SOME NEW d-ORTHOGONAL POLYNOMIAL SETS OF SHEFFER TYPE

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ABSTRACT. In this paper, we present some new Sheffer type d-orthogonal polynomial sets. Moreover, we obtain the d-dimensional functional vector ensuring the d-orthogonality of these new polynomial sets.

1. INTRODUCTION

Recently, the generalization of orthogonal polynomials called "d-orthogonal polynomials" have attracted so much attention from many authors. The well-known properties of orthogonal polynomials such as recurrence relations, Favard theorem, generating function relations and differential equations have found correspondence in this new notion. New polynomial sets which contain classical orthogonal polynomials have been created so far. Let us give a brief summary of d-orthogonal polynomials.

Let $\mathcal{P}$ be the vector space of polynomials with complex coefficients and $\mathcal{P}'$ be the vector space of all linear functionals on $\mathcal{P}$ called the algebraic dual. $\langle u, f \rangle$ is the representation of the effect of any linear functional $u \in \mathcal{P}'$ to the polynomial $f \in \mathcal{P}$. Let $\{P_n\}_{n \geq 0}$ be a polynomial set $(\deg(P_n) = n$ for all non-negative integer $n)$, and corresponding dual sequence $(u_n)_{n \geq 0}$ for polynomials taken from this set can be given by

$$\langle u_n, P_k \rangle = \delta_{nk}, \quad n, k = 0, 1, ...,$$

where δ_{nk} is the Kronecker delta.

A polynomial set $\{P_n\}_{n \geq 0}$ in $\mathcal{P}$ is said to be d-orthogonal polynomial set with respect to the d-dimensional functional vector $\Gamma = {}^t(u_0, u_1, ..., u_{d-1})$ if the following orthogonality conditions are hold

$$\begin{cases} \langle u_k, P_n P_m \rangle = 0, & m > nd + k, \\ \langle u_k, P_n P_{nd+k} \rangle \neq 0, & n \geq 0, \end{cases} \tag{1.1}$$

where $n \in \mathbb{N}_0 = \{0, 1, 2, ...\}$, d is a positive integer and $k \in \{0, 1, ..., d - 1\}$ (see [1 − 2]). Characterization of these polynomials by recurrence relations and Favard type theorem was also given in [1 − 2]. A polynomial set $\{P_n\}_{n \geq 0}$ is a d-orthogonal polynomial set if and only if it fulfills a $(d + 1)$-order recurrence relation of the type

$$xP_n(x) = \sum_{k=0}^{d+1} \alpha_{k,d}(n) P_{n-d+k}(x), \quad n \in \mathbb{N}_0, \tag{1.2}$$

with the regularity conditions $\alpha_{d+1,d}(n) \alpha_{0,d}(n) \neq 0$, $n \geq d$ and by convention $P_{-n}(x) = 0$, $n \geq 1$. Taking $d = 1$ in (1.1) and (1.2) leads us to the celebrated notion of orthogonal polynomials (see [3]).

The recurrence relation of order $d+1$ has been the main reason of deriving many d-orthogonal polynomial sets as an extension of known ones in orthogonal polynomials. Classical orthogonal polynomials such as Laguerre, Hermite and Jacobi polynomials, discrete orthogonal polynomials like Charlier, Meixner polynomials and so on were extended to the $d-$orthogonality notion and many basic properties linking with these polynomials were stated by various authors ([4 − 23]).

A polynomial set $\{P_n\}_{n \geq 0}$ is called Sheffer polynomial set if and only if it has the generating function of the form

$$A(t) e^{xH(t)} = \sum_{n=0}^{\infty} P_n(x) \frac{t^n}{n!} \tag{1.3}$$

where $A(t)$ and $H(t)$ have the power series expansions as following

$$A(t) = \sum_{k=0}^{\infty} a_k t^k, \ \ a_0 \neq 0, \ H(t) = \sum_{k=0}^{\infty} h_k t^{k+1}, \ \ h_0 \neq 0.$$

This means that $A(t)$ is invertible and $H(t)$ has a compositional inverse. There are numerous polynomial sets belong to the class of Sheffer polynomials (see [24 − 25]). Note that, for $H(t) = t$, we meet the definition of Appell polynomial sets [25] from the aspect of generating functions. That is to say, Appell polynomials can be defined by generating function of the type

$$A(t) e^{xt} = \sum_{n=0}^{\infty} P_n(x) \frac{t^n}{n!}$$

with $A(t) = \sum_{k=0}^{\infty} a_k t^k \ (a_0 \neq 0)$.

