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Paracompactness in Multiset Topological Spaces

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Abstract
In this paper, we introduce the concept of paracompactness in multiset topological spaces. We give some useful results in m-paracompact m-topological spaces.

Keywords: *multiset, m-topological spaces, paracompactness, m-paracompact m-topological spaces.*

1. Introduction

Multi-set theory was introduced by Cerf et al. (1971) and then Peterson (1976), Yager (1986) and Jena (2001) made contribution to the theory further. Blizzard (1991) brought multi-set theory a new perspective and formalized the theory. Girish and Jacob (2012), introduced m-topology for multi-sets. El-Sheikh et al. (2015) introduced separation axioms for multi-set topological spaces. Tantawy et al. (2015) studied the concept of connectedness for multi-set topological spaces. Mahanta and Samanta (2017) studied the concept of compactness for multi-set topological spaces.

2. Preliminaries

We give some basic definitions (Girish and Jacob, 2012; Sobhy et al., 2015; Mahanta and Samanta, 2017).

Definition 1. Let $C_M : X \to \mathbb{N}$ a function where X is a set and $\mathbb{N}$ the set of non-negative integers.

$$M := \{C_M(x)/x : x \in X, C_M(x) > 0\}$$

is called a multiset (or mset) drawn from X.

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A mset M drawn from a set X is said to be an empty mset, denoted by ϕ, if $C_M(x) = 0$ for every x in X.

Notation 1. It is denoted by $x \in^n M$ the fact that M is a mset drawn for a set X and x appears n times in M.

Definition 2. The support set of a mset M drawn from a set X, denoted by M^*, is defined by

$$\{x \in X : C_M(x) > 0\}.$$

Notation 2. $[M]_x$ denotes that x belongs to the M^*, and $|[M]_x|$ denotes the appearing number of x in M.

Definition 3. The set

$$[X]^m := \{M : M \text{ is a mset drawn from } X \text{ and } \forall x \in X, C_M(x) \le m\}$$

is called the multiset (or mset) space.

Definition 4. Let $M, N \in [X]^m$.
1. $M = N$ if, for every x in X, $C_M(x) = C_N(x)$ (mset equality condition),
2. $M \subseteq N$ if, for every x in X, $C_M(x) \le C_N(x)$ (submset condition),
3. $M \cup N$ is defined by $C_{M\cup N}(x) := \max\{C_M(x), C_N(x)\}$ for every x in X (mset union),
4. $M \cap N$ is defined by $C_{M\cap N}(x) := \min\{C_M(x), C_N(x)\}$ for every x in X (mset intersection),
5. $M \oplus N$ is defined by $C_{M\oplus N}(x) := \min\{m, C_M(x) + C_N(x)\}$ for every x in X (mset addition)
6. $M \oplus N$ is defined by $C_{M\ominus N}(x) := \max\{0, C_M(x) - C_N(x)\}$ for every x in X (mset subtraction).

Definition 5. Let $M \in [X]^m$. The (absolute) complement of M is the mset M^c where $C_{M^c}(x) := m - C_M(x)$ for every x in X.

Definition 6. Let $M \in [X]^m$. The power mset of M denoted by $P(M)$ is defined by

$$C_{P(M)}(N) := \begin{cases} 1 & N = \phi \\ \prod_{x \in N^*} \binom{|[M]_x|}{|[N]_x|} & N \neq \phi \end{cases}$$

where N is a submset of M.

The power set of a mset M, denoted by $P^*(M)$ is the support set of the power mset $P(M)$.

Definition 7. Let $M \subseteq [X]^m$, that is, $\mathcal{M}$ be a collection of msets in $[X]^m$, and $M^* = \{M^* : M \in \mathcal{M}\}$.
1. $\bigcup \mathcal{M}$ is defined by $C_{\bigcup \mathcal{M}}(x) := \max\{C_M(x) : M \in \mathcal{M}\}$ for every x in X (generalized mset union),
2. $\bigcap \mathcal{M}$ is defined by $C_{\bigcap \mathcal{M}}(x) := \min\{C_M(x) : M \in \mathcal{M}\}$ for every x in X (generalized mset intersection),
3. $\bigoplus \mathcal{M}$ is defined by $C_{\bigoplus \mathcal{M}}(x) := \min\{m, \sum_{M \in \mathcal{M}} C_M(x)\}$ for every x in X (generalized mset addition).

