2}$.

It is supposed that $a_5=a_{11}=a_{17}=a_{23}=0$. From the (19), it is followed as the proof of the (18) is similar and is omitted:

$$\frac{1+x_{-1}.x_{-7}.x_{-13}}{x_{-1}.x_{-7}.x_{-13}}=\sum_{j=0}^{\infty}\prod_{i=1}^{6j}\frac{1}{1+x_{6i-1}.x_{6i-7}.x_{6i-13}}>\frac{1+x_{-1}.x_{-7}.x_{-13}}{x_{-1}.x_{-7}.x_{-19}}=\sum_{j=0}^{\infty}\prod_{i=1}^{6j+1}\frac{1}{1+x_{6i-1}.x_{6i-7}.x_{6i-13}},$$
$$\frac{1+x_{-1}.x_{-7}.x_{-13}}{x_{-1}.x_{-7}.x_{-19}}=\sum_{j=0}^{\infty}\prod_{i=1}^{6j+1}\frac{1}{1+x_{6i-1}.x_{6i-7}.x_{6i-13}}>\frac{1+x_{-1}.x_{-7}.x_{-13}}{x_{-1}.x_{-13}.x_{-19}}=\sum_{j=0}^{\infty}\prod_{i=1}^{6j+2}\frac{1}{1+x_{6i-1}.x_{6i-7}.x_{6i-13}}, \quad (19)$$
$$\frac{1+x_{-1}.x_{-7}.x_{-13}}{x_{-1}.x_{-13}.x_{-19}}=\sum_{j=0}^{\infty}\prod_{i=1}^{6j+2}\frac{1}{1+x_{6i-1}.x_{6i-7}.x_{6i-13}}>\frac{1+x_{-1}.x_{-7}.x_{-13}}{x_{-7}.x_{-13}.x_{-19}}=\sum_{j=0}^{\infty}\prod_{i=1}^{6j+3}\frac{1}{1+x_{6i-1}.x_{6i-7}.x_{6i-13}},$$

thus, $x_{-19}>x_{-13}>x_{-7}>x_{-1}$.

It is supposed that $a_6=a_{12}=a_{18}=a_{24}=0$. Suppose that $x_{-18}=x_{-12}=x_{-6}=x_0=0$. from then (20), it follows, proof of the (19) is similar and will be omitted:

$$\frac{1+x_0.x_{-6}.x_{-12}}{x_0.x_{-6}.x_{-12}}=\sum_{j=0}^{\infty}\prod_{i=1}^{6j}\frac{1}{1+x_{6i}.x_{6i-6}.x_{6i-12}}>\frac{1+x_0.x_{-6}.x_{-12}}{x_0.x_{-6}.x_{-18}}=\sum_{j=0}^{\infty}\prod_{i=1}^{6j+1}\frac{1}{1+x_{6i}.x_{6i-6}.x_{6i-12}},$$
$$\frac{1+x_0.x_{-6}.x_{-12}}{x_0.x_{-6}.x_{-18}}=\sum_{j=0}^{\infty}\prod_{i=1}^{6j+1}\frac{1}{1+x_{6i}.x_{6i-6}.x_{6i-12}}>\frac{1+x_0.x_{-6}.x_{-12}}{x_0.x_{-12}.x_{-18}}=\sum_{j=0}^{\infty}\prod_{i=1}^{6j+2}\frac{1}{1+x_{6i}.x_{6i-6}.x_{6i-12}}, \quad (20)$$
$$\frac{1+x_0.x_{-6}.x_{-12}}{x_0.x_{-12}.x_{-18}}=\sum_{j=0}^{\infty}\prod_{i=1}^{6j+2}\frac{1}{1+x_{6i}.x_{6i-6}.x_{6i-12}}>\frac{1+x_0.x_{-6}.x_{-12}}{x_{-6}.x_{-12}.x_{-18}}=\sum_{j=0}^{\infty}\prod_{i=1}^{6j+3}\frac{1}{1+x_{6i}.x_{6i-6}.x_{6i-12}},$$

thus, $x_{-18}>x_{-12}>x_{-6}>x_0$. It is concluded at a contradiction which completes the proof of theorem.