In this contribution, first, we recall the characterization theorem for the d-orthogonality of Sheffer polynomial sets recently given by Ben Cheikh and Gam in [26]. Then, we try to derive new d-orthogonal polynomial sets of Sheffer type and find some of them's d-dimensional functional vector for which promises the d-orthogonality.

2. MAIN RESULTS

Characterization problems related to Sheffer polynomial sets have a deep history. Both Meixner [27] and Sheffer [28] interested in the same problem: what are the all possible forms of polynomial sets which are at the same time orthogonal and Sheffer polynomials. They stated that $A(t)$ and $H(t)$ of (1.3) should satisfy the following conditions

$$\frac{1}{H'(t)} = (1 - \alpha t)(1 - \beta t),$$

$$\frac{A'(t)}{A(t)} = \frac{\lambda_2 t}{(1 - \alpha t)(1 - \beta t)}.$$

If we discuss all possible cases in view of these two conditions, then we face with the known orthogonal polynomial sets listed below:

Case 1: $\alpha = \beta = 0 \Rightarrow$ Hermite polynomials.

Case 2: $\alpha = \beta \neq 0 \Rightarrow$ Laguerre polynomials.

Case 3: $\alpha \neq 0$, $\beta = 0 \Rightarrow$ Charlier polynomials.

Case 4: $\alpha \neq \beta$, $(\alpha, \beta \in \mathbb{R}) \Rightarrow$ Meixner polynomials.

Case 5: $\alpha \neq \beta$, (complex conjugate of each other) $\Rightarrow$ Meixner-Pollaczek polynomials.

For more information see [29]. Similar investigation was made in [30] for 2-orthogonal polynomials. Generalization of these results to d-orthogonal polynomials have been found by Ben Cheikh and Gam in [26]. The authors stated the following theorem by using the technique described in [14].

Theorem 1. *Let $\gamma_d(t) = \displaystyle\sum_{k=0}^{d} \beta_k t^k$ be a polynomial of degree d $(\beta_d \neq 0)$ and $\sigma_{d+1}(t) = \displaystyle\sum_{k=0}^{d+1} \alpha_k t^k$ be a polynomial of degree less than or equal to $d+1$. The only polynomial sets $\{P_n\}_{n \geq 0}$,which are d-orthogonal and also Sheffer polynomial set, are generated by*

$$\exp\left[\int_0^t \frac{\gamma_d(s)}{\sigma_{d+1}(s)} ds\right] \exp\left[x \int_0^t \frac{1}{\sigma_{d+1}(s)} ds\right] = \sum_{n=0}^{\infty} P_n(x) \frac{t^n}{n!} \qquad (2.1)$$

with the conditions

$$\alpha_0 (n\alpha_{d+1} - \beta_d) \neq 0, \quad n \geq 1. \qquad (2.2)$$

Moreover, the authors deduced the exact number of d-orthogonal polynomial sets of Sheffer type. For detailed information see [26]. Next, in view of Theorem 1,we derive new d-orthogonal polynomial sets of Sheffer type and find their d-dimensional functional vector.

Now, we present a new d-orthogonal polynomial set for $d \geq 2$. It seems that this polynomial set is a Laguerre type d-orthogonal polynomial set but the difference is Laguerre polynomials are not generated hence $d \geq 2$.

Application 1: Suppose that $\{P_n\}_{n \geq 0}$ is a Sheffer polynomial set possessing the generating function (2.1) relate to the couple of polynomials

$$[\gamma_d(t), \sigma_3(t)] = \left[-(1-t)^3 \, \pi'_{d-2}(t) - (\alpha+1)(1-t)^2, \ -(1-t)^3 \right]$$

where $\pi_{d-2}(t) = \sum_{k=0}^{d-2} a_k t^k$ with $a_{d-2} \neq 0$. Then, taking Theorem 1 into account, we have a new d-orthogonal polynomial set generated by

$$e^{\pi_{d-2}(t)} (1-t)^{-\alpha-1} \exp\left(x \frac{t^2 - 2t}{2(1-t)^2} \right) = \sum_{n=0}^{\infty} P_n(x) \frac{t^n}{n!}$$

for $d \geq 2$. $a_{d-2} \neq 0$ guarantees the conditions (2.2).

