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The Tanh-Coth Method for Two System of Sobolev Type Equations In Mathematical Physics

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Abstract: *Sobolev equations have been used to describe many physical phenomena and they are characterized by having mixed time and space derivatives appearing in the highest-order terms of an partial differential equation. In this work we consider two important system of Sobolev type equations namely improved Boussinesq and higher-order improved Boussinesq. By using tanh-coth method, we obtain abundant new travelling wave solutions of these important physical structures.*

Keywords: *Sobolev equation; The tanh-coth method; The system of improved Boussinesq equations; The system of higher-order improved Boussinesq equations*

Matematiksel Fizikteki Sobolev Tipi İki Denklem Sistemi için Tanh-Coth Yöntemi

Öz: *Sobolov denklemleri en yüksek mertebeden türevinde zaman ve boyuta göre türevleri beraber bulunduran denklemler olarak tanımlanır. Bu çalışmada, Boussinesq ve yüksek mertebeden geliştirilmiş Boussinesq adlı iki önemli Sobolev denklem sistemini ele aldık. Tanh-coth yöntemi kullanarak bu iki önemli denklem sisteminin bir çok yeni hareketli dalga çözümünü elde ettik.*

Anahtar Kelimeler: *Sobolev denklemi; Tanh-coth metot; Geliştirilmiş Boussinesq denklem sistemi; Yüksek mertebeden geliştirilmiş Boussinesq denklem sistemi.*

INTRODUCTION

The Tanh-Coth Method for Two System of Sobolev Type Equations in Mathematical Physics

We have focused on travelling wave solutions of the system of two improved Boussinesq equations.

The system of improved Boussinesq equations

The system of improved Boussinesq equations

$$u_{xxtt} - u_{tt} + u_{xx} + \left(f_1(u,w)\right)_{xx} = 0 \tag{1}$$

$$w_{xxtt} - w_{tt} + w_{xx} + \left(f_2(u,w)\right)_{xx} = 0$$

where f_1 and f_2 are given nonlinear functions, $u(x,t)$ and $w(x,t)$ are unknown functions and subscripts x and t indicate partial derivatives, has been derived to describe bidirectional wave propagation in several study. For instance, a transversal degree of freedom is introduced in the Toda lattice model and the initial value problem for the system (1) have been studied. Travelling wave solutions and numerical solutions of the system are obtained [1]. Khusnutdinova et al. have studied nonlinear longitudinal waves in a two-layered structure with a soft bonding layer using a relatively simple long wave model in the form of coupled Boussinesq-type equations [2]. Wattis has used (1) to find waves of general speed in a diatomic lattice by using the quasi-continuum method of approximation [3]. In [4], Godefroy has studied (1) as the Cauchy problem under certain conditions and showed that the solution for the Cauchy problem of this system blows up in finite time. Wang and Li have considered the Cauchy problem for (1) and proved the existence and uniqueness of the global solution and given sufficient conditions of blow-up of the solution in finite time by convex methods [5]. Rosenau has studied transversal degree of freedom for the propagation of non-linear wave in Toda lattice via the system (1) [6]. Sergei has analysed the system and stated that the solution of the system would blow up if there is a non-positive Hamiltonian [7]. Pego et al. have studied the stability of solitary waves of two coupled Boussinesq equations which model weakly nonlinear vibrations in a cubic lattice [8].