3. EXAMPLE

Example 3.1. Consider the following equation $x_{n+1}=\dfrac{x_{n-23}}{1+x_{n-5}x_{n-11}x_{n-17}}$.

If the initial conditions are selected as follows:

$x_{-23}=0.9999999999999999999999999$; $x_{-22}=0.9999999999999999999999998$;

$x_{-21}=0.9999999999999999999999997$; $x_{-20}=0.9999999999999999999999996$;

$x_{-19}=0.9999999999999999999999995$; $x_{-18}=0.9999999999999999999999994$;

$x_{-17}=0.9999999999999999999999993$; $x_{-16}=0.9999999999999999999999992$;

$x_{-15} = 0.99999999999999999999991$; $x_{-14} = 0.99999999999999999999999$;

$x_{-13} = 0.99999999999999999999998$; $x_{-12} = 0.99999999999999999999997$;

$x_{-11} = 0.99999999999999999999996$; $x_{-10} = 0.99999999999999999999995$;

$x_{-9} = 0.99999999999999999999994$; $x_{-8} = 0.99999999999999999999993$;

$x_{-7} = 0.99999999999999999999992$; $x_{-6} = 0.99999999999999999999991$;

$x_{-5} = 0.9999999999999999999999$; $x_{-4} = 0.9999999999999999999998$;

$x_{-3} = 0.9999999999999999999997$; $x_{-2} = 0.9999999999999999999996$;

$x_{-1} = 0.9999999999999999999995$; $x_{0} = 0.9999999999999999999994$.

The following solutions are obtained:

x_n ={0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.666667, 0.666667, 0.666667, 0.666667, 0.666667, 0.666667, 0.75, 0.75, 0.75, 0.75, 0.75, 0.75, 0.8, 0.8, 0.8, 0.8, 0.8, 0.8, 0.357143, 0.357143, 0.357143, 0.357143, 0.357143, 0.357143, 0.54902, 0.54902, 0.54902, 0.54902, 0.54902, 0.54902, 0.648305, 0.648305, 0.648305, 0.648305, 0.648305, 0.648305, 0.709774, 0.709774, 0.709774, 0.709774, 0.709774, 0.709774, 0.285114, 0.285114, 0.285114, 0.285114, 0.285114, 0.285114, 0.485345, 0.485345, 0.485345, 0.485345, 0.485345, 0.485345, 0.590325, 0.590325, 0.590325, 0.590325, 0.590325, 0.590325, 0.656173, 0.656173, 0.656173, 0.656173, 0.656173, 0.656173, 0.239995, 0.239995, 0.239995, 0.239995, 0.239995, 0.239995, 0.444063, 0.444063, 0.444063, 0.444063, 0.444063, 0.444063, 0.551741, 0.551741, 0.551741, 0.551741, 0.551741, 0.551741, 0.619732, 0.619732, 0.619732, 0.619732, 0.619732, 0.619732, 0.208358, 0.208358, 0.208358, 0.208358, 0.208358, 0.208358, 0.41453, 0.41453, 0.41453, 0.41453, 0.41453, 0.41453, 0.523709, 0.523709, 0.523709, 0.523709, 0.523709, 0.523709, 0.592913, 0.592913, 0.592913, 0.592913, 0.592913, 0.592913, 0.184597, 0.184597,...}

The graph of the solutions is given below.