Now, we deal with the case $d = 2$ i.e.: a 2-orthogonal polynomial set. Thus

$$(1-t)^{-\alpha-1} \exp\left(x \frac{t^2 - 2t}{2(1-t)^2} \right) = \sum_{n=0}^{\infty} P_n(x) \frac{t^n}{n!} \tag{2.3}$$

with conditions $\alpha + n + 1 \neq 0$, $n \geq 0$. Note that in [30], the authors found nine classes of 2-orthogonal polynomial set and this polynomial set belongs to the class E. Before finding the corresponding linear functionals u_0 and u_1 of this 2-orthogonal polynomial set, we need to state following useful lemma.

Lemma 1. ([31]) *Let $A(t)$ and $H(t)$ be two power series given as in (1.3) and*

$$H^*(t) = \sum_{k=0}^{\infty} h_k^* t^{k+1}$$

is the compositional inverse of $H(t)$ such that

$$H(H^*(t)) = H^*(H(t)) = t.$$

(i) The lowering operator σ of the polynomial set $\{P_n\}_{n \geq 0}$ generated by (1.3) is given with

$$\sigma = H^*(D), \quad D = \frac{d}{dx}.$$

(ii) The lowering operator σ of the polynomial set $\{P_n\}_{n \geq 0}$ generated by

$$A(t)(1 + \omega H(t))^{\frac{x}{\omega}} = \sum_{n=0}^{\infty} P_n(x) \frac{t^n}{n!}$$

is given by

$$\sigma = H^*(\Delta_\omega), \quad \Delta_\omega[f(x)] = \frac{f(x+\omega) - f(x)}{\omega}.$$

Theorem 2. *The polynomial set* $\{P_n\}_{n\geq 0}$ *generated by* (2.3) *are 2-orthogonal with respect to the following linear functionals*

$$\langle u_0, f\rangle = \int_0^\infty \omega_\alpha(x)\, f(x)\, dx, \quad f \in \mathcal{P},$$

$$\langle u_1, f\rangle = \int_0^\infty \left[\omega_\alpha(x) - \omega_{\alpha+1}(x)\right] f(x)\, dx, \quad f \in \mathcal{P},$$

where

$$\omega_\alpha(x) = \frac{1}{2^{\frac{\alpha+1}{2}}\Gamma\left(\frac{\alpha+1}{2}\right)} x^{\frac{\alpha-1}{2}} e^{-\frac{x}{2}},$$

and Γ *is the widely known Gamma function.*

Proof. From [32], the dual sequence corresponding to this 2-orthogonal polynomial set is given by

$$\langle u_i, f\rangle = \frac{1}{i!}\left[\frac{\sigma^i}{A(\sigma)} f(x)\right]_{x=0}, \quad i = 0, 1, \ f \in \mathcal{P},$$

where σ is the lowering operator of 2-orthogonal polynomial set generated by (2.3). The lowering operator σ of this polynomial set is

$$H(t) = \frac{t^2 - 2t}{2(1-t)^2} \Rightarrow \sigma = H^*(D) = 1 - (1 - 2D_x)^{-1/2}$$

where we use Lemma 1. Then, for $i = 0$ and $A(t) = (1-t)^{-\alpha-1}$, by using the following relation

$$2^k \left(\frac{\alpha+1}{2}\right)_k = \frac{1}{2^{\frac{\alpha+1}{2}}\Gamma\left(\frac{\alpha+1}{2}\right)} \int_0^\infty x^{\frac{\alpha-1}{2}} e^{-\frac{x}{2}} x^k dx$$

we obtain

$$\langle u_0, f\rangle = \left[(1 - 2D_x)^{-\left(\frac{\alpha+1}{2}\right)} f(x)\right]_{x=0}$$

$$= \sum_{k=0}^\infty \frac{\left(\frac{\alpha+1}{2}\right)_k 2^k}{k!} f^{(k)}(0)$$

