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A METHOD FOR SOLVING GENERAL SINGULAR EQUATIONS AND ITS APPLICATION

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In memory of Z.I.Khalilov

ABSTRACT. In this paper we present a new method for solving general singular equations of normal type. As an application of this method, we give a solution for a class of convolution-type integral equations.

1. INTRODUCTION

The fundamentals of the theory of singular integral equations of the type described were included in the work of H. Poincaré and D. Hilbert, almost directly after the development of the classical theory of integral equations by I. Fredholm. On the other hand many problems of an applied character naturally reduced to singular equations, e.g. problems of the theory of elasticity, etc.; thus often in practice these equations were arrived at by "ordinary methods" and this did not always lead to satisfactory results. However, the theory of singular integral equations has advanced considerably. This theory appears to be particularly simple and effective, if the solution of a boundary problem of the theory of functions of a complex variable, to be called as the Riemann problem, is considered. Therefore the theory of singular equations is here closely linked with the above boundary problem. The solution of the latter is used for the development of the theory of singular equations; afterwards this theory is applied to the solution of other more complicated boundary problems, in particular, to problems encountered in potential theory, the theory of elasticity and in hydromechanics. Having in mind the implications for different problems of mathematical physics, some restrictions is imposed upon the unknown and the given functions appearing in the integral equations under consideration or in the boundary conditions of the problems considered, which is largely simplify the investigation, but not affect the final theory. A number of important

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properties of singular integral equations were established by F. Noether. Studies by T. Carleman and I.I. Privalov were of great importance in the development of the theory of singular integral equations. The most complete results have been obtained by such Georgian mathematicians as N.I. Muskhelishvili [9], I.N. Vekua [13], and B.V. Khvedelidze [8].

The first consideration of abstract equations with an operator satisfying some conditions was undertaken by Z.I. Khalilov [7] with in the framework of normed rings. The theory developed there was a direct treatment of the theory of singular integral equations with continuous coefficients

$$a_1(t)\varphi(t) + \frac{a_2(t)}{\pi i}\int \frac{\varphi(\tau)d\tau}{\tau - t} + \int k(t,\tau)\varphi(\tau)d\tau = f(t) \tag{1.1}$$

with in the framework of an abstract normed ring. He consider this equation as an operator equation and defined the general singular equation of the form

$$M\varphi = A_1\varphi + A_2 S\varphi + T\varphi = f, \tag{1.2}$$

under this abstract view. Here φ is the solution function, f is the element of a Banach space, and the operators A_1, A_2, T, S are mentioned below. He also gave the regularization and solvability of these equations in view of the known results for some integral equations. A significant step in the abstract theory was made by Yu.I. Cherskii [2]. He presented a method of solving general singular equations with the help of analogous to the Riemann boundary value problem and applied his method to the general theory to singular integral equations of convolution type, less extensive cases of which are considered by the author in previous papers (see [4]). Thus he considered, as an instance of (1.2), operators A of the form

$$A\varphi = \lambda\varphi(x) + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} a(x-t)\varphi(t)dt \tag{1.3}$$

where $a(x) \in L^2(-\infty,\infty)$, $\varphi(x) \in L^2(-\infty,\infty)$, T is a compact operator and $S\varphi = \mathrm{sgn}x \cdot \varphi(x)$. With additional restrictions on $a(x)$, explicit solutions of (1.2) were obtained. In this paper we consider a different method for solving (1.2). Our method based on the connection between general singular equations and an analogous to the Riemann boundary value problem with continuous coefficient in view of Vekua's method [13] to regularize of (1.1). As an application of our method, we present an explicit solution for a class of convolution type integral equation containing (1.3). The principal advantages of our method are its simplicity and the validity for many classes of integral equations.

2. Preliminaries

In this section we recall some basic definitions and results which will be needed for our method. No proofs or specific details are given—readers can find all necessary details in [7, 2, 10].

Let $B(X)$ denote the space of all bounded linear operators on a Banach space X, and R be a subalgebra of $B(X)$.