The system of higher-order improved Boussinesq equations

$$-\beta u_{xxxxtt} + \alpha u_{xxtt} - u_{tt} + u_{xx} + \left(g_1(u,w)\right)_{xx} = 0 \tag{2}$$

$$-\beta w_{xxxxtt} + \alpha w_{xxtt} - w_{tt} + w_{xx} + \left(g_2(u,w)\right)_{xx} = 0$$

where g_1 and g_2 are given nonlinear functions, have been considered and blow-up results were obtained in [9]. Schneider and Wayne have shown that in the longwave limit the water wave problem without surface tension can be described approximately by single component form of equations (2) [10]. Duruk et al. have proved in [11] that the Cauchy problem for the single component form of (2) is globally well-posed in Sobolev spaces H^s for $s > 1/2$ under certain conditions on nonlinear term and initial data.
In this paper, we focused on travelling wave solutions of the system of two improved Boussinesq equations

$$u_{xxtt} - u_{tt} + u_{xx} + (uw)_{xx} = 0 \tag{3}$$
$$w_{xxtt} - w_{tt} + w_{xx} + (uw)_{xx} = 0$$

and the system of higher-order improved Boussinesq equations

$$-\beta u_{xxxxtt} + \alpha u_{xxtt} - u_{tt} + u_{xx} + (uw)_{xx} = 0 \tag{4}$$

$$-\beta w_{xxxxtt} + \alpha w_{xxtt} - w_{tt} + w_{xx} + (uw)_{xx} = 0$$

where α and β are positive constants, $u(x,t)$ and $w(x,t)$ are unknown functions. We described outline of the tanh-coth method in following section and derived various exact travelling wave solutions of these physical structures in section 3 and 4 by using tanh-coth method. Finally, we summarized our conclusions in section 5.

OUTLINE OF THE TANH-COTH METHOD

Wazwaz has summarized the tanh method in the following manner:

i. First consider a general form of nonlinear equation

$$P(u, u_t, u_x, u_{xx}, \dots) = 0. \tag{5}$$

ii. To find the traveling wave solution of Eq. (5), the wave variable $\xi = x + y + z + \cdots - Vt$ is introduced so that

$$u(x,t) = U(\mu\xi). \tag{6}$$

Based on this one may use the following changes

$$\frac{\partial}{\partial t} = -V\frac{d}{d\xi}$$

$$\frac{\partial}{\partial x} = \mu\frac{d}{d\xi}$$

$$\frac{\partial^2}{\partial x^2} = \mu^2\frac{d^2}{d\xi^2} \tag{7}$$
$$\frac{\partial^3}{\partial x^3} = \mu^3\frac{d^3}{d\xi^3}$$

and so on for other derivatives. Using (7) changes the PDE (5) to an ODE

$$Q(U, U', U'', \dots) = 0 \tag{8}$$

iii. If all terms of the resulting ODE contain derivatives in ξ, then by integrating this equation, and by considering the constant of integration to be zero, one obtains a simplified ODE.

iv. A new independent variable

$$Y = tanh(\mu\xi) \tag{9}$$

is introduced that leads to the change of derivatives:

$$\begin{aligned}
\frac{d}{d\xi} &= \mu(1-Y^2)\frac{d}{dY} \\
\frac{d^2}{d\xi^2} &= -2\mu^2 Y(1-Y^2)\frac{d}{dY} + \mu^2(1-Y^2)^2\frac{d^2}{dY^2} \\
\frac{d^3}{d\xi^3} &= 2\mu^3(1-Y^2)(3Y^2-1)\frac{d}{dY} - 6\mu^3 Y(1-Y^2)^2\frac{d^2}{dY^2} + \mu^3(1-Y^2)^3\frac{d^3}{dY^3} \\
\frac{d^4}{d\xi^4} &= -8\mu^4 Y(1-Y^2)(3Y^2-2)\frac{d}{dY} + 4\mu^4(1-Y^2)^2(9Y^2-2)\frac{d^2}{dY^2} - 12\mu^4 Y(1-Y^2)^3\frac{d^3}{dY^3} \\
&\quad + \mu^4(1-Y^2)^4\frac{d^4}{dY^4}
\end{aligned} \tag{10}$$

where other derivatives can be derived in a similar manner.

v. The ansatz of the form

$$U(\mu\xi) = S(Y) = \sum_{k=0}^{M} a_k Y^k + \sum_{k=1}^{M} b_k Y^{-k} \tag{11}$$

is introduced where M is a positive integer, in most cases, that will be determined. If M is not an integer, then a transformation formula is used to overcome this difficulty. Substituting (10) and (11) into the ODE (8) yields an equation in powers of Y.

