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SOME INEQUALITIES FOR POSITIVE MULTILINEAR MAPPINGS

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Abstract. This paper devoted to obtaining some inequalities for positive multilinear mappings. More precisely, we present some Kantorovich and arithmetic-geometric mean inequalities for this kind of mappings. Our results improve earlier results by Kian and Dehghani.

1. Introduction

Let $M_n(\mathbb{C}) = M_n$ be the algebra of all $n \times n$ complex matrices and assume that M and m are scalars and I denotes the identity matrix. We write $A \geq 0$ to mean that the matrix A is positive semidefinite matrix and identify $A \geq B$ with $A - B \geq 0$. Likewise, we write $A > 0$ to refer that A is a positive definite matrix. The operator norm is denoted by $\|\cdot\|$.

A linear map $\Phi : M_n(\mathbb{C}) \to M_k(\mathbb{C})$ is called positive if $\Phi(A) \geq 0$, whenever $A \geq 0$. Also Φ is strictly positive if $\Phi(A) > 0$, whenever $A > 0$ and Φ is called unital if $\Phi(I) = I$. A real-valued continuous function f defined on $[0,\infty)$ is called matrix monotone if $f(A) \geq f(B)$ for $A \geq B \geq 0$. It is well known that $f(t) = t^r$ $(0 \leq r \leq 1)$ is a matrix monotone function, namely

$$A \geq B \Longrightarrow A^p \geq B^p \quad \text{for } 0 \leq p \leq 1.$$

Although,

$$A \geq B \Longrightarrow A^p \geq B^p \quad \text{for } 1 \leq p$$

is not true in general.

If $\Phi : M_n \longrightarrow M_p$ is a unital positive linear mapping, then Kadison's inequality states that $\Phi^2(A) \leq \Phi(A^2)$ for every Hermitian matrix A and Choi's inequality says that $\Phi^{-1}(A) \leq \Phi(A^{-1})$ for every strictly positive matrix A, see [4]. There have been a lot of works in which counterparts of these inequalities are presented. Especially see [8, 9].

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A mapping $\Phi : M_n^k := M_n \times \ldots \times M_n \to M_p$ is said to be multilinear if it is linear in each of its variable. A multilinear mapping $\Phi : M_n^k \longrightarrow M_p$ is called positive if $\Phi(A_1, ..., A_k) \geq 0$ whenever $A_i \geq 0$ for $i = 1, ..., k$. It is called strictly positive if $A_i > 0$ for $i = 1, ..., k$ implies that $\Phi(A_1, ..., A_k) > 0$ and Φ is called unital if $\Phi(I, ..., I) = I$; see [5].

Recently, Dehghani et al. [5] obtained an extension of the Choi's inequality and Kadison's inequality for positive multilinear mappings:

Lemma 1. *If $\Phi : M_n^k \longrightarrow M_p$ is a unital positive multilinear mapping, then*

$$\Phi^{-1}(A_1, ..., A_k) \leq \Phi\left(A_1^{-1}, ..., A_k^{-1}\right) \tag{1}$$

and

$$\Phi^2(A_1, ..., A_k) \leq \Phi\left(A_1^2, ..., A_k^2\right) \tag{2}$$

for all strictly positive matrices $A_i \in M_n$ $(i = 1, ..., k)$.

In the same paper, the authors presented an Pólya–Szegö type inequality for strictly positive multilinear mappings as follows: If $(A_1, ...A_k)$ and $(B_1, ..., B_k)$ are $k-$tuples of positive matrices with $0 < mI \leq A_i, B_i \leq MI$ $(i = 1, ..., k)$ for some positive real numbers $m < M$, then

$$\Phi(A_1, ..., A_k) \sharp \Phi(B_1, ..., B_k) \leq \frac{M^k + m^k}{2M^{\frac{k}{2}} m^{\frac{k}{2}}} \Phi(A_1 \sharp B_1, ..., A_k \sharp B_k) \tag{3}$$

where $A \sharp B = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^{\frac{1}{2}} A^{\frac{1}{2}}$ is called geometric mean of A, B. In [3], Kian and Dehghani presented a Kantorovich type inequality for positive multilinear mappings which is a counterpart of (1) as follows:

Lemma 2. *If $A_i \in M_n$ $(i = 1, ..., k)$ are positive matrices with $0 < mI \leq A_i \leq MI$ for some scalars $m < M$ and $\Phi : M_n^k \longrightarrow M_p$ is a unital positive multilinear mapping, then*

$$\Phi\left(A_1^{-1}, ..., A_k^{-1}\right) \leq \frac{\left(M^k + m^k\right)^2}{4M^k m^k} \Phi^{-1}(A_1, ..., A_k). \tag{4}$$