Definition 1. *A set $F \subset B(X)$ is called a* regular class *if the following conditions are satisfied:*

(i) *For each $T \in F$, $(I+T)$ satisfies Riesz–Schauder's theorems, where I is the identity operator (for Riesz–Schauder's theorems, see* [14, p. 283]*).*
(ii) *If $A \in R$ and $T, T_1 \in F$, then $AT, TA, T+T_1, TT_1 \in F$.*

Definition 2. *An operator $S \in B(X)$ is called a* singular operator *if the following conditions are satisfied:*

(i) $S^2 = I$;
(ii) $S \neq \mp I$;
(iii) *If $T \in F$, then $ST, TS \in F$;*
(iv) *If $A \in R$, then $(SA - AS) \in F$.*

Definition 3. *An operator of the form*

$$M = A_1 + A_2 S + T, \tag{2.1}$$

where $A_1, A_2 \in R$, S is a singular operator, and $T \in F$, is called a general singular operator *on X.*

The operator

$$M^* = A_1^* + S^* A_2^* + T^*, \tag{2.2}$$

is called *an adjoint general singular equation* on the adjoint space X^*, where A_1^*, A_2^*, S^*, T^* are operators on X^* adjoint respectively to A_1, A_2, S, T. Remark that the class F^* of all T^* where $T \in F$, the operators S^* and M^* have the same properties as F, S, M.

The operator (2.1) is called a *normal type* if $(A_1 + A_2)^{-1}$ and $(A_1 - A_2)^{-1}$ exist and belong to R. It is clear that if the operator (2.1) is of normal type, then so is the adjoint operator (2.2).

The equation

$$M\varphi = A_1\varphi + A_2 S\varphi + T\varphi = f, \tag{2.3}$$

where φ is the solution function and $f \in X$, is called a *general singular equation.* If M is an operator of normal type, then (2.3) is also said to be of normal type. An equation of the form

$$M^o\varphi = A_1\varphi + A_2 S\varphi = f \tag{2.4}$$

is called the *characteristic equation* of (2.3), and

$$M^*\psi = \overline{f} \tag{2.5}$$

is called the *adjoint singular equation* of (2.3). Khalilov [7] generalized Noether's theorems, which are known for integral equations, to the general singular equations of normal type:

(i) The number of linearly independent solutions of $M\varphi = 0$ and $M^*\psi = 0$ is finite.
(ii) In order for solutions of $M\varphi = f$ (respectively, $M^*\psi = \overline{f}$) to exist, it is necessary and sufficient that $\psi_0(f) = 0$ for every solution ψ_0 of $M^*\psi = 0$ (respectively, $\overline{f}(\varphi_0) = 0$ for every solution φ_0 of $M\varphi = 0$).
(iii) The number of linearly independent solutions of $M\varphi = 0$ minus the number of linearly independent solutions of $M^*\psi = 0$ depends only on A_1, A_2, S.

The fundamental example for the general singular equations of normal type is the singular equation with Cauchy kernel has of the form

$$M\varphi \equiv a(t)\varphi(t) + b(t)S\varphi(t) + T\varphi(t) = f(t), \quad t \in \Gamma,$$

where $a(t)$ and $b(t)$ are continuous functions on a closed Lyapunov curve Γ. The so-called singular integral

$$S\varphi \equiv \frac{1}{\pi i}\int_{\Gamma} \frac{\varphi(\tau)}{\tau - t}\mathrm{d}\tau, \quad t \in \Gamma$$

is understood in the sense of the Cauchy principle value. The reader is referred to [3, 9, 10] for the fundamental theory of this type equation.