vi. To determine the parameter M, the linear terms of highest order in the resulting equation with the highest order nonlinear terms are balanced. With M determined, one collects the all coefficients of powers of Y in the resulting equation where these coefficients have to vanish. This will give a system of algebraic equations involving the a_k and b_k, $(k = 0, \ldots, M)$, V, and μ. Having determined these parameters, knowing that M is a positive integer in most cases, and using (11) one obtains an analytic solution in a closed form.

THE SOLUTIONS OF THE SYSTEM OF IMPROVED BOUSSINESQ EQUATIONS

The system of improved Boussinesq equations is given by

$$\begin{aligned}
u_{xxtt} - u_{tt} + u_{xx} + (uw)_{xx} &= 0 \\
w_{xxtt} - w_{tt} + w_{xx} + (uw)_{xx} &= 0.
\end{aligned} \tag{12}$$

Using the wave variable $\xi = x - Vt$ then by integrating this equation twice, and considering the constants of integration to be zero, the system (12) is carried to a system of ODEs

$$V^2U'' - V^2U + U + UW = 0 \tag{13}$$

$V^2W'' - V^2W + W + UW \ = \ 0.$

Balancing V^2U'' with UW and V^2W'' with UW in (13) gives

$$2M \ = \ M + N \tag{14}$$

$$2N \ = \ M + N,$$

so that

$$M = N. \tag{15}$$

We take $M = N = 2$ and consider solutions in the form

$$U(\mu\xi) = S(Y) = \sum_{k=0}^{2} a_k Y^k + \sum_{k=1}^{2} b_k Y^{-k} \tag{16}$$

$$W(\mu\xi) = S(Y) = \sum_{k=0}^{2} c_k Y^k + \sum_{k=1}^{2} d_k Y^{-k}.$$

Substituting (16) into the two components of (13), and collecting the coefficients of Y gives two systems of algebraic equations for $a_0, a_1, a_2, b_1, b_2, c_0, c_1, c_2, d_1, d_2, V$ and μ:

$6a_2V^2\mu^2 + c_2a_2 = 0, 6c_2V^2\mu^2 + c_2a_2 = 0, 2a_1V^2\mu^2 + c_1a_2 + c_2a_1 = 0, 2c_1V^2\mu^2 + c_1a_2 + c_2a_1 = 0$

$a_2 - a_2V^2 - 8a_2V^2\mu^2 + c_0a_2 + c_1a_1 + c_2a_0 = 0, c_2 - c_2V^2 - 8c_2V^2\mu^2 + c_0a_2 + c_1a_1 + c_2a_0 = 0$

$a_1 - a_1V^2 - 2a_1V^2\mu^2 + d_1a_2 + b_1c_2 + c_0a_1 + c_1a_0 = 0, c_1 - c_1V^2 - 2c_1V^2\mu^2 + d_1a_2 + b_1c_2 + c_0a_1 + c_1a_0 = 0$

$a_0 - V^2a_0 + d_1a_1 + d_2a_2 + b_1c_1 + b_2c_2 + c_0a_0 + 2V^2b_2\mu^2 + 2V^2a_2\mu^2 = 0$

$c_0 - V^2c_0 + d_1a_1 + ba_2 + b_1c_1 + b_2c_2 + c_0a_0 + 2V^2d_2\mu^2 + 2V^2c_2\mu^2 = 0$