Definition 8. Let $M \in [X]^m$ and $\tau \subseteq P^*(M)$. τ is called a multiset topology (or m-topology) on M, an ordered pair (M, τ) a multiset topological space (or m-topological space) if τ satisfies the following conditions:
1. $\emptyset, M \in \tau$,
2. For every $\mathcal{G} \subseteq \tau$, $\bigcup \mathcal{G} \in \tau$,
3. For every finite $\mathcal{G} \subseteq \tau$, $\bigcap \mathcal{G} \in \tau$.

Let (M, τ) be a m-topological space. Each mset $G \in \tau$ is called an open mset of M.

Definition 9. Let $M \in [X]^m$, (M, τ) be a m-topological space. A submset N of M with m-topology

$$\tau_N := \{N \cap U : U \in \tau\}$$

is called a subspace of M.

Definition 10. Let $M \in [X]^m$ and (M, τ) be a m-topological space. A submset $N \subseteq M$ is called a closed submset if $M \ominus N$ is an open mset.

Theorem 1. Let $M \in [X]^m$ and (M, τ) be a m-topological space. The followings hold:
1. The msets M, $\emptyset$ are closed msets.
2. The intersection of arbitrarly many closed submsets of M is a closed mset.
3. The union of finitely many closed submsets of M is a closed mset.

Definition 11. Let $M \in [X]^m$ and (M, τ) be a m-topological space. A neighborhood of a mset $A \subseteq M$ is a submset N of M such that there exists an open mset U such that $A \subseteq U \subseteq N$. A neighborhood of an element $x \in^k M$ is a submset N of M such that there exists an open mset U such that $x \in^k U \subseteq N$.

Also, a neighborhood is called an open neighborhood if it belongs to τ.

Definition 12. Let $M \in [X]^m$, $A \subseteq M$ and (M, τ) be a m-topological space.
1. The interior of A, denoted by $Int(A)$, is defined by

$$C_{Int(A)}(x) := \max\{C_G(x) : G \text{ is open mset and } G \subseteq A\} \text{ for every } x \in X,$$

or equivalently,

$$C_{Int(A)}(x) := C_{\bigcup\{C_G(x) : G \text{ is open mset and } G \subseteq A\}} \text{ for every } x \in X,$$

2. The closure of A, denoted by $Cl(A)$, is defined by

$$C_{Cl(A)}(x) := \min\{C_K(x) : K \text{ is closed mset and } A \subseteq K\} \text{ for every } x \in X,$$

or equivalently,

$$C_{Cl(A)}(x) := C_{\bigcap\{C_K(x) : K \text{ is closed mset and } A \subseteq K\}} \text{ for every } x \in X,$$

3. An element of $k/x \in M$ is called a limit point of an mset A if every neighborhood of k/x intersects A in some point with non-zero multiplicity other than k/x itself. We denote the mset of all limit points of A by A'.

Theorem 2. Let $M \in [X]^m$, $A \subseteq M$, $x \in^k M$ and (M, τ) be a m-topological space. Then $x \in^k Cl(A)$ if and only if every open mset U containing k/x intersects A.

Theorem 3. Let $M \in [X]^m$, $A, B \subseteq M$ and (M, τ) be a m-topological space. Then the following properties hold: $\forall x \in X$,
1. $C_A(x) \leq C_B(x) \Rightarrow C_{Int(A)}(x) \leq C_{Int(B)}(x)$,
2. $C_A(x) \leq C_B(x) \Rightarrow C_{Cl(A)}(x) \leq C_{Cl(B)}(x)$,
3. $C_{Int(A \cap B)}(x) = \min\{C_{Int(A)}(x), C_{Int(B)}(x)\}$,
4. $C_{Cl(A \cup B)}(x) = \max\{C_{Cl(A)}(x), C_{Cl(B)}(x)\}$.

Definition 13. Let $M \in [X]^m$. A collection $\mathcal{C} \subseteq P^*(M)$ is said to cover M, or to be a cover of M if, $\forall x \in X$,

$$C_M(x) \leq C_{\bigcup \mathcal{C}}(x).$$

Definition 14. Let $M \in [X]^m$, $\mathcal{C}$ be a cover of M. A subcollection $\mathcal{C}^*$ of $\mathcal{C}$ is called a subcover of $\mathcal{C}$ for M that covers M if it is a cover of M.

