



RELATIVE CONTROLLABILITY RESULTS FOR NONLINEAR HIGHER ORDER FRACTIONAL DELAY INTEGRODIFFERENTIAL SYSTEMS WITH TIME VARYING DELAY IN CONTROL

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ABSTRACT. This paper is concerned with the controllability of nonlinear higher order fractional delay integrodifferential equations with time varying delay in control, which involved Caputo derivatives of any different orders. A formula for the solution expression of the system is derived by using Laplace transform. A necessary and sufficient condition for the relative controllability of linear fractional delay dynamical systems with time varying delays in control is proved, and a sufficient condition for the corresponding nonlinear integrodifferential equation has obtained. Examples has given to verify the results.

1. INTRODUCTION

In this article, we investigate the relative controllability results for following higher order fractional delay integrodifferential equation:

$$\left\{ \begin{array}{l} {}^C D^\alpha x(t) = Ax(t) + Bx(t-h) + \sum_{i=0}^M C_i u(\rho_i(t)) \\ \quad + f\left(t, x(t), {}^C D^\beta x(t), \int_0^t g(t, s, x(s)) ds, u(t)\right), \quad t \in J = [0, T], \\ x(t) = \phi(t), \quad x'(t) = \phi'(t), \dots, x^{(p)}(t) = \phi^{(p)}(t), \quad -h < t \leq 0, \end{array} \right.$$

where $p-1 < \alpha \leq p$, $q-1 < \beta \leq q$, $q \leq p-1$, $x \in R^n$, $u \in R^m$ and A, B are $n \times n$ matrices and C_i are $n \times m$ matrices for $i = 0, 1, 2, \dots, M$ and f is continuous function. To solve this problem by using the main tools such as Mittag-Leffler matrix function, Grammain matrix and Schaefer's fixed point theorem.

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The controllability of dynamical system plays a vital role in the mathematical control theory. Now a days various controllability problems for different types of fractional dynamical systems has been studied by several authors[6, 15, 17, 25, 26, 27]. Fractional order models express the behavior of many real life process more precisely than integer order ones. The various types of fractional differential equations has gained importance and popularity due to its vast applications in different fields including bioengineering[19, 21], signal processing [22], frequency dependent damping behavior of many visco-elastic materials[1, 2], filter design, circuit theory [28], dynamics of interfaces between nanoparticles and substrates [7, 10], continuum and statistical mechanics [20], the nonlinear oscillation of earthquakes[9] and robotics [28] .

Moreover, the models which representing the control processes frequently involve delays in state or control variables[11]. Delay is one of the general phenomenon in practical system, which has an crucial effect on the stability and performance of system. Theory of dynamical systems with delays in control is a necessary to distinguish between the relative controllability (relative approximate controllability) and controllability (approximate controllability). The relative controllability of fractional dynamical systems with delays in control was investigated by many researchers[3, 4, 5, 18] .

However, so far, little attention reports on the relative controllability of fractional delay dynamical systems [12, 13, 14] but no relevant work has been reported on the higher order dynamical systems. Inspired by this fact, the objective of this paper is to study the relative controllability of nonlinear higher order fractional delay integrodifferential equation with time varying delay in control via the Schaefer's fixed point theorem.

The paper is organized as follows. In Section 1, the background, motivations and objective of this paper has been discussed. Some preliminary facts, formula and notations are recalled in the Section 2. In Section 3, definition for relative controllability, solution representation of linear delay systems and established necessary and sufficient conditions for the relative controllability of fractional delay dynamical system with time varying delay in control are provided. In Section 4, relative controllability was established for corresponding nonlinear fractional delay integrodifferential equation. Finally, in Section 5, numerical examples are provided to illustrate the effectiveness of our results.

2. PRELIMINARIES

In this section we shall provide some basic definitions.

Definition 1. [24] *Let f be a real-or complex-valued function of the variable $t > 0$ and let s be a real or complex parameter. The Laplace transform of f is defined as*

$$F(s) = \int_0^{\infty} e^{-st} f(t) dt \quad \text{for } \operatorname{Re}(s) > 0$$

Definition 2. [16] *The Caputo fractional derivative of order $\alpha > 0$, $n - 1 < \alpha \leq n$, is defined as*

$${}^C D_{0+}^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - s)^{n-\alpha-1} f^{(n)}(s) ds$$

where $f^{(n)}(s) = \frac{d^n f}{ds^n}$ and the function $f(t)$ has absolutely continuous derivative up to order $n - 1$. For the brevity, Caputo fractional derivative ${}^C D_{0+}^\alpha$ is taken as ${}^C D^\alpha$.

The Laplace transform of Caputo derivative is

$$L[{}^C D^\alpha x(t)](s) = s^\alpha L[x(t)](s) - \sum_{k=0}^{n-1} x^{(k)}(0) s^{\alpha-1-k}, \quad n - 1 < \alpha \leq n.$$

Definition 3. [23] *The Mittag-Leffler functions of various types are defined as*

$$E_\alpha(z) = E_{\alpha,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad z \in C, \quad Re(\alpha) > 0,$$

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z, \beta \in C, \quad Re(\alpha) > 0,$$

$$E_{\alpha,\beta}^\gamma(-\lambda t^\alpha) = \sum_{k=0}^{\infty} \frac{(\gamma)_k (-\lambda)^k}{k! \Gamma(\alpha k + \beta)} t^{\alpha k}$$

where $(\gamma)_n$ is a Pochhammer symbol which is defined as $\gamma(\gamma + 1)\dots(\gamma + n - 1)$ and $(\gamma)_n = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}$. The Laplace transforms of Mittag-Leffler functions are defined as

$$L[E_{\alpha,1}(\pm \lambda t^\alpha)](s) = \frac{s^{\alpha-1}}{(s^\alpha \pm \lambda)}, \quad Re(\alpha) > 0,$$

$$L[t^{\beta-1} E_{\alpha,\beta}(\pm \lambda t^\alpha)](s) = \frac{s^{\alpha-\beta}}{(s^\alpha \pm \lambda)}, \quad Re(\alpha) > 0, \quad Re(\beta) > 0,$$

$$L[t^{\beta-1} E_{\alpha,\beta}^\gamma(\pm \lambda t^\alpha)](s) = \frac{s^{\alpha\gamma-\beta}}{(s^\alpha \pm \lambda)^\gamma}, \quad Re(s) > 0, \quad Re(\beta) > 0, \quad |\lambda s^{-\alpha}| < 1.$$