$b_1 - b_1V^2 - 2b_1V^2\mu^2 + d_1a_0 + ba_1 + b_1c_0 + b_2c_1 = 0$

$d_1 - d_1V^2 - 2d_1V^2\mu^2 + d_1a_0 + d_2a_1 + b_1c_0 + b_2c_1 = 0$

$b_2 - b_2V^2 - 8b_2V^2\mu^2 + d_1b_1 + d_2a_0 + b_2c_0 = 0$

$d_2 - d_2V^2 - 8d_2V^2\mu^2 + d_1b_1 + d_2a_0 + b_2c_0 = 0$

$2b_1V^2\mu^2 + d_2b_1 + d_1b_2 = 0, 2d_1V^2\mu^2 + d_2b_1 + d_1b_2 = 0$

$$6b_2V^2\mu^2 + d_2b_2 = 0, 6d_2V^2\mu^2 + d_2b_2 = 0 \tag{17}$$

Solving these systems leads to the following sets:

$$a_0 = 6\mu^2, b_2 = d_2 = -6\mu^2, c_0 = 2\mu^2, V = \pm 1 \tag{18}$$

$$a_0 = c_0 = \frac{2\mu^2}{4\mu^2+1}, b_2 = d_2 = -\frac{6\mu^2}{4\mu^2+1}, V = \pm\frac{1}{\sqrt{4\mu^2+1}} \tag{19}$$

$$a_0 = 2\mu^2, b_2 = d_2 = -6\mu^2, c_0 = 6\mu^2, V = \pm 1 \tag{20}$$

$$a_0 = c_0 = -\frac{6\mu^2}{4\mu^2-1}, b_2 = d_2 = \frac{6\mu^2}{4\mu^2-1}, V = \pm\frac{1}{\sqrt{-4\mu^2+1}}, |\mu| < \frac{1}{2} \tag{21}$$

$$a_0 = 6\mu^2, a_2 = c^2 = -6\mu^2, c_0 = 2\mu^2, V = \pm 1 \tag{22}$$

$$a_0 = c_0 = \frac{2\mu^2}{4\mu^2+1}, a_2 = c_2 = -\frac{6\mu^2}{4\mu^2+1}, V = \pm\frac{1}{\sqrt{4\mu^2+1}} \tag{23}$$

$$a_0 = c_0 = -\frac{6\mu^2}{4\mu^2-1}, a_2 = c_2 = \frac{6\mu^2}{4\mu^2-1}, V = \pm\frac{1}{\sqrt{-4\mu^2+1}}, |\mu| < \frac{1}{2} \tag{24}$$

$$a_0 = c_0 = -\frac{4\mu^2}{16\mu^2+1}, a_2 = b_2 = c_2 = d_2 = -\frac{6\mu^2}{16\mu^2+1}, V = \pm\frac{1}{\sqrt{16\mu^2+1}} \tag{25}$$

$$a_0 = -4\mu^2, a_2 = b_2 = c_2 = d_2 = -6\mu^2, c_0 = 12\mu^2, V = \pm 1 \tag{26}$$

$$a_0 = c_0 = -\frac{12\mu^2}{16\mu^2-1}, a_2 = b_2 = c_2 = d_2 = -\frac{12\mu^2}{16\mu^2-1}, V = \pm\frac{1}{\sqrt{-16\mu^2+1}}, |\mu| < \frac{1}{4} \tag{27}$$

where μ is left as a free parameter and $a_1 = b_1 = c_1 = d_1 = 0$. Consequently, we obtain the following travelling wave solutions:

$$\begin{aligned} u_1(x,t) &= 6\mu^2 - 6\mu^2 coth^2\mu(x \mp t) \\ w_1(x,t) &= 2\mu^2 - 6\mu^2 coth^2\mu(x \mp t) \end{aligned} \tag{28}$$

$$\begin{aligned} u_2(x,t) &= \frac{2\mu^2}{4\mu^2+1} - \frac{6\mu^2}{4\mu^2+1} coth^2\mu\left(x \mp \frac{1}{\sqrt{4\mu^2+1}}t\right) \\ w_2(x,t) &= \frac{2\mu^2}{4\mu^2+1} - \frac{6\mu^2}{4\mu^2+1} coth^2\mu\left(x \mp \frac{1}{\sqrt{4\mu^2+1}}t\right) \end{aligned} \tag{29}$$