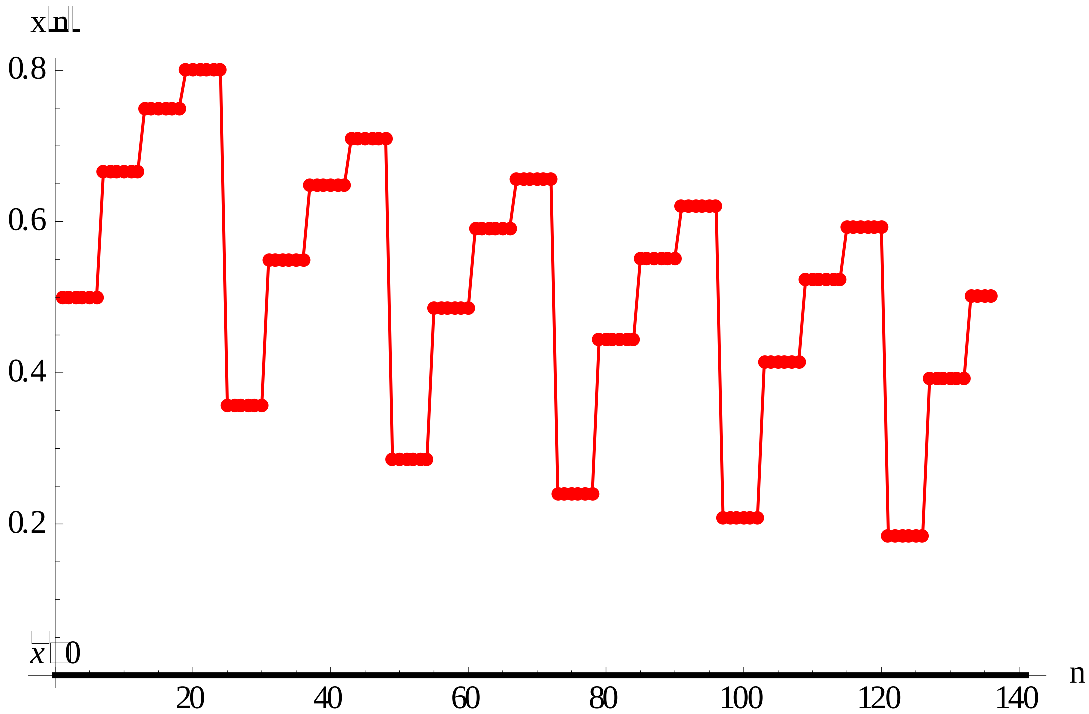

Figure 3.1. x(n) graph of the solutions.

ACKNOWLEDGEMENTS

This article was produced from Oğul B. PhD Thesis.

REFERENCES:

[1] Amleh A.M., Grove E.A., Ladas G., Georgiou D.A., On the recursive sequence $x_{n+1} = \alpha + \frac{x_{n-1}}{x_n}$, J. Math. Anal. Appl., 233, no. 2, (1999), 790-798.

[2] Cinar C., On the positive solutions of the difference equation $x_{n+1} = \frac{x_{n-1}}{1 + ax_n x_{n-1}}$, Appl. Math. Comp., 158 (3), (2004), 809–812.

[3] Cinar C., On the positive solutions of the difference equation $x_{n+1} = \frac{x_{n-1}}{-1 + ax_n x_{n-1}}$, Appl. Math. Comp., 158 (3), (2004), 793–797.

[4] Cinar C., On the positive solutions of the difference equation $x_{n+1} = \frac{ax_{n-1}}{1 + bx_n x_{n-1}}$, Appl. Math. Comp., 156 (3), (2004), 587–590.

[5] DeVault R., Ladas G., Schultz W.S., On the recursive sequence $x_{n+1} = \frac{A}{x_n} + \frac{1}{x_{n-2}}$, Proc. Amer. Math. Soc., 126 no. 11, (1998), 3257-3261.

[6] Elabbasy E. M., El-Metwally H., Elsayed E. M., On the difference equation $x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-1}}$, Advances in Difference Equation, (2006), 1-10.

[7] Elabbasy E. M., El-Metwally H., Elsayed E. M., Qualitative behavior of higher order difference equation, Soochow Journal of Mathematics, 33(4), (2007), 861-873.

[8] Elabbasy E. M., El-Metwally H., Elsayed E. M., Global attractivity and periodic character of a fractional difference equation of order three, Yokohama Mathematical Journal, 53, (2007), 89-100.

[9] Elabbasy E. M., El-Metwally H., Elsayed E. M., On the difference equation $x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \Pi_{i=0}^{k} x_{n-i}}$, J. Conc. Appl. Math., 5(2), (2007), 101-113.

[10] Elabbasy E. M. and Elsayed E. M., On the Global Attractivity of Difference Equation of Higher Order, Carpathian Journal of Mathematics, 24 (2), (2008), 45–53.