Lemma 1. [8] *The Mittag-leffler matrix function derivative of order $p(p \in N)$ is defined as*

$$\left(\frac{d}{dt}\right)^{(p)} (t^{\alpha-1} E_{\alpha-\beta,\alpha}(At^{\alpha-\beta})) = t^{\alpha-p-1} E_{\alpha-\beta,\alpha-p}(At^{\alpha-\beta}), \quad (p \in N).$$

3. LINEAR DELAY SYSTEMS

Consider the linear fractional delay dynamical system with time varying delay in control of the form

$${}^C D^\alpha x(t) = Ax(t) + Bx(t - h) + \sum_{i=0}^M C_i u(\rho_i(t)), \quad t \in J = [0, T],$$

$$x(t) = \phi(t), \quad x'(t) = \phi'(t), \dots, x^{(p)}(t) = \phi^{(p)}(t), \quad -h < t \leq 0, \quad (1)$$

where $p - 1 < \alpha \leq p$, A, B are $n \times n$ matrices and C_i are $n \times m$ matrices for $i = 0, 1, 2, \dots, M$. Assume the following conditions:

(H1) The functions $\rho_i(t) : J \rightarrow R, i = 0, 1, 2, \dots, M$, are twice continuously differentiable and strictly increasing in J . Moreover

$$\rho_i(t) \leq t, i = 0, 1, 2, \dots, M, \text{ for } t \in J.$$

(H2) Introduce the time lead functions $r_i(t) : [\rho_i(0), \rho_i(T)] \rightarrow [0, T], i = 0, 1, 2, \dots, M$, such that $r_i(\rho_i(t)) = t$ for $t \in J$. Further $\rho_0(t) = t$ and for $t = T$. The following inequalities holds

$$\rho_M(T) \leq \rho_{M-1}(T) \leq \dots \rho_{m+1}(T) \leq 0 = \rho_m(T) < \rho_{m-1}(T) = \dots = \rho_1(T) = \rho_0(T) = T. \quad (2)$$

Definition 4. The set $y(t) = \{x(t), \psi(t, s)\}$ where $\psi(t, s) = u(s)$ for $s \in [\min h_i(t), t]$ is said to be the complete state of the system (1) at time t .

Definition 5. System (1) is said to be relatively controllable on $[0, T]$ if for every complete state $y(t)$ and every $x_1 \in R^n$, there exists a control $u(t)$ defined on $[0, T]$, such that the solution of system (1) satisfies $x(T) = x_1$.

The solution of system (1) can be written as [26, 27]

$$\begin{aligned} x(t) = & \sum_{k=0}^{p-1} x^k(0)t^k \Phi_{\alpha, 1+k}(t) + B \int_{-h}^0 (t-s-h)^{\alpha-1} \Phi_{\alpha, \alpha}(t-s-h)\phi(s)ds \\ & + \int_0^t (t-s)^{\alpha-1} \Phi_{\alpha, \alpha}(t-s) \sum_{k=0}^M C_k u(\rho_k(s))ds \end{aligned} \quad (3)$$

Using the time lead function $r_i(t)$, the solution is of the form

$$\begin{aligned} x(t) = & \sum_{k=0}^{p-1} x^k(0)t^k \Phi_{\alpha, 1+k}(t) + B \int_{-h}^0 (t-s-h)^{\alpha-1} \Phi_{\alpha, \alpha}(t-s-h)\phi(s)ds \\ & + \sum_{i=0}^M \int_{\rho_i(0)}^{\rho_i(t)} (t-r_i(s))^{\alpha-1} \Phi_{\alpha, \alpha}(t-r_i(s))C_i \dot{r}_i(s)u(\rho_i(s))ds \end{aligned} \quad (4)$$

The solution (4) is expressed as

$$x(t) = x(t; \phi) + \sum_{i=0}^M \int_{\rho_i(0)}^{\rho_i(t)} (t-r_i(s))^{\alpha-1} \Phi_{\alpha, \alpha}(t-r_i(s))C_i \dot{r}_i(s)u(\rho_i(s))ds$$

where

$$x(t; \phi) = \sum_{k=0}^{p-1} x^k(0)t^k \Phi_{\alpha, 1+k}(t) + B \int_{-h}^0 (t-s-h)^{\alpha-1} \Phi_{\alpha, \alpha}(t-s-h)\phi(s)ds$$

Now using the inequality (2) the above equation we get

$$\begin{aligned} x(t) = & x(t; \phi) + \sum_{i=0}^m \int_{\rho_i(0)}^0 (t - r_i(s))^{\alpha-1} \Phi_{\alpha,\alpha}(t - r_i(s)) C_i \dot{r}_i(s) \psi(s) ds \\ & + \sum_{i=0}^m \int_0^t (t - r_i(s))^{\alpha-1} \Phi_{\alpha,\alpha}(t - r_i(s)) C_i \dot{r}_i(s) u(s) ds \\ & + \sum_{i=m+1}^M \int_{\rho_i(0)}^{\rho_i(t)} (t - r_i(s))^{\alpha-1} \Phi_{\alpha,\alpha}(t - r_i(s)) C_i \dot{r}_i(s) \psi(s) ds \end{aligned}$$

Further simplify we get

$$x(t) = x(t; \phi) + \sigma(t) + \sum_{i=0}^m \int_0^t (t - r_i(s))^{\alpha-1} \Phi_{\alpha,\alpha}(t - r_i(s)) C_i \dot{r}_i(s) u(s) ds \quad (5)$$

where

$$\begin{aligned} \sigma(t) = & \sum_{i=0}^m \int_{\rho_i(0)}^0 (t - r_i(s))^{\alpha-1} \Phi_{\alpha,\alpha}(t - r_i(s)) C_i \dot{r}_i(s) \psi(s) ds \\ & + \sum_{i=m+1}^M \int_{\rho_i(0)}^{\rho_i(t)} (t - r_i(s))^{\alpha-1} \Phi_{\alpha,\alpha}(t - r_i(s)) C_i \dot{r}_i(s) \psi(s) ds \end{aligned}$$

The controllability Grammian matrix and the control function are defined by as follows

$$\begin{aligned} W(0, T) = & \sum_{i=0}^m \int_0^T (T - r_i(s))^{\alpha-1} \Phi_{\alpha,\alpha}(T - r_i(s)) C_i \dot{r}_i(s) \\ & \times [(T - r_i(s))^{\alpha-1} \Phi_{\alpha,\alpha}(T - r_i(s)) C_i \dot{r}_i(s)]^* ds \end{aligned}$$

and

$$u(t) = [(T - r_i(t))^{\alpha-1} \Phi_{\alpha,\alpha}(T - r_i(t)) C_i \dot{r}_i(t)]^* W^{-1} [x_1 - x(T; \phi) - \sigma(T)] \quad (6)$$

where the * indicates the matrix transpose.