Definition 15. Let $M \in [X]^m$, $\mathcal{C}$ be a cover of M and τ a multiset topology on M. A cover $\mathcal{C}$ is called an open cover of M if $\mathcal{C} \subseteq \tau$.

Definition 16. Let $M \in [X]^m$ and (M,τ) be a m-topological space. Then M is called m-compact if, for every open cover $\mathcal{U}$ of M, there exists a finite subcover $\mathcal{V}$ of $\mathcal{U}$ for M.

Definition 17. Let $M \in [X]^m$ and (M,τ) be a m-topological space.
1. (M,τ) is called m-T_1 space if, for every $x_1 \in^{k_1} M, x_2 \in^{k_2} M$ such that $x_1 \neq x_2$, there exists open sets G, H such that $x_1 \in^{k_1} G \not\ni^{k_2} x_2$ and $x_1 \notin^{k_1} H \ni^{k_2} x_2$.
2. (M,τ) is called m-T_2 space or Hausdorff space if, for every $x_1 \in^{k_1} M, x_2 \in^{k_2} M$ such that $x_1 \neq x_2$, there exists open sets G, H such that $x_1 \in^{k_1} G$, $x_2 \in^{k_2} H$ and $G \cap H = \emptyset$.
3. (M,τ) is called m-regular space if, for every $x \in^k M$ and every closed mset F such that $x \notin^k F$, there exists open sets G, H such that $F \subseteq G$, $x \in^k H$ and $G \cap H = \emptyset$.
4. (M,τ) is called m-T_3 space if it is m-regular and m-T_1 space.
5. (M,τ) is called m-normal space if, for every pair of disjoint closed msets F_1, F_2, there exists open sets G, H such that $F_1 \subseteq G$, $F_2 \subseteq H$ and $G \cap H = \emptyset$.
6. (M,τ) is called m-T_4 space if it is m-normal and m-T_1 space.

3. M-Paracompact Multiset Topologies

Definition 18. Let $M \in [X]^m$, $\mathcal{W}$ be a cover of M. A cover $\mathcal{T}$ of M is called a refinement of $\mathcal{W}$ if, for every mset T in $\mathcal{T}$, there exists some mset W in $\mathcal{W}$ such that

$$C_T(x) \leq C_W(x), \forall x \in X.$$

$\mathcal{T}$ is called an open refinement of $\mathcal{W}$ if $\mathcal{T} \subseteq \tau$. We call $\mathcal{T}$ a closed refinement of $\mathcal{W}$ if $\mathcal{T}$ is a collection of closed msets.

Definition 19. Let $M \in [X]^m$ and (M,τ) be a m-topological space. A collection $\mathcal{W} \subseteq P^*(M)$ is called locally finite if each $k/x \in M$ has an U open neighborhood (which intersects only finitely many msets in $\mathcal{W}$) such that, for every mset V in only a finite subcollection $\mathcal{V}$ of $\mathcal{W}$,

$$C_{U\cap V}(y) > 0, \exists y \in X.$$

Proposition 1. Let $M \in [X]^m$, (M,τ) be a m-topological space and $\mathcal{W} \subseteq P^*(M)$. If $\mathcal{W}$ is locally finite, then

$$\bigcup Cl(\mathcal{W}) = Cl(\bigcup \mathcal{W})$$

where $Cl(\mathcal{W}) := \{Cl(W) : W \in \mathcal{W}\}$.

Proof. Let $M \in [X]^m$ be a mset, (M,τ) a m-topological space and $\mathcal{W} \subseteq P^*(M)$ locally finite. From Definition 7(1), for each $W \in \mathcal{W}$, $C_W(x) \leq C_{\bigcup\mathcal{W}}(x)$, $\forall x \in X$. Then, from Definition 3(2), for each $W \in \mathcal{W}$, we have $C_{Cl(W)}(x) \leq C_{\bigcup Cl(W)}(x)$, $\forall x \in X$. Then $\max\{C_{Cl(W)} : W \in \mathcal{W}\}$ is not greater than $C_{Cl(\bigcup \mathcal{W})}$ for every $x \in X$. Thus, from Definition 7(1) and Definition 4(2), $\bigcup Cl(\mathcal{W}) \subseteq Cl(\bigcup \mathcal{W})$.