We now recall the method of Cherskii [2] for a solution of the general singular equation of normal type:

Let X_+, X_- be the subspaces of X consisting of the elements $\varphi_+, \varphi_- \in X$ satisfying

$$\varphi_+ - S\varphi_+ = 0, \quad \varphi_- + S\varphi_- = 0,$$

respectively. Similarly, let X^*_+, X^*_- be the subspaces of X^* such that the elements $\overline{\varphi}_+, \overline{\varphi}_-$ satisfying

$$\overline{\varphi}_+ + S^*\overline{\varphi}_+ = 0, \quad \overline{\varphi}_- - S^*\overline{\varphi}_- = 0,$$

respectively. It is clear that $X_+ \cap X_- = \{0\}$ and $X^*_+ \cap X^*_- = \{0\}$. It is easy to see that, for all $\varphi \in X$, there exists a unique $\varphi_+ \in X_+$ and $\varphi_- \in X_-$ such that the equalities

$$\varphi = \varphi_+ - \varphi_-, \quad S\varphi = \varphi_+ + \varphi_- \tag{2.6}$$

are satisfied. Let Ψ_+ and Ψ_- be invertible operators in $B(X)$ with
(i) $\Psi_+\varphi_+ \in X_+$ and $\Psi_+^{-1}\varphi_+ \in X_+$;
(ii) $\Psi_-\varphi_- \in X_-$ and $\Psi_-^{-1}\varphi_- \in X_-$.

and let U be an invertible operator in $B(X)$ so that
(i) $U\varphi_+ \in X_+$ for all $\varphi_+ \in X_+$;
(ii) $U^{-1}\varphi_- \in X_-$ for all $\varphi_- \in X_-$;
(iii) There exists a unique $h_+ \in X_+$ with respect to the scalar multiplication such that $h_+ \neq 0$ and $U^{-1}h_+ \in X_+$;
(iv) There exists a unique $\overline{h}_- \in X^*_-$ with respect to the scalar multiplication such that $\overline{h}_- \neq 0$ and $U^*\overline{h}_- \in X^*_+$.

The following theorems are fundamental for both Cherskii's solution scheme and our method.

Theorem 1. [2, p.281, Theorem 4] *For $\kappa > 1$, the elements $h_+, Uh_+, \ldots, U^{\kappa-1}h_+$ are linearly independent.*

Theorem 2. [2, p.281, Theorem 5] *Let φ_+ be of the form*

$$\varphi_+ = U^\kappa \varphi_- + f_-, \quad \kappa > 0.$$

Then,

$$\varphi_+ = \sum_{k=0}^{\kappa-1} c_k U^k h_+,$$

where the c_k's are arbitrary constants.

Definition 4. *If an operator $A \in B(X)$ has the representation*

$$A = \Psi_+ U^\kappa \Psi_-^{-1},$$

then the number $\kappa \in \mathbb{Z}$ is called the index *of A, and is denoted by $indA$. It is shown that the index of the adjoint operator A^* is equal in magnitude and opposite in sign.*

In view of this definition and Equation (2.6), the characteristic equation (2.4) is reduced to analogous to the Riemann boundary value problem:

$$\Psi_+^{-1}\varphi_+ - g_+ = U^\kappa \Psi_-^{-1}\varphi_- - g_-, \tag{2.7}$$

where $\kappa = \text{ind}\{(A_1 + A_2)^{-1}(A_1 - A_2)\}$, $g_+ = (1/2)(\Psi_+^{-1}g + S\Psi_+^{-1}g)$ and $g_- = (1/2)(-\Psi_+^{-1}g + S\Psi_+^{-1}g)$. Solving the characteristic equation (2.4) is then equivalent to solving Equation (2.7). A solution of the characteristic equation (2.4) as follows:

(i) If $\kappa = 0$, then since $X_+ \cap X_- = \{0\}$, Equation (2.4) has the unique solution:

$$\varphi_+ = \Psi_+ g_+, \quad \varphi_- = \Psi_- g_-.$$

If the equation is homogeneous, i.e., $f = 0$, then it has the trivial solution $\varphi = 0$.

(ii) If $\kappa > 0$, then in view of Theorem (2), we can obtain

$$\varphi_+ = \Psi_+ \left(g_+ + \sum_{k=0}^{\kappa-1} c_k U^k h_+ \right), \quad \varphi_- = \Psi_- U^{-\kappa} \left(g_- + \sum_{k=0}^{\kappa-1} c_k U^k h_+ \right)$$

i.e., the equation has κ linearly independent solutions.