Theorem 1. *The linear control system (1) is relatively controllable on $[0, T]$ if and only if the controllability Grammian matrix*

$$\begin{aligned} W(0, T) = & \sum_{i=0}^m \int_0^T (T - r_i(s))^{\alpha-1} \Phi_{\alpha,\alpha}(T - r_i(s)) C_i \dot{r}_i(s) \\ & \times [(T - r_i(s))^{\alpha-1} \Phi_{\alpha,\alpha}(T - r_i(s)) C_i \dot{r}_i(s)]^* ds \end{aligned}$$

is positive definite, for some $T > 0$.

Proof. We first prove that the necessity. Suppose that the Grammian matrix W is not positive definite. Then W is singular and so its inverse does not exist, and there exists a nonzero y such that

$$0 = y^* W y = y^* \sum_{i=0}^m \int_0^T (T - r_i(s))^{\alpha-1} \Phi_{\alpha,\alpha}(T - r_i(s)) C_i \dot{r}_i(s) \times [(T - r_i(s))^{\alpha-1} \Phi_{\alpha,\alpha}(T - r_i(s)) C_i \dot{r}_i(s)]^* y ds$$

and so, for $s \in [0, T]$

$$y^* \sum_{i=0}^m \int_0^T (T - r_i(s))^{\alpha-1} \Phi_{\alpha,\alpha}(T - r_i(s)) C_i \dot{r}_i(s) ds = 0 \quad (7)$$

Now choose $y = x(T; 0)$ and take $\psi(s) = 0$. Since system (1) is relatively controllable, there exists a control $u \in R^m$ such that it steers the complete state $y(0) = \{\phi(0), \phi'(0), \dots, \phi^{(p)}(0), \psi(0)\}$ to the origin in the interval J , it follows that

$$\begin{aligned} x(T) &= x(T; \phi) + \sigma(T) + \sum_{i=0}^m \int_0^T (T - r_i(s))^{\alpha-1} \Phi_{\alpha,\alpha}(T - r_i(s)) C_i \dot{r}_i(s) u(s) ds \\ &= y + \sum_{i=0}^m \int_0^T (T - r_i(s))^{\alpha-1} \Phi_{\alpha,\alpha}(T - r_i(s)) C_i \dot{r}_i(s) u(s) ds \\ &= 0 \end{aligned}$$

for which we get,

$$0 = y^* y + \sum_{i=0}^m \int_0^T y^* (T - r_i(s))^{\alpha-1} \Phi_{\alpha,\alpha}(T - r_i(s)) C_i \dot{r}_i(s) u(s) ds \quad (8)$$

It follows from (7) and (8) that $y^* y = 0$. This is a contradiction to $y \neq 0$. Thus W is nonsingular.

Next, we show that the sufficiency. Suppose that W is positive definite, that is, it is nonsingular and so its inverse is well-defined. Where the complete state $y(0) = \{\phi(0), \phi'(0), \dots, \phi^{(p)}(0), \psi(0)\}$ and the vector $x_1 \in R^n$ are chosen arbitrarily. Substituting the control function (6) in (5) we have

$$\begin{aligned} x(T) &= x(T; \phi) + \sigma(t) + \sum_{i=0}^m \int_0^T (T - r_i(s))^{\alpha-1} \Phi_{\alpha,\alpha}(T - r_i(s)) C_i \dot{r}_i(s) \\ &\quad \times [(T - r_i(t))^{\alpha-1} \Phi_{\alpha,\alpha}(T - r_i(t)) C_i \dot{r}_i(t)]^* W^{-1} [x_1 - x(T; \phi) - \sigma(T)] ds \\ &= x_1 \end{aligned}$$

This means that control $u(t)$ transfer the initial state $y(0)$ to the desired vector $x_1 \in R^n$ at time T . Hence the system (1) is relatively controllable. \square

4. INTEGRODIFFERENTIAL SYSTEMS

Consider the nonlinear fractional delay Integro-differential systems with time varying delays in control of the form

$$\begin{aligned}
 {}^C D^\alpha x(t) &= Ax(t) + Bx(t-h) + \sum_{i=0}^M C_i u(\rho_i(t)) \\
 &+ f\left(t, x(t), {}^C D^\beta x(t), \int_0^t g(t, s, x(s)) ds, u(t)\right), \quad t \in J = [0, T], \\
 x(t) &= \phi(t), \quad x'(t) = \phi'(t), \dots, x^{(p)}(t) = \phi^{(p)}(t), \quad -h < t \leq 0, \quad (9)
 \end{aligned}$$

where $p - 1 < \alpha \leq p$, $q - 1 < \beta \leq q$, $q \leq p - 1$, $x \in R^n$, $u \in R^m$ and A, B are $n \times n$ matrices and C_i are $n \times m$ matrices for $i = 0, 1, 2, \dots, M$ and $f : J \times R^n \times R^n \times R^n \times R^m \rightarrow R^n$ is continuous function. The solution of (9) is given as

$$\begin{aligned}
 x(t) &= x(t; \phi) + \sigma(t) + \sum_{i=0}^m \int_0^t (t - r_i(s))^{\alpha-1} \Phi_{\alpha, \alpha}(t - r_i(s)) C_i \dot{r}_i(s) u(s) ds \\
 &+ \int_0^t (t - s)^{\alpha-1} \Phi_{\alpha, \alpha}(t - s) f\left(s, x(s), {}^C D^\beta x(s), \int_0^t g(t, s, x(s)) ds, u(s)\right) ds, u(s) ds
 \end{aligned}$$

Further we assume the following hypotheses

(H3) For each $t \in J$, the function $f(t, \cdot, \cdot, \cdot, \cdot) : J \times R^n \times R^n \times R^n \times R^m \rightarrow R^n$ is continuous and the function $f(\cdot, x(\cdot), y(\cdot), z(\cdot), u(\cdot)) : J \rightarrow R^n$ is strongly measurable for each $x, y, z \in R^n, u \in R^m$.

(H4)

$$\left\| f\left(t, x(t), {}^C D^\beta x(t), \int_0^t g(t, s, x(s)) ds, u(t)\right) \right\| \leq K$$

where $t \in J$, $x \in R^n$, $u \in R^m$, $K \in R$.