$$\begin{aligned} u_3(x,t) &= 2\mu^2 - 6\mu^2 coth^2\mu(x \mp t) \\ w_3(x,t) &= 6\mu^2 - 6\mu^2 coth^2\mu(x \mp t) \end{aligned} \tag{30}$$

$$\begin{aligned} u_4(x,t) &= -\frac{6\mu^2}{4\mu^2-1} + \frac{6\mu^2}{4\mu^2-1} coth^2\mu\left(x \mp \frac{1}{\sqrt{-4\mu^2+1}}t\right), |\mu| < \frac{1}{2} \\ w_4(x,t) &= -\frac{6\mu^2}{4\mu^2-1} + \frac{6\mu^2}{4\mu^2-1} coth^2\mu\left(x \mp \frac{1}{\sqrt{-4\mu^2+1}}t\right), |\mu| < \frac{1}{2} \end{aligned} \tag{31}$$

$$\begin{aligned} u_5(x,t) &= 6\mu^2 - 6\mu^2 tanh^2\mu(x \mp t) \\ w_5(x,t) &= 2\mu^2 - 6\mu^2 tanh^2\mu(x \mp t) \end{aligned} \tag{32}$$

$$u_6(x,t) = \frac{2\mu^2}{4\mu^2+1} - \frac{6\mu^2}{4\mu^2+1} tanh^2\mu\left(x \mp \frac{1}{\sqrt{4\mu^2+1}}t\right)$$
$$w_6(x,t) = \frac{2\mu^2}{4\mu^2-1} + \frac{6\mu^2}{4\mu^2+1} tanh^2\mu\left(x \mp \frac{1}{\sqrt{4\mu^2+1}}t\right) \tag{33}$$

$$u_7(x,t) = -\frac{6\mu^2}{4\mu^2-1} + \frac{6\mu^2}{4\mu^2-1} tanh^2\mu\left(x \mp \frac{1}{\sqrt{-4\mu^2+1}}t\right), |\mu| < \frac{1}{2}$$
$$w_7(x,t) = -\frac{6\mu^2}{4\mu^2-1} + \left(\frac{6\mu^2}{4\mu^2-1}\right) tanh^2\mu\left(x \mp \frac{1}{\sqrt{-4\mu^2+1}}t\right), |\mu| < \frac{1}{2} \tag{34}$$

$$u_8(x,t) = -\frac{4\mu^2}{16\mu^2+1} - \frac{6\mu^2}{16\mu^2+1}\left\{tanh^2\mu\left(x \mp \frac{1}{\sqrt{16\mu^2+1}}t\right) + coth^2\mu\left(x \mp \frac{1}{\sqrt{16\mu^2+1}}t\right)\right\}$$
$$\begin{aligned} w_8(x,t) = &-\frac{4\mu^2}{16\mu^2+1} \\ &- \frac{6\mu^2}{16\mu^2+1}\left\{tanh^2\mu\left(x \mp \frac{1}{\sqrt{16\mu^2+1}}t\right) + coth^2\mu\left(x \mp \frac{1}{\sqrt{16\mu^2+1}}t\right)\right\} \end{aligned} \tag{35}$$

$$u_9(x,t) = -4\mu^2 - 6\mu^2\{tanh^2\mu(x \mp t) + coth^2\mu(x \mp t)\}$$
$$\begin{aligned} w_9(x,t) = 12\mu^2 - 6\mu^2\{&tanh^2\mu(x \mp t) \\ &+ coth^2\, \mu(x \mp t) \end{aligned} \tag{36}$$

$$u_{10}(x,t) = -\frac{12\mu^2}{16\mu^2-1}\left\{1 + tanh^2\mu\left(x \mp \frac{1}{\sqrt{-16\mu^2+1}}t\right) + coth^2\mu\left(x \mp \frac{1}{\sqrt{-16\mu^2+1}}t\right)\right\}, |\mu| < \frac{1}{4}$$

$$\begin{aligned} w_{10}(x,t) = -\frac{12\mu^2}{16\mu^2-1}\Bigg\{&1 + tanh^2\mu\left(x \mp \frac{1}{\sqrt{-16\mu^2+1}}t\right) \\ &+ coth^2\mu\left(x \mp \frac{1}{\sqrt{-16\mu^2+1}}t\right)\Bigg\}, |\mu| < \frac{1}{4} \end{aligned} \tag{37}$$