(iii) If $\kappa < 0$, then the solution is

$$\varphi_+ = \Psi_+ g_+, \quad \varphi_- = \Psi_- U^{-\kappa} g_-.$$

However, φ_- is in X_- if and only if $\Psi_- U^{-\kappa} g_-$ is in X_-. Therefore, a necessary and sufficient condition for the characteristic equation (2.4) to be solvable is that

$$(I + S)U^{-\kappa} g_- = 0.$$

Under the this condition, Equation (2.4) has a unique solution. Moreover, if the equation is homogeneous, then it has the trivial solution $\varphi = 0$.

Note that the solution of the adjoint of (2.4)

$$(M^o)^*\overline{\varphi} = A_1^*\overline{\varphi} + S^* A_2^*\overline{\varphi} = \overline{f} \tag{2.8}$$

was found in a similar fashion in Cherskii's paper [2].

3. A New Method for Solving General Singular Equations

In this section we present a different method from that of Cherskii for a solution of the general singular equations of normal type (2.3).

We follow the terminology and notation of Section 1.

We start to construct the following operators via general singular operator of normal type (2.1) and its adjoint operator (2.2).

$$M_1 = \frac{1}{2}[(A_1 + A_2)^{-1} + U^\kappa (A_1 - A_2)^{-1}] + \frac{1}{2}[(A_1 + A_2)^{-1} - U^\kappa (A_1 - A_2)^{-1}]S + T_1,$$

$$M_2 = \frac{1}{2}[(A_1^* + A_2^*)^{-1} + (U^{-\kappa})^* (A_1^* - A_2^*)^{-1}] + \frac{1}{2}[(A_1^* + A_2^*)^{-1} - (U^{-\kappa})^* (A_1^* - A_2^*)^{-1}]S^* + T_2,$$

where $\kappa \in \mathbb{Z}$ and U is an operator defined in Section 1.

Definition 5. *The number $\kappa \in \mathbb{Z}$ is said to be an* index *of the general singular operator M of normal type if the homogeneous equations $M_1\varphi = 0$ and $M_2\psi = 0$ have only the trivial solution, i.e., $Ker M_1 = \{0\}$ and $Ker M_2 = \{0\}$.*

Lemma 1. *If $Ker M_1 = \{0\}$, i.e., the homogeneous equation $M_1\varphi = 0$ has only trivial solution ($\varphi = 0$), then the equations $M\varphi = f$ and*

$$M_1 M\varphi = \frac{1}{2}(I + S)\varphi + \frac{1}{2}U^\kappa (I - S)\varphi + T_3\varphi = M_1 f \tag{3.1}$$

have the same solutions. Similarly, if $Ker M_2 = \{0\}$, i.e., the homogeneous equation $M_2\psi = 0$ has only the trivial solution ($\psi = 0$), then the equations $M^*\psi = \overline{f}$ and

$$M_2 M^*\psi = \frac{1}{2}(I + S^*)\psi + \frac{1}{2}(U^{-\kappa})^*(I - S^*)\psi + T_4\psi = M_2\overline{f} \tag{3.2}$$

have the same solutions.

The proof is simple.

As a result of Lemma (1),Solving Equation (2.3) is then equivalent to solving (3.1). Using the equalities (2.6), we show that the characteristic equation

$$(M_1 M)^o\varphi = \frac{1}{2}(I + S)\varphi + \frac{1}{2}U^\kappa (I - S)\varphi = g, \tag{3.3}$$

where $g = M_1 f$, is equivalent to the equation

$$\varphi_+ = U^\kappa \varphi_- + g, \tag{3.4}$$

which has a solution according to [11]. The following equation is then easily obtained:

$$\varphi_+ - g_+ = U^\kappa \varphi_- - g_-, \tag{3.5}$$

where $g_+ = \frac{1}{2}(g+Sg)$ and $g_- = \frac{1}{2}(-g+Sg)$. Three cases are considered, according to $\kappa < 0,\ \kappa = 0,\ \kappa > 0$

(i) In case of $\kappa = 0$, since $X_+ \cap X_- = \{0\}$, (3.5) implies that (3.3) has the unique solution:

$$\varphi_+ = g_+, \quad \varphi_- = g_-. \tag{3.6}$$

Moreover, if the equation is homogeneous, i.e., $g = 0$, then it has only the trivial solution.