(H5) Let

$$m_1 = \sup\{\|x(t; \phi)\|, t \in J\}, \quad m_2 = \sup\{\|(t - s)^{\alpha-1} \Phi_{\alpha, \alpha}(t - s)\|, t, s \in J\},$$

$$m_3 = \sup\{\|(t - s)^{\alpha-p-1} \Phi_{\alpha, \alpha-p}(t - s)\|, t, s \in J\},$$

$$m_4 = \sup\{\|\sigma(t)\|\}, \quad m_5 = \sup\{\|\sigma_1(t)\|\}$$

$$m_6 = a_i b_i \|C_i\|, \quad m_7 = m_1 + m_5 + L_2 \|W^{-1}\| \left[\|x_1\| + m_1 + m_4 + m_2 K \right]$$

$$a_i = \sup\|(T - r_i(s))^{\alpha-1} \Phi_{\alpha, \alpha}(T - r_i(s))\|, \quad b_i = \sup\|\dot{r}_i(s)\|, \quad i = 0, 1, 2, \dots, M$$

$$d_i = \sup\|(T - r_i(s))^{\alpha-p-1} \Phi_{\alpha, \alpha-p}(T - r_i(s))\|, \quad L_1 = \sum_{i=0}^m \int_0^T a_i^2 b_i^2 \|C_i\| \|C_i^*\| ds$$

$$L_2 = \sum_{i=0}^m \int_0^T a_i d_i b_i^2 \|C_i\| \|C_i^*\| ds.$$

Theorem 2. Assume that hypotheses (H3)–(H5) hold and the linear system (1) is relatively controllable on J , then the nonlinear system (9) is relatively controllable on J .

Proof. Consider the space $X = \{x : x^{(p)} \in C(J, R^n), {}^C D^\beta x \in C(J, R^n) \text{ and } u(t) \in R^m\}$ be a Banach space endowed with norm $\|x\|^x = \max_{t \in J} \{\|x\|, \|{}^C D^\beta x\|, \|u\|\}$. Define the control for an arbitrary function $x(\cdot)$ by using the hypotheses

$$u(t) = [(T - r_i(t))^{\alpha-1} \Phi_{\alpha,\alpha}(T - r_i(t)) C_i \dot{r}_i(t)]^* W^{-1} \left[x_1 - x(T; \phi) - \sigma(T) - \int_0^T (T-s)^{\alpha-1} \Phi_{\alpha,\alpha}(T-s) f \left(s, x(s), {}^C D^\beta x(s), \int_0^T g(T, s, x(s)) ds, u(s) \right) ds \right]$$

Now we shall show that the nonlinear operator $F : X \rightarrow X$ has a fixed point. By using the control function, the nonlinear operator F is defined as

$$(Fx)(t) = x(t; \phi) + \sigma(t) + \sum_{i=0}^m \int_0^t (t - r_i(s))^{\alpha-1} \Phi_{\alpha,\alpha}(t - r_i(s)) C_i \dot{r}_i(s) u(s) ds + \int_0^t (t-s)^{\alpha-1} \Phi_{\alpha,\alpha}(t-s) f \left(s, x(s), {}^C D^\beta x(s), \int_0^t g(t, s, x(s)) ds, u(s) \right) ds$$

This fixed point is then a solution of (9). Substituting the control $u(t)$ in the above equation we get

$$(Fx)(T) = x(T; \phi) + \sigma(T) + \int_0^T (T-s)^{\alpha-1} \Phi_{\alpha,\alpha}(T-s) \times f \left(s, x(s), {}^C D^\beta x(s), \int_0^T g(t, s, x(s)) ds, u(s) \right) ds + \sum_{i=0}^m \int_0^T (T - r_i(s))^{\alpha-1} \Phi_{\alpha,\alpha}(T - r_i(s)) C_i \dot{r}_i(s) \times [(T - r_i(t))^{\alpha-1} \Phi_{\alpha,\alpha}(T - r_i(t)) C_i \dot{r}_i(t)]^* \times W^{-1} \left[x_1 - x(T; \phi) - \sigma(T) - \int_0^T (T - \xi)^{\alpha-1} \Phi_{\alpha,\alpha}(T - \xi) \times f \left(\xi, x(\xi), {}^C D^\beta x(\xi), \int_0^T g(T, \xi, x(\xi)) d\xi, u(\xi) \right) d\xi \right] ds$$

Clearly, $(Fx)(T) = x_1$ this implies that at the time T , control u steers the system from the initial state x_0 to x_1 , if we can obtain a fixed point of the nonlinear operator F .

The first step is to obtain a priori bound of the set $\zeta(F) = \{x \in X : x = \lambda Fx, \lambda \in (0, 1)\}$.

Let $x \in \zeta(F)$, then $x = \lambda Fx$ for some $0 < \lambda < 1$. Then for each $t \in J$, we have

$$x(t) = \lambda x(t; \phi) + \lambda \sigma(t) + \lambda \int_0^t (t-s)^{\alpha-1} \Phi_{\alpha,\alpha}(t-s)$$

$$\begin{aligned} & \times f\left(s, x(s), {}^C D^\beta x(s), \int_0^t g(t, s, x(s)) ds, u(s)\right) ds \\ & + \lambda \sum_{i=0}^m \int_0^t (t-r_i(s))^{\alpha-1} \Phi_{\alpha,\alpha}(t-r_i(s)) C_i \dot{r}_i(s) \\ & \quad \times [(T-r_i(t))^{\alpha-1} \Phi_{\alpha,\alpha}(T-r_i(t)) C_i \dot{r}_i(t)]^* \\ & \quad \times W^{-1} \left[x_1 - x(T; \phi) - \sigma(T) - \int_0^T (T-\xi)^{\alpha-1} \Phi_{\alpha,\alpha}(T-\xi) \right. \\ & \quad \left. \times f\left(\xi, x(\xi), {}^C D^\beta x(\xi), \int_0^T g(T, \xi, x(\xi)) d\xi, u(\xi)\right) d\xi \right] ds \end{aligned}$$

$$\begin{aligned} \|x(t)\| & \leq m_1 + m_4 + m_2 \int_0^T K ds \\ & \quad + \sum_{i=0}^m \int_0^T a_i^2 b_i^2 \|C_i\| \|C_i^*\| \|W^{-1}\| \left[\|x_1\| + m_1 + m_4 + m_2 \int_0^T K d\xi \right] ds \\ & \leq m_1 + m_4 + m_2 K T + L_1 \|W^{-1}\| [\|x_1\| + m_1 + m_4 + m_2 K T] \end{aligned}$$