Conversely, assume $x \in^k Cl(\bigcup \mathcal{W})$. Then, from the definition of multiset, $C_{Cl(\bigcup \mathcal{W})}(x) = k$. Since $\mathcal{W}$ is locally finite, we find an open mset U of k/x such that for every mset T in only a finite subcollection $\mathcal{T}$ of $\mathcal{W}$, there exists some $y \in X$ such that $C_{U\cap T}(y) > 0$. Assume $C_{\bigcup Cl(\mathcal{W})}(x) < k$ which implies $x \notin^k \bigcup Cl(\mathcal{W})$. Then, from Definition 7(1), for every $W \in \mathcal{W}$, $C_{Cl(W)}(x) < k$ and so $x \notin^k Cl(W)$. Set $V := U \ominus \bigcup Cl(\mathcal{T})$ where $Cl(\mathcal{T}) := \{Cl(T) : T \in \mathcal{T}\}$. From Definition 12(2) and Theorem 1(2), $\bigcup Cl(\mathcal{T})$ is a closed mset. Therefore, V is an open neighborhood of k/x since $V = U \ominus \bigcup Cl(\mathcal{T}) = U \cap (\bigcup Cl(\mathcal{T}))^c$.

On the other hand, the intersection of V with each mset W in $\mathcal{W}$ is an empty mset. Therefore V does not intersect $\bigcup \mathcal{W}$, contrary to $x \in^k Cl(\bigcup \mathcal{W})$. Then we have reached this contradiction because of the assumption that $x \notin^k Cl(\bigcup \mathcal{W})$. So $x \in^k \bigcup Cl(\mathcal{W})$. Thus $Cl(\bigcup \mathcal{W}) \subseteq \bigcup Cl(\mathcal{W})$.

Definition 20. Let $M \in [X]^m$ and (M,τ) be a m-topological space. M is called m-paracompact if every open cover of M has a locally finite refinement that covers M.

Proposition 2. Let $M \in [X]^m$, $\mathcal{W}$, $\mathcal{T}$ be covers of M. If $\mathcal{T}$ is a subcover of $\mathcal{W}$ then $\mathcal{T}$ is also a refinement of $\mathcal{W}$.

Proof. Let $M \in [X]^m$, $\mathcal{W}$ be a cover of M and $\mathcal{T}$ a subcover of $\mathcal{W}$. Then, $\mathcal{T} \subseteq \mathcal{W}$, that is, every mset T in $\mathcal{T}$ is also in $\mathcal{W}$. If we take the mset W as T, then we say that for every mset $T \in \mathcal{T}$, there exists $W \in \mathcal{W}$ such that $T \subseteq W$. Thus, $\mathcal{T}$ is a refinement of $\mathcal{W}$.

Conclusion 1. Let $M \in [X]^m$ and (M,τ) be a m-topological space. If M is m-compact then M is also m-paracompact.

Theorem 4. Let $M \in [X]^m$, $A \subseteq M$ and (M,τ) be a m-paracompact m-topological space. If A is closed then A is m-paracompact as a subspace of M.

Proof. $M \in [X]^m$ be a mset, (M,τ) be a m-topological space, $A \subseteq M$ and $\mathcal{U}$ be an open cover of A. Since A is a subspace of M, from Definition 9, for every $U \in \mathcal{U}$, there exists a τ-open mset V_U such that $U = V_U \cap A$. Let $\mathcal{V}$ be a collection which consists of the mset A^c and these msets V_U.

$\mathcal{V}$ is an open cover of M since these msets V_U are τ-open msets and A is an τ-closed mset. Then $\mathcal{V}$ has a locally finite refinement, we say $\mathcal{W}$, because M is m-paracompact. Let $a \in A$. Since $\mathcal{W}$ is locally finite, $a \in X$ has an open neighborhood G whose intersection with each msets W in only a finite subcollection $\mathcal{S}$ of $\mathcal{W}$ is non-empty, that is, there exists an open neighborhood G of $a \in X$ such that $G \cap W \neq \emptyset$ for every msets W in only a finite subcollection $\mathcal{S}$ of $\mathcal{W}$.