With the same assumptions of Lemma 2, Kian and Dehghani obtained

$$\Phi^p\left(A_1^{-1}, ..., A_k^{-1}\right) \leq \left(\frac{\left(M^k + m^k\right)^2}{4^{\frac{2}{p}} M^k m^k}\right)^p \Phi^{-p}(A_1, ..., A_k) \quad \text{for } p \geq 2. \tag{5}$$

Notice that the inequality

$$\Phi(A_1, ..., A_k) + M^k m^k \Phi\left(A_1^{-1}, ..., A_k^{-1}\right) \leq \left(M^k + m^k\right) I \tag{6}$$

holds for every unital positive multilinear mappings. By taking $0 < m^2 I \leq A_i^2 \leq M^2 I$ in the inequality (6) and using inequality (2), we can write the following inequality which will be a important tool for getting our results

$$\Phi^2(A_1, ..., A_k) + M^{2k} m^{2k} \Phi^2\left(A_1^{-1}, ..., A_k^{-1}\right) \leq \left(M^{2k} + m^{2k}\right) I. \tag{7}$$

In the paper [3], Kian and Dehgani proved that if $(A_1, ..., A_k)$ and $(B_1, ..., B_k)$ are $k-$tuples of positive matrices with $0 < mI \leq A_i, B_i \leq MI$ $(i = 1, ..., k)$ for some positive real numbers $m < M$, then

$$\Phi^2\left(\frac{A_1+B_1}{2}, ..., \frac{A_k+B_k}{2}\right) \leq \left(\frac{\left(M^k+m^k\right)^2}{4M^km^k}\right)^2 \Phi^2\left(A_1\sharp B_1, ..., A_k\sharp B_k\right). \tag{8}$$

In this paper, we will present some operator inequalities for positive unital multilinear mappings which are generalization of the inequality (8) and improvement of the inequality (5) for $p \geq 4$. Our idea throughout the paper is similar to the study of Fu and He [10] and Zhang [6] for positive linear maps . Moreover, we will give a squared version of the inequality (3) .

2. Main Results

Let's give some well known lemmas before we give the main theorems of this paper.

Lemma 3. (i) [2, Theorem 1] *Let $A, B > 0$. Then the following norm inequality holds:*

$$\|AB\| \leq \frac{1}{4}\|A+B\|^2.$$

(ii) [1, Theorem 3] *Let A and B be positive operators. Then*

$$\|A^r + B^r\| \leq \|(A+B)^r\| \quad \textit{for } 1 \leq r \leq \infty.$$

Theorem 4. *Let $A_i \in M_n$ with $0 < m \leq A_i \leq M$ for some positive real numbers $m < M$ $(i = 1, ..., k)$. If $\Phi : M_n^k \to M_l$ is a unital positive multilinear mapping, then for $4 \leq p < \infty$*

$$\Phi^p\left(A_1^{-1}, ..., A_k^{-1}\right) \leq \left(\frac{M^{2k}+m^{2k}}{4^{\frac{2}{p}}M^km^k}\right)^p \Phi^{-p}\left(A_1, ..., A_k\right). \tag{9}$$

Proof. The matrix inequality (9) is equivalent to

$$\left\|\Phi^{\frac{p}{2}}\left(A_1^{-1}, ..., A_k^{-1}\right)\Phi^{\frac{p}{2}}\left(A_1, ..., A_k\right)\right\| \leq \frac{1}{4}\frac{\left(M^{2k}+m^{2k}\right)^{\frac{p}{2}}}{M^{\frac{kp}{2}}m^{\frac{kp}{2}}}.$$

Compute

$$\begin{aligned}
& \left\| \Phi^{\frac{p}{2}} \left(A_1^{-1}, ..., A_k^{-1} \right) M^{\frac{kp}{2}} m^{\frac{kp}{2}} \Phi^{\frac{p}{2}} \left(A_1, ..., A_k \right) \right\| \\
\leq \ & \frac{1}{4} \left\| \Phi^{\frac{p}{2}} \left(A_1, ..., A_k \right) + M^{\frac{kp}{2}} m^{\frac{kp}{2}} \Phi^{\frac{p}{2}} \left(A_1^{-1}, ..., A_k^{-1} \right) \right\|^2 \\
& \text{(by Lemma 3 } (i)) \\
\leq \ & \frac{1}{4} \left\| \left(\Phi^2 \left(A_1, ..., A_k \right) + M^{2k} m^{2k} \Phi^2 \left(A_1^{-1}, ..., A_k^{-1} \right) \right)^{\frac{p}{4}} \right\|^2 \\
& \text{(by Lemma 3 } (ii)) \\
= \ & \frac{1}{4} \left\| \Phi^2 \left(A_1, ..., A_k \right) + M^{2k} m^{2k} \Phi^2 \left(A_1^{-1}, ..., A_k^{-1} \right) \right\|^{\frac{p}{2}} \\
\leq \ & \frac{1}{4} \left\| \left(M^{2k} + m^{2k} \right) I \right\|^{\frac{p}{2}} \quad \text{(by (7))} \\
\leq \ & \frac{1}{4} \left(M^{2k} + m^{2k} \right)^{\frac{p}{2}}.
\end{aligned}$$