and then

$$\begin{aligned} \|u(t)\| & \leq \left\| [(T-r_i(t))^{\alpha-1} \Phi_{\alpha,\alpha}(T-r_i(t)) C_i \dot{r}_i(t)]^* W^{-1} \left[x_1 - x(T; \phi) - \sigma(T) \right. \right. \\ & \quad \left. \left. - \int_0^T (T-s)^{\alpha-1} \Phi_{\alpha,\alpha}(T-s) \right. \right. \\ & \quad \left. \left. \times f\left(s, x(s), {}^C D^\beta x(s), \int_0^T g(T, s, x(s)) ds, u(s)\right) ds \right] \right\| \\ & \leq a_i b_i \|C_i^*\| \|W^{-1}\| [\|x_1\| + m_1 + m_4 + m_2 K T] \\ & \leq m_6 \|W^{-1}\| [\|x_1\| + m_1 + m_4 + m_2 K T] \end{aligned}$$

by Lemma (1), we have

$$\begin{aligned} x^{(p)}(t) & = x(t; \phi) + \sum_{i=0}^m \int_{\rho_i(0)}^0 (t-r_i(s))^{\alpha-p-1} \Phi_{\alpha,\alpha-p}(t-r_i(s)) C_i \dot{r}_i(s) \psi(s) ds \\ & \quad + \sum_{i=m+1}^M \int_{\rho_i(0)}^{\rho_i(t)} (t-r_i(s))^{\alpha-p-1} \Phi_{\alpha,\alpha-p}(t-r_i(s)) C_i \dot{r}_i(s) \psi(s) ds \\ & \quad + \sum_{i=0}^m \int_0^t (t-r_i(s))^{\alpha-p-1} \Phi_{\alpha,\alpha-p}(t-r_i(s)) C_i \dot{r}_i(s) u(s) ds \\ & \quad + \int_0^t (t-s)^{\alpha-p-1} \Phi_{\alpha,\alpha-p}(t-s) f(s, x(s), {}^C D^\beta x(s), \int_0^t g(t, s, x(s)) ds, u(s)) ds \end{aligned}$$

$$\begin{aligned}
 x^{(p)}(t) = & x(t; \phi) + \sigma_1(t) + \int_0^t (t-s)^{\alpha-p-1} \Phi_{\alpha, \alpha-p}(t-s) \\
 & \times f\left(s, x(s), {}^C D^\beta x(s), \int_0^t g(t, s, x(s)) ds, u(s)\right) ds \\
 & + \sum_{i=0}^m \int_0^t (t-r_i(s))^{\alpha-p-1} \Phi_{\alpha, \alpha-p}(t-r_i(s)) C_i \dot{r}_i(s) \\
 & \times [(T-r_i(t))^{\alpha-1} \Phi_{\alpha, \alpha}(T-r_i(t)) C_i \dot{r}_i(t)]^* \\
 & \times W^{-1} \left[x_1 - x(T; \phi) - \sigma(T) - \int_0^T (T-\xi)^{\alpha-1} \Phi_{\alpha, \alpha}(T-\xi) \right. \\
 & \left. \times f\left(\xi, x(\xi), {}^C D^\beta x(\xi), \int_0^T g(T, \xi, x(\xi)) d\xi, u(\xi)\right) d\xi \right] ds
 \end{aligned}$$

where

$$\begin{aligned}
 \sigma_1(t) = & \sum_{i=0}^m \int_{\rho_i(0)}^0 (t-r_i(s))^{\alpha-p-1} \Phi_{\alpha, \alpha-p}(t-r_i(s)) C_i \dot{r}_i(s) \psi(s) ds \\
 & + \sum_{i=m+1}^M \int_{\rho_i(0)}^{\rho_i(t)} (t-r_i(s))^{\alpha-p-1} \Phi_{\alpha, \alpha-p}(t-r_i(s)) C_i \dot{r}_i(s) \psi(s) ds
 \end{aligned}$$

So

$$\begin{aligned}
 \|x^{(p)}(t)\| \leq & m_1 + m_5 + m_3KT + L_2 \|W^{-1}\| [\|x_1\| + m_1 + m_4 + m_2KT] \\
 = & M_0.
 \end{aligned}$$

Thus,

$$\begin{aligned}
 \|{}^C D^\beta x(t)\| \leq & \left\| \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} x^{(p)}(s) ds \right\| \\
 \leq & \left\| \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} M_0 ds \right\|
 \end{aligned}$$

Hence it follows that there exist a constant K_0 , such that

$$\|x\|^x = \max_{t \in J} \{ \|x\|, \|{}^C D^\beta x(t)\|, \|u\| \} \leq K_0$$

and the set $\zeta(F)$ is bounded.

Next we show that F is completely continuous operator.

Let $B_\nu = \{x \in X; \|x\|^x \leq \nu\}$, then we first prove that F maps bounded sets B_ν into equicontinuous family. Let $x \in B_\nu$ and $t_1, t_2 \in J, 0 < t_1 < t_2 \leq T$,

$$\begin{aligned}
 \|(Fx)(t_2) - (Fx)(t_1)\| \leq & \|x(t_2; \phi) - x(t_1; \phi)\| + \|\sigma(t_2) - \sigma(t_1)\| \\
 & + \int_{t_1}^{t_2} \|(t_2-s)^{\alpha-1} \Phi_{\alpha, \alpha}(t_2-s)\| K ds
 \end{aligned}$$

$$\begin{aligned}
 & + \int_0^{t_1} \left\| (t_2 - s)^{\alpha-1} \Phi_{\alpha,\alpha}(t_2 - s) - (t_1 - s)^{\alpha-1} \Phi_{\alpha,\alpha}(t_1 - s) \right\| K ds \\
 & + \sum_{i=0}^m \int_{t_1}^{t_2} \left\| (t_2 - r_i(s))^{\alpha-1} \Phi_{\alpha,\alpha}(t_2 - r_i(s)) \right\| \|C_i\| \|C_i^*\| \|\dot{r}_i(s)\| \|\dot{r}_i^*(s)\| \\
 & \quad \times \left\| \left[(T - r_i(s))^{\alpha-1} \Phi_{\alpha,\alpha}(T - r_i(s)) \right]^* \right\| \|W^{-1}\| \\
 & \quad \times \left[\|x_1\| + m_1 + m_4 + m_2 \int_0^T (T - \xi)^{\alpha-1} K d\xi \right] ds \\
 & + \sum_{i=0}^m \int_0^{t_1} \left\| (t_2 - r_i(s))^{\alpha-1} \Phi_{\alpha,\alpha}(t_2 - r_i(s)) - (t_1 - r_i(s))^{\alpha-1} \Phi_{\alpha,\alpha}(t_1 - r_i(s)) \right\| \\
 & \quad \times \|C_i\| \|C_i^*\| \|\dot{r}_i(s)\| \|\dot{r}_i^*(s)\| \left\| \left[(T - r_i(s))^{\alpha-1} \Phi_{\alpha,\alpha}(T - r_i(s)) \right]^* \right\| \\
 & \quad \times \|W^{-1}\| \left[\|x_1\| + m_1 + m_4 + m_2 \int_0^T (T - \xi)^{\alpha-1} K d\xi \right] ds \tag{10}
 \end{aligned}$$