The system of higher-order improved Boussinesq equations

The system of higher-order improved Boussinesq equations reads:

$$-\beta u_{xxxxtt} + \alpha u_{xxtt} - u_{tt} + u_{xx} + (uw)_{xx} = 0 \tag{38}$$
$$-\beta w_{xxxxtt} + \alpha w_{xxtt} - w_{tt} + w_{xx} + (uw)_{xx} = 0,$$

where α and β are positive constants, $u(x,t)$ and $w(x,t)$ are unknown functions. Using the wave variable $\xi = x - Vt$ then by integrating twice, considering the constants of integration to be zero we find

$$-\beta V^2 U^{(4)} + \alpha V^2 U'' - V^2 U + U + UW = 0 \tag{39}$$
$$-\beta V^2 W^{(4)} + \alpha V^2 W'' - V^2 W + W + UW = 0.$$

Balancing the linear term of highest order with the nonlinear terms, we find

$$M + 4 \;=\; M + N \tag{40}$$

$$N + 4 \;=\; M + N$$

so that $M = N = 4$. This gives the solutions in the form:

$$U(\mu\xi) = S(Y) = \sum_{k=0}^{4} a_k Y^k + \sum_{k=1}^{4} b_k Y^{-k} \tag{41}$$

$$W(\mu\xi) = S(Y) = \sum_{k=0}^{4} c_k Y^k + \sum_{k=1}^{4} d_k Y^{-k}.$$

Substituting (41) into the two components of (39), and collecting the coefficients of Y gives two systems of algebraic equations for a_i,b_i,$(i = 0,\ldots,4)$, V and μ:

$b_4a_4 - 840V^2a_4\beta\mu^4 = 0, b_4a_4 - 840V^2b_4\beta\mu^4 = 0, b_3a_4 - 360a_3\beta V^2\mu^4 + b_4a_3 = 0,$

$b_3a_4 - 360b_3\beta V^2\mu^4 + b_4a_3 = 0$

$b_2a_4 + b_3a_3 + b_4a_2 + 20V^2a_4\alpha\mu^2 + 2080V^2a_4\beta\mu^4 - 120V^2a_2\beta\mu^4 = 0$

$b_2a_4 + b_3a_3 + b_4a_2 - 120V^2b_2\beta\mu^4 + 20V^2b_4\alpha\mu^2 + 2080V^2b_4\beta\mu^4 = 0$

$b_2a_3 + b_3a_2 + b_4a_1 + b_1a_4 + 12V^2a_3\alpha\mu^2 + 816V^2a_3\beta\mu^4 - 24V^2a_1\beta\mu^4 = 0$

$b_2a_3 + b_3a_2 + b_4a_1 + b_1a_4 + 12V^2b_3\alpha\mu^2 + 816V^2b_3\beta\mu^4 - 24V^2b_1\beta\mu^4 = 0$

$a_4 - V^2a_4 + b_2a_2 + b_3a_1 + b_4a_0 + b_0a_4 + b_1a_3 - 32V^2a_4\alpha\mu^2 - 1696V^2a_4\beta\mu^4 + 6V^2a_2\alpha\mu^2 + 240V^2a_2\beta\mu^4 = 0$

$b_4 - V^2b_4 + b_2a_2 + b_3a_1 + b_4a_0 + b_0a_4 + b_1a_3 + 6V^2b_2\alpha\mu^2 + 240V^2a\beta\mu^4 - 32V^2b_4\alpha\mu^2 - 1696V^2b_4\beta\mu^4 = 0$