So

$$\left\| \Phi^{\frac{p}{2}} \left(A_1^{-1}, ..., A_k^{-1} \right) \Phi^{\frac{p}{2}} \left(A_1, ..., A_k \right) \right\| \leq \frac{\left(M^{2k} + m^{2k} \right)^{\frac{p}{2}}}{4 M^{\frac{kp}{2}} m^{\frac{kp}{2}}}.$$

Thus inequality (9) holds. □

Remark 5. *It is obvious that inequality* (9) *is tighter than inequality* (5) *for* $p \geq 4$.

Now, let's give the generalization of the inequality (8).

Theorem 6. *Let* $A_i,\ B_i \in M_n$ *with* $0 < m \leq A_i, B_i \leq M$ *for some positive real numbers* $m < M$ $(i = 1, ..., k)$. *If* $\Phi : M_n^k \to M_l$ *is a unital positive multilinear mapping, then for* $2 \leq p < \infty$

$$\Phi^p \left(\frac{A_1 + B_1}{2}, ..., \frac{A_k + B_k}{2} \right) \leq \left(\frac{\left(M^k + m^k \right)^2}{4^{\frac{2}{p}} M^k m^k} \right)^p \Phi^p \left(A_1 \sharp B_1, ..., A_k \sharp B_k \right). \quad (10)$$

Proof. The claimed inequality is equivalent to

$$\left\| \Phi^{\frac{p}{2}} \left(\frac{A_1 + B_1}{2}, ..., \frac{A_k + B_k}{2} \right) M^{\frac{kp}{2}} m^{\frac{kp}{2}} \Phi^{-\frac{p}{2}} \left(A_1 \sharp B_1, ..., A_k \sharp B_k \right) \right\| \leq \frac{1}{4} \left(M^k + m^k \right)^p.$$

By computation, we have

$$\begin{aligned}
& \left\|\Phi^{\frac{p}{2}}\left(\frac{A_1+B_1}{2},...,\frac{A_k+B_k}{2}\right)M^{\frac{kp}{2}}m^{\frac{kp}{2}}\Phi^{-\frac{p}{2}}\left(A_1\sharp B_1,...,A_k\sharp B_k\right)\right\| \\
\leq\ & \frac{1}{4}\left\|\Phi^{\frac{p}{2}}\left(\frac{A_1+B_1}{2},...,\frac{A_k+B_k}{2}\right)+M^{\frac{kp}{2}}m^{\frac{kp}{2}}\Phi^{-\frac{p}{2}}\left(A_1\sharp B_1,...,A_k\sharp B_k\right)\right\|^2 \\
& \text{(by Lemma 3 } (i)) \\
\leq\ & \frac{1}{4}\left\|\left(\Phi\left(\frac{A_1+B_1}{2},...,\frac{A_k+B_k}{2}\right)+M^k m^k\Phi\left(\left(A_1\sharp B_1\right)^{-1},...,\left(A_k\sharp B_k\right)^{-1}\right)\right)^{\frac{p}{2}}\right\|^2 \\
& \text{(by Lemma 3 } (ii)) \\
=\ & \frac{1}{4}\left\|\Phi\left(\frac{A_1+B_1}{2},...,\frac{A_k+B_k}{2}\right)+M^k m^k\Phi\left(\left(A_1\sharp B_1\right)^{-1},...,\left(A_k\sharp B_k\right)^{-1}\right)\right\|^p.
\end{aligned}$$