So,

$$\begin{aligned}
 \|(Fu)(t_2) - (Fu)(t_1)\| & \leq \left\| (T - r_i(t_2))^{\alpha-1} \Phi_{\alpha,\alpha}^*(T - r_i(t_2)) \right. \\
 & \quad \left. - (T - r_i(t_1))^{\alpha-1} \Phi_{\alpha,\alpha}^*(T - r_i(t_1)) \right\| \\
 & \times \|C_i^*\| \|\dot{r}_i^*(t_2) - \dot{r}_i^*(t_1)\| \|W^{-1}\| \left[\|x_1\| + m_1 + m_4 + m_2 \int_0^T (T - s)^{\alpha-1} K ds \right] \tag{11}
 \end{aligned}$$

and

$$\begin{aligned}
 \|(Fx)^{(p)}(t)\| & \leq m_1 + m_5 + m_3 \int_0^t K ds + L_2 \|W^{-1}\| \left[\|x_1\| + m_1 + m_4 + m_2 K T \right] \\
 & \leq m_7 + m_3 \int_0^t K ds
 \end{aligned}$$

Hence, it follows that

$$\begin{aligned}
 \|{}^C D^\beta(Fx)(t_2) - {}^C D^\beta(Fx)(t_1)\| & = \left\| \frac{1}{\Gamma(1-\beta)} \int_0^{t_2} (t_2 - s)^{-\beta} (Fx)^{(p)}(s) ds \right. \\
 & \quad \left. - \frac{1}{\Gamma(1-\beta)} \int_0^{t_1} (t_1 - s)^{-\beta} (Fx)^{(p)}(s) ds \right\| \\
 & \leq \frac{m_7}{\Gamma(2-\beta)} (t_2^{1-\beta} - t_1^{1-\beta}) + \frac{m_3}{\Gamma(1-\beta)} \int_{t_1}^{t_2} (t_2 - s)^{-\beta} \left(\int_0^s K d\xi \right) ds \\
 & \quad + \frac{m_3}{\Gamma(2-\beta)} \int_0^{t_1} \left((t_2 - s)^{1-\beta} - (t_2 - t_1)^{1-\beta} - (t_1 - s)^{1-\beta} \right) K ds \tag{12}
 \end{aligned}$$

Clearly (10),(11) and (12) tend to zero when $t_2 \rightarrow t_1$. Then F maps B_ν into an equicontinuous family of functions and hence the family FB_ν is uniformly bounded.

Next we show that F is a compact operator. Obviously, the closure of FB_ν is compact. Let $0 \leq t \leq T$ be fixed and ϵ a real number satisfying $0 < \epsilon < t$. For $x \in B_\nu$, we define

$$\begin{aligned} (F_\epsilon x)(T) &= x(t; \phi) + \sigma(t) \\ &+ \int_0^{t-\epsilon} (t-s)^{\alpha-1} \Phi_{\alpha,\alpha}(t-s) f\left(s, x(s), {}^C D^\beta x(s), \int_0^t g(t,s,x(s))ds, u(s)\right) ds \\ &+ \sum_{i=0}^m \int_0^{t-\epsilon} (t-r_i(s))^{\alpha-1} \Phi_{\alpha,\alpha}(t-r_i(s)) C_i \dot{r}_i(s) \left[(T-r_i(t))^{\alpha-1} \Phi_{\alpha,\alpha}(T-r_i(t)) C_i \dot{r}_i(t) \right]^* \\ &\quad \times W^{-1} \left[x_1 - x(T; \phi) - \sigma(T) - \int_0^T (T-\xi)^{\alpha-1} \Phi_{\alpha,\alpha}(T-\xi) \right. \\ &\quad \left. \times f(\xi, x(\xi), {}^C D^\beta x(\xi), \int_0^T g(T,\xi,x(\xi))d\xi, u(\xi))d\xi \right] ds \end{aligned}$$

To acquire the bounded and equicontinuous property of F_ϵ we can using the previous method. Hence we have,

$$S_\epsilon(t) = \{(F_\epsilon x)(t), x \in B_\nu\}$$

is relatively compact in X for every $0 < \epsilon < t$, and hence for every $x \in B_\nu$,

$$\begin{aligned} \|(Fx)(t) - (F_\epsilon x)(t)\| &\leq \int_{t-\epsilon}^t \|(t-s)^{\alpha-1} \Phi_{\alpha,\alpha}(t-s)\| K ds \\ &+ \sum_{i=0}^m \int_{t-\epsilon}^t \|(t-r_i(s))^{\alpha-1} \Phi_{\alpha,\alpha}(t-r_i(s))\| \\ &\quad \times \|C_i\| \|C_i^*\| \|\dot{r}_i(s)\| \|\dot{r}_i^*(s)\| \left\| \left[(T-r_i(s))^{\alpha-1} \Phi_{\alpha,\alpha}(T-r_i(s)) \right]^* \right\| \\ &\quad \times \|W^{-1}\| \left[\|x_1\| + m_1 + m_4 + m_2 \int_0^T K d\xi \right] ds \end{aligned}$$

Also,

$$\begin{aligned} \|(Fx)^{(p)}(t) - (F_\epsilon x)^{(p)}(t)\| &\leq \int_{t-\epsilon}^t \|(t-s)^{\alpha-p-1} \Phi_{\alpha,\alpha-p}(t-s)\| K ds \\ &+ \sum_{i=0}^m \int_{t-\epsilon}^t \|(t-r_i(s))^{\alpha-p-1} \Phi_{\alpha,\alpha-p}(t-r_i(s))\| \\ &\quad \times \|C_i\| \|C_i^*\| \|\dot{r}_i(s)\| \|\dot{r}_i^*(s)\| \left\| \left[(T-r_i(s))^{\alpha-1} \Phi_{\alpha,\alpha}(T-r_i(s)) \right]^* \right\| \\ &\quad \times \|W^{-1}\| \left[\|x_1\| + m_1 + m_4 + m_2 \int_0^T K d\xi \right] ds \end{aligned}$$

Since $\|(F_x)(t) - (F_\epsilon x)(t)\| \rightarrow 0$ and $\|(F_x)^{(p)}(t) - (F_\epsilon x)^{(p)}(t)\| \rightarrow 0$ as $\epsilon \rightarrow 0$, therefore, in the same way, we have

$$\begin{aligned} & \|{}^C D^\beta(F_x)(t) - {}^C D^\beta(F_\epsilon x)(t)\| \\ & \leq \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} \|(F_x)^{(p)}(t) - (F_\epsilon x)^{(p)}(t)\| ds \rightarrow 0, \text{ as } \epsilon \rightarrow 0. \end{aligned}$$

Hence, the relatively compact sets $S_\epsilon(t)$ are arbitrary close to the set $\{(F_x)(t), x \in B_\nu\}$. By the Arzela-ascoli theorem the set $\{(F_x)(t), x \in B_\nu\}$ is compact in X .