(ii) In case of $\kappa > 0$, Theorem (2) gives us the following solution:

$$\varphi_+ = g_+ + \sum_{k=0}^{\kappa-1} c_k U^k h_+, \qquad \varphi_- = U^{-\kappa}\left(g_- + \sum_{k=0}^{\kappa-1} c_k U^k h_+\right), \tag{3.7}$$

where c_k's are arbitrary constants. Thus,(3.3) has κ linearly independent solutions.

(iii) In case of $\kappa < 0$, we obtain the solution

$$\varphi_+ = g_+, \quad \varphi_- = U^{-\kappa} g_- \tag{3.8}$$

from (3.5). It is clear that φ_- is in X_- if and only if $U^{-\kappa} g_-$ is in X_-. It follows that a necessary and sufficient condition for (3.3) to be solvable is that there exists some $\psi \in X$ such that

$$U^{-\kappa} g_- = \frac{1}{2}(-\psi + S\psi).$$

Under this condition, (3.3) has a unique solution. If the equation is homogeneous, it has only the trivial solution.

Similarly, we note that the solution of $(M_2M^*)^o\psi = l$ is found, where $l = M_2\overline{f}$. Finally, the results deduced above are given as the following theorem.

Theorem 3. *The general solutions of the characteristic equations* $(M_1M)^o\varphi = g$ *and* $(M_2M^*)^o\psi = l$ *are as follows:*

(i) *If* $\kappa = 0$, *then the equations* $(M_1M)^o\varphi = g$ *and* $(M_2M^*)^o\psi = l$ *have a unique solution. In case of the equations being homogeneous, then they have only the trivial solution.*

(ii) *If* $\kappa > 0$, *then the equation* $(M_1M)^o\varphi = g$ *has exactly* κ *linearly independent solutions. A necessary and sufficient condition for the solvability of* $(M_2M^*)^o\psi = l$ *is that there exists some* $\xi \in X^*$ *such that*

$$(U^{-\kappa})^* l_+ = \frac{1}{2}(\xi + S^*\xi).$$

Under this condition, the equation has a unique solution.

(iii) *If $\kappa < 0$, then the equation $(M_1 M)^o \varphi = g$ is solvable if and only if there exists some $\psi \in X$ such that*

$$U^{-\kappa} g_- = \frac{1}{2}(-\psi + S\psi).$$

Provided this condition is satisfied, the equation has a unique solution. If the equation is homogeneous, it has only the trivial solution. However, the equation $(M_2 M^*)^o \psi = l$ has exactly $(-\kappa)$ linearly independent solutions.

Remark 1. *One may ask whether our index is the same as Cherskii's. It is easily seen that both definitions are different but the indexes are the same in magnitude and as a number. In addition, the elements $\Psi_+, \Psi_+^{-1}, \Psi_-, \Psi_-^{-1}, U, \kappa$, and h_+ are necessary for Cherskii's solution scheme but for our method, it is only needed U, κ, and h_+ to solve the general singular equation of normal type. Thus, we can say that our method is simpler than Cherskii's method.*

4. A Solution for a Class of Convolution-Type Integral Equations

In this section, we apply our method to obtain a solution for a class of convolution-type singular integral equations. For consistency, let $L^2 = L^2(-\infty, +\infty)$ and $L^\infty = L^\infty(-\infty, +\infty)$.

Let X (defined in Section 1) be the space L_2 and $V\varphi \equiv \Phi(x)$ denote the Fourier transform of a function $\varphi \in L^2$. Let R, the subalgebra of X, be a set of operators A of the form

$$A\varphi \equiv \lambda\varphi(x) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} a(x-t)\varphi(t)\mathrm{d}t, \tag{4.1}$$

where $a(x)$ is in L^2 and $Va = \Theta(x)$ is a continuous and bounded function in L^2. As we indicated in the first section, Cherskii [2] applied his method to this type of integral equations with Va satisfies the Hölder condition. But our method is valid for a larger class of convolution-type integral equations-the case of Va is a continuous and bounded function.