$$= \frac{1}{2^{\frac{\alpha+1}{2}}\Gamma\left(\frac{\alpha+1}{2}\right)} \int_0^\infty x^{\frac{\alpha-1}{2}} e^{-\frac{x}{2}} \left(\sum_{k=0}^\infty \frac{f^{(k)}(0)}{k!} x^k\right) dx$$

$$= \int_0^\infty \omega_\alpha(x) f(x)\, dx.$$

where
$$\omega_\alpha (x) = \frac{1}{2^{\frac{\alpha+1}{2}} \Gamma \left(\frac{\alpha+1}{2} \right)} x^{\frac{\alpha-1}{2}} e^{-\frac{x}{2}}$$

and $(a)_n$ is the Pochhammer's symbol defined by the rising factorial
$$(a)_n = a(a+1)...(a+n-1), \quad n \geq 1,$$
$$(a)_0 = 1.$$

Furthermore, we calculate in a similar manner for $i = 1$

$$\langle u_1, f \rangle = \left[\sum_{r=0}^{1} \binom{1}{r} (-1)^r (1 - 2D_x)^{-\left(\frac{\alpha+r+1}{2}\right)} f(x) \right]_{x=0}$$

$$= \sum_{r=0}^{1} \binom{1}{r} (-1)^r \sum_{k=0}^{\infty} \frac{\left(\frac{\alpha+r+1}{2} \right)_k 2^k}{k!} f^{(k)}(0)$$

$$= \int_0^\infty \left[\sum_{r=0}^{1} \binom{1}{r} (-1)^r \frac{1}{2^{\frac{\alpha+r+1}{2}} \Gamma \left(\frac{\alpha+r+1}{2} \right)} x^{\frac{\alpha+r-1}{2}} \right] e^{-\frac{x}{2}} f(x) \, dx$$

$$= \int_0^\infty [\omega_\alpha (x) - \omega_{\alpha+1} (x)] f(x) \, dx.$$

This finishes the proof.   □

Next, we express a new Meixner type d-orthogonal polynomial set and we find its d-dimensional functional vector.

Application 2: Let $\{P_n\}_{n \geq 0}$ be a Sheffer polynomial set generated by (2.1) according to the couple of polynomials

$$[\gamma_d (t), \sigma_{d+1} (t)] = \left[\frac{dc\beta}{c-1} \left[(1-t)^d + \frac{c-1}{dc} \left[1 - (1-t)^d \right] \right], \right.$$
$$\left. \frac{c}{c-1} (1-t) \left[(1-t)^d + \frac{c-1}{dc} \left[1 - (1-t)^d \right] \right] \right]$$

with the restrictions
$$\begin{cases} c \neq \left\{ 0, \frac{1}{1-d}, 1 \right\} \\ \beta \neq -\frac{n}{d} \ , \ n \geq 0 \end{cases} . \tag{2.4}$$

After some computations, Theorem 1 allows us to introduce the following new d-orthogonal polynomial set with

$$(1-t)^{-\beta d} \left(1 + \frac{c-1}{dc} \left[\frac{1}{(1-t)^d} - 1 \right] \right)^x = \sum_{n=0}^{\infty} P_n (x) \frac{t^n}{n!}. \tag{2.5}$$

The conditions (2.2) are satisfied from restrictions (2.4). It seems that this Meixner type d-orthogonal polynomial set is the first explicit one among others in the literature.

Theorem 3. *The d-dimensional functional vectors, for which the d-orthogonality of the polynomial set generated by (2.5) holds, are*

$$\langle u_r, f \rangle = \frac{1}{r!} \sum_{i=0}^{r} \binom{r}{i} (-1)^i \sum_{j=0}^{\infty} \frac{\left(\beta + \frac{i}{d}\right)_j \left(\frac{dc}{1-c}\right)^j}{\left(1 - \frac{dc}{c-1}\right)^{\beta + \frac{i}{d} + j}} \frac{f(j)}{j!} \qquad (2.6)$$

where $r = 0, 1, ..., d - 1$ and $f \in \mathcal{P}$.