$a_3 - V^2a_3 + b_2a_1 + b_3a_0 + b_0a_3 + b_1a_2 - 18V^2a_3\alpha\mu^2 - 576V^2a_3\beta\mu^4 + 2V^2a_1\alpha\mu^2 + 40V^2a_1\beta\mu^4 = 0$

$b_3 - V^2b_3 + b_2a_1 + b_3a_0 + b_0a_3 + b_1a_2 - 18V^2b_3\alpha\mu^2 - 576V^2b_3\beta\mu^4 + 2V^2b_1\alpha\mu^2 + 40V^2b_1\beta\mu^4 = 0$

$a_2 - V^2a_2 + b_2a_0 + b_0a_2 + b_1a_1 + 12V^2a_4\alpha\mu^2 + 480V^2a_4\beta\mu^4 - 8V^2a_2\alpha\mu^2 - 136V^2a_2\beta\mu^4 = 0$

$b_2 - V^2b_2 + b_2a_0 + b_0a_2 + b_1a_1 - 8V^2b_2\alpha\mu^2 - 136V^2b_2\beta\mu^4 + 12V^2b_4\alpha\mu^2 + 480V^2b_4\beta\mu^4 = 0$

$a_1 - V^2a_1 + b_0a_1 + b_1a_0 + 6V^2a_3\alpha\mu^2 + 120V^2a_3\beta\mu^4 - 2V^2a_1\alpha\mu^2 - 16V^2a_1\beta\mu^4 = 0$

$b_1 - V^2b_1 + b_0a_1 + b_1a_0 + 6V^2b_3\alpha\mu^2 + 120V^2b_3\beta\mu^4 - 2V^2b_1\alpha\mu^2 - 16V^2b_1\beta\mu^4 = 0$

$a_0 - V^2a_0 + b_0a_0 - 24V^2a_4\beta\mu^4 + 2V^2a_2\alpha\mu^2 + 16V^2a_2\beta\mu^4 = 0$

$b_0 - V^2b_0 + b_0a_0 + 2V^2b_2\alpha\mu^2 + 16V^2b_2\beta\mu^4 - 24V^2b_4\beta\mu^4 = 0$

(42)

Using Maple or Mathematica we get the following four solutions:

$$a_0 = a_4 = b_4 = \frac{105\alpha^2}{338\beta}, a_2 = b_2 = -\frac{105\alpha^2}{169\beta}, b_0 = \frac{33\alpha^2}{338\beta}, V = \pm 1, \mu = \pm\frac{1}{26}\sqrt{\frac{13\alpha}{\beta}} \tag{43}$$

$$a_0 = b_0 = \frac{33\alpha^2}{2(36\alpha^2+169\beta)}, a_2 = b_2 = -\frac{105\alpha^2}{36\alpha^2+169\beta}, a_4 = b^4 = \frac{105\alpha^2}{36\alpha^2+169\beta}, \tag{44}$$

$$V = \pm 13\sqrt{\frac{\beta}{36\alpha^2+169\beta}}, \mu = \pm\frac{1}{26}\sqrt{\frac{13\alpha}{\beta}}$$

$$a_0 = \frac{33\alpha^2}{338\beta}, a_2 = b_2 = -\frac{105\alpha^2}{169\beta}, a_4 = b_4 = \frac{105\alpha^2}{338\beta}, V = \pm 1, \mu = \pm\frac{1}{26}\sqrt{\frac{13\alpha}{\beta}} \tag{45}$$

$$a_0 = a_4 = b_0 = b_4 = -\frac{105\alpha^2}{2(36\alpha^2-169\beta)}, a_2 = b_2 = \frac{105\alpha^2}{36\alpha^2-169\beta}, \tag{46}$$