By using well-known properties of the Fourier transforms in L^2 (see [6, p.445]), we obtain the following equality on the Fourier transform of $A\varphi \in L^2$.

$$V(A\varphi) \equiv \lambda V\varphi(x) + Va(x) \cdot V\varphi(x) = \lambda\Phi(x) + \Theta(x) \cdot \Phi(x) \tag{4.2}$$

In view of this equality, we have a necessary and sufficient condition for A being continuous which is that Va is in L^∞ from [5, Problem 53]. In addition, since V is a unitary operator, we obtain

$$||A|| \leq |\lambda| + ||Va||_\infty. \tag{4.3}$$

Let the regular class F be a collection of all compact operators T on L^2 and the singular operator S be

$$S\varphi \equiv \mathrm{sgn}x \cdot \varphi(x).$$

Let us first show that S is a singular operator:

(i) $S^2\varphi \equiv (\mathrm{sgn}x)^2 \cdot \varphi(x) = \varphi(x)$;

(ii) $S \neq \pm I$;

(iii) $TS \in F$, $ST \in F$ for all $T \in F$;

(iv) Suppose $A \in R$. Then

$$(SA - AS)\varphi \equiv \mathrm{sgn}x\left[\lambda\varphi(x) + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} a(x-t)\varphi(t)\mathrm{d}t\right]$$
$$-\left[\lambda\varphi(x) + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} a(x-t)\mathrm{sgn}t\varphi(t)\mathrm{d}t\right]$$

From [12, p.120], we have that

$$V(\mathrm{sgn}x\varphi(x)) \equiv \frac{1}{\pi i}\int_{-\infty}^{+\infty} \frac{V\varphi}{t-x}\mathrm{d}t,$$

and then we find that

$$(SA - AS)\varphi = V^{-1}TV\varphi,$$

where

$$T\varphi \equiv \frac{1}{\pi i}\int_{-\infty}^{+\infty} [\Theta(t) - \Theta(x)]\frac{\varphi(t)}{t-x}\mathrm{d}t.$$

If we take the operators

$$A_i\varphi \equiv \lambda_i\varphi(x) + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} a_i(x-t)\varphi(t)\mathrm{d}t \quad (i = 1, 2)$$

as the coefficients of the operator (2.1), then the general singular equation (2.3) becomes the following convolution-type integral equation of the form

$$M\varphi \equiv (\lambda_1 + \lambda_2\mathrm{sgn}x)\varphi(x) + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} a_1(x-t)\varphi(t)\mathrm{d}t \tag{4.4}$$
$$+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} a_2(x-t)\mathrm{sgn}t\varphi(t)\mathrm{d}t + T\varphi(x) = f(x)$$

where $f(x)$ is in L^2 and T is a compact operator on L^2.

From inequality (4.3),

$$||A_i|| \leq |\lambda_i| + ||\Theta_i||_\infty \ (i = 1, 2),$$

where $\Theta_i(x) = Va_i$ $(i = 1, 2)$ holds. This leads to the following inequality

$$||M^o|| \leq |\lambda_1| + |\lambda_2| + ||\Theta_1||_\infty + ||\Theta_2||_\infty. \tag{4.5}$$

To solve Equation (4.4), we need the operators $(A_1 + A_2)^{-1}$, $(A_1 - A_2)^{-1}$, U, the function h_+, and the index κ according to our method. First of all, let us find