By operator arithmetic-geometric mean inequality

$$\begin{aligned}
\leq\ & \frac{1}{4}\left\|\Phi\left(\frac{A_1+B_1}{2},...,\frac{A_k+B_k}{2}\right)+M^k m^k\Phi\left(\frac{A_1^{-1}+B_1^{-1}}{2},...,\frac{A_k^{-1}+B_k^{-1}}{2}\right)\right\|^p \\
=\ & \frac{1}{4}\frac{1}{2^{pk}}\left\|\Phi\left(A_1+B_1,...,A_k+B_k\right)+M^k m^k\Phi\left(A_1^{-1}+B_1^{-1},...,A_k^{-1}+B_k^{-1}\right)\right\|^p \\
\leq\ & \frac{1}{4}\frac{1}{2^{pk}}\Big\|\ \Phi\left(A_1,A_2,...,A_k\right)+M^k m^k\Phi\left(A_1^{-1},A_2^{-1},...,A_k^{-1}\right)+\Phi\left(B_1,A_2,...,A_k\right)+ \\
& +M^k m^k\Phi\left(B_1^{-1},A_2^{-1},...,A_k^{-1}\right)+...+\Phi\left(B_1,B_2,...,B_k\right) \\
& +M^k m^k\Phi\left(B_1^{-1},B_2^{-1},...,B_k^{-1}\right)\ \Big\|^p \\
\leq\ & \frac{1}{4}\frac{1}{2^{pk}}2^{pk}\left(M^k+m^k\right)^p \quad \text{(by (6))} \\
\leq\ & \frac{1}{4}\left(M^k+m^k\right)^p.
\end{aligned}$$

So

$$\left\|\Phi^{\frac{p}{2}}\left(\frac{A_1+B_1}{2},...,\frac{A_k+B_k}{2}\right)\Phi^{-\frac{p}{2}}\left(A_1\sharp B_1,...,A_k\sharp B_k\right)\right\|\leq\frac{1}{4}\frac{\left(M^k+m^k\right)^p}{M^{\frac{kp}{2}}m^{\frac{kp}{2}}}.$$

Thus (8) holds. □

Remark 7. *Inequality* (8) *is a special case of Theorem* 6 *by taking* $p=2$. *Thus* (10) *is a generalization of* (8).

Finally, let's give squared version of (3). For our object, we need the following lemma (see [7, Theorem 6]).

Lemma 8. *Let* $A,B\in M_n$ *such that* $0<A\leq B$ *and* $0<m\leq A\leq M$. *Then*

$$A^2\leq K\left(h\right)B^2,$$

where $K(h) = \frac{(h+1)^2}{4h}$ *with* $h = \frac{M}{m}$.

Theorem 9. *Let* A_i *and* B_i *be positive matrices with* $0 < mI \leq A_i, B_i \leq MI$ $(i = 1, ..., k)$ *for some positive real numbers* $m < M$ *and* Φ *be a strictly positive unital multilinear map. Then*

$$\left(\Phi\left(A_1, ..., A_k\right) \sharp \Phi\left(B_1, ..., B_k\right)\right)^2 \leq \left(\frac{\left(M^k + m^k\right)^2}{4M^k m^k}\right)^2 \Phi^2\left(A_1 \sharp B_1, ..., A_k \sharp B_k\right). \quad (11)$$

Proof. We have

$$\Phi\left(A_1, ..., A_k\right) \sharp \Phi\left(B_1, ..., B_k\right) \leq \frac{M^k + m^k}{2M^{\frac{k}{2}} m^{\frac{k}{2}}} \Phi\left(A_1 \sharp B_1, ..., A_k \sharp B_k\right).$$

Since $m^k \leq \Phi\left(A_1, ..., A_k\right) \sharp \Phi\left(B_1, ..., B_k\right) \leq M^k$, by applying Lemma 8 we get the inequality (11). □

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References

[1] Ando, T. and Zhan, X., Norm inequalities related to operator monotone functions, *Math. Ann.* 315 (1999), 771–780.
[2] Bhatia, R. and Kittaneh, F., Notes on matrix arithmetic geometric mean inequalities, Linear Algebra Appl. 308 (2000), 203–211.
[3] Kian, M. and Dehghani, M., Extension of the Kantorovich inequality for positive multilinear mappings, *Filomat* 31:20 (2017), 6473-6481.
[4] Bhatia, R., Positive definite matrices, Princeton Universtiy Press, Princeton 2007.
[5] Dehghani, M., Kian M. and Seo, Y., Developed matrix inequalities via positive multilinear mappings, *Linear Algebra Appl.* 484 (2015) 63-85.
[6] Zhang, P., More operator inequalities for positive linear maps, Banach J. Math. Anal. 9 (2015) 166–172.
[7] Fujii, M. and Izumino, S., Nakamoto, R. and Seo, Y., Operator inequalities related to Cauchy-Schwarz and Hölder-McCarthy inequalities, *Nihonkai Math. J.* 8 (1997), 117–122.
[8] Marshall, A.W. and Olkin, I., Matrix versions of Cauchy and Kantorovich inequalities, *Aequationes Math.* 40 (1990), 89–93.
[9] Mond, B. and Pecaric, J.E., Converses of Jensen's inequality for linear maps of operators, *An. Univ. Vest Timis. Ser. Mat. Inform.* 31(2), (1993), 223-228.
[10] Fu X. and He, C., Some operator inequalities for positive linear maps, Linear and Multilinear Algebra 63 (2015), 571–577.

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