$$V = \frac{\pm 13\sqrt{-(36\alpha^2-169\beta)\beta}}{36\alpha^2-169\beta}, \qquad \mu = \pm\frac{1}{26}\sqrt{\frac{13\alpha}{\beta}}, 36\alpha^2 < 169\beta$$

where $a_1 = a_3 = b_1 = b_3 = 0$. As a result we find following solutions:

$$u_1(x,t) = \frac{105\alpha^2}{338\beta}\{1 - 2tanh^2\mu(x \mp t) + tanh^4\mu(x \mp t)\} \tag{47}$$

$$w_1(x,t) = \frac{33\alpha^2}{338\beta} + \frac{105\alpha^2}{338\beta}\{-2tanh^2\mu(x \mp t) + tanh^4\mu(x \mp t)\}$$

$$u_2(x,t) = \frac{33\alpha^2}{2(36\alpha^2+169\beta)} - \frac{105\alpha^2}{36\alpha^2+169\beta}tanh^2\mu\left(x \mp 13\sqrt{\frac{\beta}{36\alpha^2+169\beta}}t\right) + \frac{105\alpha^2}{36\alpha^2+169\beta}tanh^4\mu\left(x \mp 13\sqrt{\frac{\beta}{36\alpha^2+169\beta}}t\right)$$

(48)

$$w_2(x,t) = \frac{33\alpha^2}{2(36\alpha^2+169\beta)} + -\frac{105\alpha^2}{36\alpha^2+169\beta}tanh^2\mu\left(x \mp 13\sqrt{\frac{\beta}{36\alpha^2+169\beta}}t\right) + \frac{105\alpha^2}{36\alpha^2+169\beta}tanh^4\mu\left(x \mp 13\sqrt{\frac{\beta}{36\alpha^2+169\beta}}t\right)$$

$$u_3(x,t) = \frac{33\alpha^2}{338\beta} + \frac{105\alpha^2}{169\beta}\{-tanh^2\mu(x \mp t) + tanh^4\mu(x \mp t)\} \tag{49}$$

$$w_3(x,t) = \frac{33\alpha^2}{338\beta} + \frac{105\alpha^2}{169\beta}\{-tanh^2\mu(x \mp t) + tanh^4\mu(x \mp t)\}$$

$$u_4(x,t) = -\frac{105\alpha^2}{2(36\alpha^2-169\beta)} + \frac{105\alpha^2}{36\alpha^2-169\beta}tanh^2\mu\left(x \mp 13\frac{\sqrt{(-36\alpha^2+169\beta)\beta}}{36\alpha^2-169\beta}t\right) - \frac{105\alpha^2}{2(36\alpha^2-169\beta)}tanh^4\mu\left(x \mp 13\frac{\sqrt{(-36\alpha^2+169\beta)\beta}}{36\alpha^2-169\beta}t\right), 169\beta > 36\alpha^2 \tag{50}$$

$$w_4(x,t) = -\frac{105\alpha^2}{2(36\alpha^2-169\beta)} + \frac{105\alpha^2}{36\alpha^2-169\beta}tanh^2\mu\left(x \mp 13\frac{\sqrt{(-36\alpha^2+169\beta)\beta}}{36\alpha^2-169\beta}t\right) - \frac{105\alpha^2}{2(36\alpha^2-169\beta)}tanh^4\mu\left(x \mp 13\frac{\sqrt{(-36\alpha^2+169\beta)\beta}}{36\alpha^2-169\beta}t\right), 169\beta > 36\alpha^2$$

Moreover, we obtained some complex solutions and will not be considered these solutions in this work.

CONCLUSION

In this article, with aid of the Maple and Mathematica, the tanh--coth method has been successfully implemented to find new traveling wave solutions for two Sobolev type system of equations, namely, the system of improved Boussinesq equations and the system of higher-order improved Boussinesq equations and exact solutions are obtained. The solutions show that the tanh-coth method is a powerful mathematical tool for solving Sobolev type partial differential equations.

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