Finally to show that F is continuous. Let $\{x_n\}$ be a sequence in X such that $\|x_n - x\| \rightarrow 0$ and $\|u_n - u\| \rightarrow 0$ as $n \rightarrow \infty$. Then for each n and $t \in J$, there is an integer k , $\|x_n\| \leq k$, $\|{}^C D^\beta x_n\| \leq k$, $\|u_n\| \leq k$. Hence, $\|x\| \leq k$, $\|{}^C D^\beta x\| \leq k$, $\|u\| \leq k$ and $x, {}^C D^\beta x, u \in X$. By (H3),

$$\begin{aligned} & f\left(t, x_n(t), {}^C D^\beta x_n(t), \int_0^t g(t, s, x_n(s)) ds, u_n(t)\right) \\ & \rightarrow f\left(t, x(t), {}^C D^\beta x(t), \int_0^t g(t, s, x(s)) ds, u(t)\right) \end{aligned}$$

for all $t \in J$ and since

$$\begin{aligned} & \left\| f\left(t, x_n(t), {}^C D^\beta x_n(t), \int_0^t g(t, s, x_n(s)) ds, u_n(t)\right) \right. \\ & \quad \left. - f\left(t, x(t), {}^C D^\beta x(t), \int_0^t g(t, s, x(s)) ds, u(t)\right) \right\| \leq 2K, \end{aligned}$$

By the Dominated convergence theorem we have

$$\begin{aligned} & \|(F_{x_n})(t) - (F_x)(t)\| \leq \int_0^T \|(t-s)^{\alpha-1} \Phi_{\alpha,\alpha}(t-s) \\ & \quad \times \left[f\left(s, x_n(s), {}^C D^\beta x_n(s), \int_0^t g(t, s, x_n(s)) ds, u_n(s)\right) \right. \\ & \quad \left. - f\left(s, x(s), {}^C D^\beta x(s), \int_0^t g(t, s, x(s)) ds, u(s)\right) \right] ds \\ & + \sum_{i=0}^m \int_0^T \|(t-r_i(s))^{\alpha-1} \Phi_{\alpha,\alpha}(t-r_i(s)) C_i \dot{r}_i(s) [(T-r_i(t))^{\alpha-1} \Phi_{\alpha,\alpha}(T-r_i(t)) C_i \dot{r}_i(t)]^* \\ & \times W^{-1} \left[x_1 - x(T; \phi) - \sigma(T) - \int_0^T (T-\xi)^{\alpha-1} \Phi_{\alpha,\alpha}(T-\xi) \right. \\ & \quad \times \left[f\left(\xi, x_n(\xi), {}^C D^\beta x_n(\xi), \int_0^T g(T, \xi, x_n(\xi)) d\xi, u_n(\xi)\right) d\xi \right. \\ & \quad \left. \left. - f\left(\xi, x(\xi), {}^C D^\beta x(\xi), \int_0^T g(T, \xi, x(\xi)) d\xi, u(\xi)\right) d\xi \right] \right] ds \end{aligned}$$

and

$$\begin{aligned} & \| (Fu_n)(t) - (Fu)(t) \| \leq \left\| \left[(T - r_i(t))^{\alpha-1} \Phi_{\alpha,\alpha}(T - r_i(t)) C_i \dot{r}_i(t) \right]^* \right. \\ & W^{-1} \left[x_1 - x(T; \phi) - \sigma(T) - \int_0^T (T - s)^{\alpha-1} \Phi_{\alpha,\alpha}(T - s) \right. \\ & \quad \times \left[f \left(s, x_n(s), {}^C D^\beta x_n(s), \int_0^T g(T, s, x_n(s)) ds, u_n(s) \right) \right. \\ & \quad \left. \left. - f \left(s, x(s), {}^C D^\beta x(s), \int_0^T g(T, s, x(s)) ds, u(s) \right) \right] ds \right\| \end{aligned}$$

Clearly $\| (Fx_n)(t) - (Fx)(t) \| \rightarrow 0$ and $\| (Fu_n)(t) - (Fu)(t) \| \rightarrow 0$ as $n \rightarrow \infty$. Also,

$$\begin{aligned} & \left\| (Fx_n)^{(p)}(t) - (Fx)^{(p)}(t) \right\| \leq \int_0^t (t - s)^{\alpha-p-1} \Phi_{\alpha,\alpha-p}(t - s) \\ & \quad \times \left[f \left(s, x_n(s), {}^C D^\beta x_n(s), \int_0^t g(t, s, x_n(s)) ds, u_n(s) \right) \right. \\ & \quad \left. - f \left(s, x(s), {}^C D^\beta x(s), \int_0^t g(t, s, x(s)) ds, u(s) \right) \right] ds \\ & \quad + \sum_{i=0}^m \int_0^t (t - r_i(s))^{\alpha-p-1} \Phi_{\alpha,\alpha-p}(t - r_i(s)) C_i \dot{r}_i(s) \\ & \quad \times \left[(T - r_i(t))^{\alpha-1} \Phi_{\alpha,\alpha}(T - r_i(t)) C_i \dot{r}_i(t) \right]^* W^{-1} \\ & \quad \times \left[x_1 - x(T; \phi) - \sigma(T) - \int_0^T (T - \xi)^{\alpha-1} \right. \\ & \quad \times \Phi_{\alpha,\alpha}(T - \xi) \left[f \left(\xi, x_n(\xi), {}^C D^\beta x_n(\xi), \int_0^T g(T, \xi, x_n(\xi)) d\xi, u_n(\xi) \right) \right. \\ & \quad \left. \left. - f \left(\xi, x(\xi), {}^C D^\beta x(\xi), \int_0^T g(T, \xi, x(\xi)) d\xi, u(\xi) \right) \right] d\xi \right] ds \\ & \hspace{15em} \rightarrow 0, \text{ as } n \rightarrow \infty. \end{aligned}$$

Thus F is continuous. Eventually, we proved in the first step, the set $\zeta(F) = \{x \in X; x = \lambda Fx, \lambda \in (0, 1)\}$ is bounded. By Schaefer's theorem, the operator F has a fixed point in X . This fixed point is then the solution of (9). Hence the system (9) is controllable on $[0, T]$. \square

5. EXAMPLE

In this section we give the numerical examples to illustrate the theoretical results.