Set $\mathcal{W}_A := \{W \cap A : W \in \mathcal{W}, W \cap A \neq \emptyset\}$ and $\mathcal{S}_A := \{W \cap A : W \in \mathcal{S}, W \cap A \neq \emptyset\}$. Then there exists an open neighborhood G of $a \in A$ such that $G \cap W \neq \emptyset$ for every msets W in only the finite subcollection $\mathcal{S}_A$ of $\mathcal{W}_A$ and so $\mathcal{W}_A$ is locally finite. Since $\mathcal{W}$ is a refinement of $\mathcal{V}$, for every mset $W \in \mathcal{W}$, there exists some mset V in $\mathcal{V}$ such that $W \subseteq V$, that is, $C_W(x) \leq C_V(x)$ for every $x \in X$. In the case $V = V_U$, we have that $W \cap A \subseteq V_U \cap A = U \in \mathcal{U}$. In the case $V = A^c$, since $W \subseteq A^c$, for any $U \in \mathcal{U}$, $W \cap A = \emptyset \subseteq U$. So, $\mathcal{W}_A$ is a locally finite refinement of $\mathcal{U}$. Thus, A is m-paracompact as a subspace of M.

Theorem 5. Let $M \in [X]^m$ and (M, τ) be a m-topological space. If M is m-paracompact Hausdorff then M is m-normal.

Proof. Let $M \in [X]^m$ and (M, τ) be a m-topological space. Let $x \in^k M$ and F be a closed mset such that $x \notin^k F$, Since M is Hausdorff, for every $y \in^m F$, there exists an open neighborhood U_y such that $x \notin Cl(U_y)$. Let $\mathcal{U}$ be a collection of open mset F^c and these open msets U_y. Then $\mathcal{U}$ is an open cover of M. Let $\mathcal{W}$ is a locally finite refinement of $\mathcal{U}$. Let $\mathcal{W}'$ be a collection of msets $W \in \mathcal{W}$ such that $W \cap F \neq \emptyset$. Therefore, $\mathcal{W}'$ covers F. Set $V := \bigcup \mathcal{W}' \supseteq F$. Since $\mathcal{W}$ is a refinement of $\mathcal{U}$, for every $W \in \mathcal{W}'$, there exists $y \in F$ such that $W \subseteq U_y$ and so $Cl(W) \subseteq Cl(U_y)$. Then, for every $W \in \mathcal{W}'$, $x \notin Cl(W)$. Therefore, $x \notin \bigcup Cl(\mathcal{W}')$ where $Cl(\mathcal{W}') := \{Cl(W) : W \in \mathcal{W}'\}$. Since $\mathcal{W}$ is locally finite, from Proposition 1, $x \notin \bigcup Cl(\mathcal{W}') = Cl(\bigcup \mathcal{W}') = Cl(V)$. Thus, M is regular.

Let A, B be disjoint closed msubsets of M. Then, for every $y \in^m F$, there exists an open neighborhood U_y such that $A \cap Cl(U_y) = \emptyset$. Let $\mathcal{U}$ be a collection of open mset F^c and these open msets U_y. Then $\mathcal{U}$ covers M. Let $\mathcal{W}$ is a locally finite refinement of $\mathcal{U}$. Let $\mathcal{W}'$ be a collection of msets $W \in \mathcal{W}$ such that $W \cap F \neq \emptyset$. Therefore, $\mathcal{W}'$ covers F. Set $V := \bigcup \mathcal{W}' \supseteq F$. Since $\mathcal{W}$ is a refinement of $\mathcal{U}$, for every $W \in \mathcal{W}'$, there exists $y \in F$ such that $W \subseteq U_y$ and so $Cl(W) \subseteq Cl(U_y)$. Then, for every $W \in \mathcal{W}'$, $A \cap Cl(W) = \emptyset$. Therefore, $\emptyset = A \cap (\bigcup Cl(\mathcal{W})) = A \cap Cl(\bigcup \mathcal{W}) = A \cap Cl(V)$ where $Cl(\mathcal{W}') := \{Cl(W) : W \in \mathcal{W}'\}$. Hence, M is m-normal.

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