Proof. Lemma 1 helps us to find the lowering operators of the *d*-orthogonal polynomial set generated by (2.5) with

$$H(t) = \frac{c-1}{dc} \left[\frac{1}{(1-t)^d} - 1 \right] \Rightarrow \sigma = H^*(\Delta) = 1 - \left(1 - \frac{dc\Delta}{1-c}\right)^{-\frac{1}{d}}$$

where $\Delta f(x) = f(x+1) - f(x)$. Thus, again from [32] if we use the dual sequence corresponding to this polynomial set, we conclude that for $r = 0, 1, ..., d - 1$ and $f \in \mathcal{P}$

$$
\begin{aligned}
\langle u_r, f \rangle &= \frac{1}{r!} \left[\frac{\sigma^r}{A(\sigma)} f(x) \right]_{x=0} \\
&= \frac{1}{r!} \left[\sum_{i=0}^{r} \binom{r}{i} (-1)^i \left(1 - \frac{dc\Delta}{1-c}\right)^{-\left(\beta + \frac{i}{d}\right)} f(x) \right]_{x=0} \\
&= \frac{1}{r!} \sum_{i=0}^{r} \binom{r}{i} (-1)^i \sum_{k=0}^{\infty} \frac{\left(\beta + \frac{i}{d}\right)_k}{k!} \left(\frac{dc}{1-c}\right)^k \Delta^k f(0). \qquad (2.7)
\end{aligned}
$$

Substituting the fact

$$\Delta^k f(0) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(j)$$

into (2.7) and after shifting indices, we obtain

$$\langle u_r, f \rangle = \frac{1}{r!} \sum_{i=0}^{r} \binom{r}{i} (-1)^i \sum_{j=0}^{\infty} \left\{ \sum_{k=0}^{\infty} \frac{\left(\beta + \frac{i}{d}\right)_{k+j} (-1)^k \left(\frac{dc}{1-c}\right)^k}{k!} \right\} \frac{\left(\frac{dc}{1-c}\right)^j f(j)}{j!}.$$

$$(2.8)$$

The equality (2.8) leads us to get the desired result by applying the following property of the Pochhammer's symbol

$$\left(\beta + \frac{i}{d}\right)_{k+j} = \left(\beta + \frac{i}{d}\right)_j \left(\beta + \frac{i}{d} + j\right)_k.$$

$$\square$$

Remark 1. *For $d = 1$, (2.5) reduces to the well known generating function of Meixner polynomial set and (2.6) becomes the following linear functional for Meixner polynomials*

$$\langle u_0, f \rangle = (1 - c)^\beta \sum_{j=0}^{\infty} \frac{(\beta)_j}{j!} \frac{c^j}{f(j)} \qquad (2.9)$$

with $0 < c < 1$ and $\beta > 0$. Meixner polynomial set is orthogonal with respect to the linear functional given by (2.9).

Application 3: Suppose that $\{P_n\}_{n \geq 0}$ is a Sheffer polynomial set represented by (2.1) associated to the couple of polynomials

$$
\begin{aligned}
[\gamma_d(t), \sigma_3(t)] \quad = \quad & \left[-\frac{c}{c-1}(1-t)\left[(1-t)^2 + \frac{c-1}{2c}\left[(1-t)^2 - 1\right]\right] \pi'_{d-2}(t) \right. \\
& \left. -\frac{\beta c}{c-1}\left[(1-t)^2 + \frac{c-1}{2c}\left[(1-t)^2 - 1\right]\right], \right. \\
& \left. -\frac{c}{c-1}(1-t)\left[(1-t)^2 + \frac{c-1}{2c}\left[(1-t)^2 - 1\right]\right] \right]
\end{aligned}
$$

where $\pi_{d-2}(t) = \sum_{k=0}^{d-2} a_k t^k$ with $a_{d-2} \neq 0$ and $c \neq \{0, \frac{1}{3}, 1\}$. Taking Theorem 1 into account, we derive a new d-orthogonal polynomial set for $d \geq 2$ with

$$e^{\pi_{d-2}(t)} (1-t)^{-\beta} \left(1 + \frac{c-1}{2c} \frac{t^2 - 2t}{(1-t)^2}\right)^x = \sum_{n=0}^{\infty} P_n(x) \frac{t^n}{n!} \qquad (2.10)$$

The conditions (2.2) are satisfied since $a_{d-2} \neq 0$ and $c \neq \{0, \frac{1}{3}, 1\}$. These d-orthogonal polynomial sets (2.10) can not generate an orthogonal polynomial set since $d \geq 2$. But for $d = 2$, one can study the properties of this Meixner type 2-orthogonal polynomial set.

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