[11] Elsayed E. M., On the Solution of Recursive Sequence of Order Two, Fasciculi Mathematici, 40, (2008), 5–13.

[12] Elsayed E. M., Dynamics of a Recursive Sequence of Higher Order, Communications on Applied Nonlinear Analysis, 16 (2), (2009), 37–50.

[13] Elsayed E. M., Solution and atractivity for a rational recursive sequence, Discrete Dynamics in Nature and Society, (2011), 17.

[14] Elsayed E. M., On the solution of some difference equation, Europan Journal of Pure and Applied Mathematics, 4 (3), (2011), 287–303.

[15] Elsayed E. M., On the Dynamics of a higher order rational recursive sequence, Communications in Mathematical Analysis, 12 (1), (2012), 117–133.

[16] Elsayed E. M., Solution of rational difference system of order two, Mathematical and Computer Modelling, 55, (2012), 378–384.

[17] Gibbons C. H., Kulenović M. R. S. and Ladas G., *On the recursive sequence* $x_{n+1} = \frac{\alpha + \beta x_{n-1}}{\chi + x_n}$, Math. Sci. Res. Hot-Line, 4, 2, (2000), 1-11.

[18] Kulenović M.R.S., Ladas G., Sizer W.S., On the recursive sequence $x_{n+1} = \frac{\alpha x_n + \beta x_{n-1}}{\chi x_n + \delta x_{n-1}}$ Math. Sci. Res. Hot-Line, 2, 5, (1998), 1-16.

[19] Stevic S., On the recursive sequence $x_{n+1} = \frac{x_{n-1}}{g(x_n)}$, Taiwanese J. Math., 6, 3, (2002), 405-414.

[20] Şimşek D., Çınar C. and Yalçınkaya İ., On the recursive sequence $x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}}$, Int. J. Contemp. Math. Sci., 1, 9-12, (2006), 475-480.

[21] Şimşek D., Çınar C., Karataş R., Yalçınkaya İ., On the recursive sequence $x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2}}$, Int. J. Pure Appl. Math., 27, 4, (2006), 501-507.

[22] Şimşek D., Çınar C., Karataş R., Yalçınkaya İ., "n the recursive sequence $x_{n+1} = \frac{x_{n-5}}{1 + x_{n-1}x_{n-3}}$, Int. J. Pure Appl. Math., 28, 1, (2006), 117-124.

[23] Şimşek D., Çınar C., Yalçınkaya İ., On The Recursive Sequence $x_{n+1}=\frac{x_{n-(5k+9)}}{1+x_{n-4}x_{n-9}...x_{n-(5k+4)}}$ Taiwanese Journal of Mathematics, Vol. 12, 5, (2008), 1087-1098.

[24] Şimşek D., Doğan A., On A Class of Recursive Sequence, Manas Journal of Engineering, 2, 1, (2014), 16-22.

[25] Şimşek D., Eröz M., Solutions of The Rational Difference Equations $x_{n+1}=\frac{x_{n-3}}{1+x_n x_{n-1} x_{n-2}}$, Manas Journal of Engineering, 4, 1, (2016), 12-20.

[26] Şimşek D., Oğul B., Solutions of The Rational Difference Equations $x_{n+1}=\frac{x_{n-(2k+1)}}{1+x_{n-k}}$, Manas Journal of Engineering, 5, 3, (2017), 57-68.

[27] Şimşek D., Oğul B., Abdullayev F., Solutions of The Rational Difference Equations $x_{n+1}=\frac{x_{n-11}}{1+x_{n-2}.x_{n-5}x_{n-8}}$, AIP Conference Proceedings, 1880(1), 040003, (2017).

[28] Şimşek D., Abdullayev F., On The Recursive Sequence $x_{n+1}=\frac{x_{n-(4k+3)}}{1+\prod_{t=0}^{2} x_{n-(k+1)t-k}}$, Journal of Mathematics Sciences, 6, 222, (2017), 762-771.

[29] Yang X., Chen B., Megson M.G. and Evans J.D., Global attractivity in a recursive sequence, Appl.Math. Comput., 158, no. 3, (2004), 667-682.