Example 1. Consider the problem of linear higher order fractional delay dynamical system with time varying delay in control of the form

$${}^C D^{\frac{3}{2}}x(t) = Ax(t) + Bx(t - 2) + C_0u(t) + C_1u(t - 1) \tag{13}$$

where $\alpha = \frac{3}{2}$, $h = 2$, $\rho = 1$, $x(t) = \phi(t)$, $x'(t) = \phi'(t) \in R^2$, $u(t) = \psi(t)$, $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $C_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $C_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The solution of the system (13) can be written as

$$\begin{aligned} x(t) &= \sum_{n=0}^{[t]} B^n(t-n)^{\frac{3}{2}n} E_{\frac{3}{2}, \frac{3}{2}n+1}^{n+1}(A(t-n)^{\frac{3}{2}}) + \sum_{n=0}^{[t]} B^n(t-n)^{\frac{3}{2}n+1} E_{\frac{3}{2}, \frac{3}{2}n+2}^{n+1}(A(t-n)^{\frac{3}{2}}) \\ &+ B \sum_{n=0}^{[t]} B^n \int_{-2}^0 (t-s-n-2)^{\frac{3}{2}n+\frac{1}{2}} E_{\frac{3}{2}, \frac{3}{2}(n+1)}^{n+1}(A(t-s-n-2)^{\frac{3}{2}}) \phi(s) ds \\ &+ \sum_{n=0}^{[t]} \sum_{i=0}^1 B^n C_i \int_0^{t-n} (t-r_i(s)-n)^{\frac{3}{2}n+\frac{1}{2}} E_{\frac{3}{2}, \frac{3}{2}(n+1)}^{n+1}(A(t-r_i(s)-n)^{\frac{3}{2}}) \dot{r}_i(s) u(s) ds \end{aligned}$$

Now consider the controllability on $[0, 2]$ where $[t] = 0$, we have

$$\begin{aligned} x(t) &= E_{\frac{3}{2}, 1}(At^{\frac{3}{2}}) + E_{\frac{3}{2}, 2}(At^{\frac{3}{2}}) + B \int_{-2}^0 (t-s-2)^{\frac{1}{2}} E_{\frac{3}{2}, \frac{3}{2}}(A(t-s-2)^{\frac{3}{2}}) \phi(s) ds \\ &+ \sum_{i=0}^1 B^n C_i \int_0^t (t-r_i(s))^{\frac{1}{2}} E_{\frac{3}{2}, \frac{3}{2}}(A(t-r_i(s))^{\frac{3}{2}}) \dot{r}_i(s) u(s) ds \\ x(t) &= E_{\frac{3}{2}, 1}(At^{\frac{3}{2}}) + E_{\frac{3}{2}, 2}(At^{\frac{3}{2}}) + Bt^{\frac{1}{2}} E_{\frac{3}{2}, \frac{5}{2}}(At^{\frac{3}{2}}) \\ &+ \sum_{i=0}^1 B^n C_i \int_0^t (t-r_i(s))^{\frac{1}{2}} E_{\frac{3}{2}, \frac{3}{2}}(A(t-r_i(s))^{\frac{3}{2}}) \dot{r}_i(s) u(s) ds \end{aligned}$$

The Grammian matrix is defined by

$$\begin{aligned} W &= \sum_{i=0}^1 \int_0^2 \left[(2-r_i(s))^{\frac{1}{2}} E_{\frac{3}{2}, \frac{3}{2}}(A(2-r_i(s))^{\frac{3}{2}}) C_i \dot{r}_i(s) \right] \\ &\times \left[(2-r_i(s))^{\frac{1}{2}} E_{\frac{3}{2}, \frac{3}{2}}(A(2-r_i(s))^{\frac{3}{2}}) C_i \dot{r}_i(s) \right]^* ds \end{aligned}$$

where $r_i(s)$ is a time lead function which is defined by $r_0(s) = s$ and $r_1(s) = s - 1$. Then the Grammian matrix can be written as

$$\begin{aligned} W &= \int_0^2 \left[(2-s)^{\frac{1}{2}} E_{\frac{3}{2}, \frac{3}{2}}(A(2-s)^{\frac{3}{2}}) C_0 \right] \left[(2-s)^{\frac{1}{2}} E_{\frac{3}{2}, \frac{3}{2}}(A(2-s)^{\frac{3}{2}}) C_0 \right]^* ds \\ &+ \int_0^2 \left[(2-s+1)^{\frac{1}{2}} E_{\frac{3}{2}, \frac{3}{2}}(A(2-s+1)^{\frac{3}{2}}) C_1 \right] \left[(2-s+1)^{\frac{1}{2}} E_{\frac{3}{2}, \frac{3}{2}}(A(2-s+1)^{\frac{3}{2}}) C_1 \right]^* ds \end{aligned}$$

Evaluating it, we get

$$W = \begin{pmatrix} 1.2044 \times 10^3 & 0 \\ 0 & 0.2104 \times 10^3 \end{pmatrix}.$$

Thus $\det(W) > 0$, which implies it is positive definite for any $T > 0$. Therefore, by Theorem 1, the linear system (13) is controllable $[0, 2]$.

Example 2. Consider the nonlinear fractional delay integrodifferential equation with time varying delay in control of the form

$$\begin{aligned} {}^C D^{\frac{3}{2}}x(t) &= Ax(t) + Bx(t-2) + C_0u(t) + C_1u(t-1) \\ &+ f\left(t, x(t), {}^C D^\beta x(t), \int_0^t g(t, s, x(s))ds, u(t)\right) \end{aligned} \quad (14)$$

where $\alpha = \frac{3}{2}$, $h = 2$, $\rho = 1$, $x(t) = \phi(t)$, $x'(t) = \phi'(t) \in R^2$, $u(t) = \psi(t)$, $A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $C_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $C_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$,

$$\begin{aligned} f\left(t, x(t), {}^C D^\beta x(t), \int_0^t g(t, s, x(s))ds, u(t)\right) \\ = f\left(\frac{x_1(t)\sin t}{x_1^2(t) + x_2^2(t)} + \frac{\int_0^t \sin(s)e^{x_1(s-1)} ds}{1 + x_1^2(t) + {}^C D^\beta x_1^2(t) + u(t)}\right) \end{aligned}$$

It is easy to verify that the nonlinear function f satisfies the condition in Theorem 2, and hence the fractional delay integrodifferential equation (14) is controllable on $[0, T]$.

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