$(A_1)^{-1}$ and then it is similarly obtained $(A_1+A_2)^{-1}$ and $(A_1-A_2)^{-1}$. From [12], we have

$$VA_1A_2\varphi \equiv [\lambda_1+\Theta_1(x)][\lambda_2+\Theta_2(x)]\Phi(x).$$

It follows that a necessary and sufficient condition for the existence of $(A_1)^{-1}$ in R is that $\lambda_1+\Theta_1(x)$ does not vanish. In fact, in the case of $|\lambda_1+\Theta_1(x)|>0$, taking $\Theta_2(x) = -\lambda_1^{-1}\Theta_1(x)[\lambda_1+\Theta_1(x)]^{-1}$ and $\lambda_2=\lambda_1^{-1}$, we find that

$$[\lambda_1+\Theta_1(x)][\lambda_2+\Theta_2(x)] \equiv 1$$

and then, by Cherskii's paper [2], we have

$$A_1^{-1}\varphi \equiv \frac{\varphi(x)}{\lambda_1} + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} b(x-t)\varphi(t)\mathrm{d}t,$$

where $b(x) = V^{-1}\{-\lambda_1^{-1}\Theta_1(x)[\lambda_1+\Theta_1(x)]^{-1}\}$. Since $\Theta_2(x)$ (defined above) is continuous and bounded in L^2, it is easy to see that A_1^{-1} is in R. Hence, general singular equation (4.4) is of normal type.

Let X_+ and X_- be collections of all elements $\varphi \in L^2$ such that $\varphi(x)=0$ when $x<0$ and $x>0$, respectively. As in [2], we choose

$$U\varphi \equiv \varphi(x) - 2\mathrm{e}^{-x}\int_{-\infty}^{x} \mathrm{e}^{t}\varphi(t)\mathrm{d}t,$$

where $\varphi \in L^2$, and

$$h_+ = (1+\operatorname{sgn} x)\mathrm{e}^{-x}.$$

The validity of conditions (i)–(iv) in Section 1 is easily verified.

Suppose that $a_i(x)$ $(i=1,2)$ are functions such that $\Theta_i(x)$ is in the L^∞-closure of the set where its elements satisfy the Hölder condition in L^2. Therefore, for any given $\varepsilon>0$, there exist functions $a'_{i,\varepsilon} \in L^2$ $(i=1,2)$ corresponding to $a_i(x)$ $(i=1,2)$ such that $\Theta'_{i,\varepsilon}(x) = Va'_{i,\varepsilon}$ satisfy the Hölder condition, and

$$||\Theta_i - \Theta'_{i,\varepsilon}||_\infty < \frac{\varepsilon}{2} \ (i=1,2).$$

Hence, the following general singular equation is formed via the functions $a'_{i,\varepsilon}(x)$:

$$M'\varphi \equiv A'_1\varphi(x) + A'_2S\varphi(x) + T_1\varphi(x) = f'(x), \tag{4.6}$$

where T_1 is a compact operator, and the coefficients of the equation have the form

$$A'_i\varphi \equiv \lambda_i\varphi(x) + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} a'_{i,\varepsilon}(x-t)\varphi(t)\mathrm{d}t \ (i=1,2).$$

the Equation (4.6) was solved by Cherskii in [2], and he gave the index of M' as the index of the function $1+\Theta'(x)$, where $\Theta'(x) = Va'$. Here, $a'(x)$ is a function corresponding to $A' = (A'_1 + A'_2)^{-1}(A'_1 - A'_2)$ of the form

$$A'\varphi \equiv \lambda'\varphi(x) + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} a'(x-t)\varphi(t)\mathrm{d}t.$$

It follows that $\Theta'(x) = 2(\lambda_2\Theta_1(x) - \lambda_1\Theta_2(x))$ holds. Using inequality (4.3), we can obtain

$$||M^o - (M'^o|| \leq ||\Theta_1 - \Theta'_1||_\infty + ||\Theta_2 - \Theta'_2||_\infty < \varepsilon.$$

It follows that

$$\mathrm{ind}M = \mathrm{ind}M' = \mathrm{ind}(1 + \Theta'(x))$$

from [1, Theorem 4] and Remark (1). A more detailed account is given by Helemskii [6, Chapter 3]. Eventually, by Theorem (3), we have a solution of (4.4) using U, h_+, κ obtained above.

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