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ON $r-$ DYNAMIC COLORING OF THE FAMILY OF BISTAR GRAPHS

G.NANDINI, M.VENKATACHALAM, AND S.GOWRI

Abstract. An r-dynamic coloring of a graph G is a proper coloring c of the vertices such that $|c(N(v))| \geq min\{r, d(v)\}$, for each $v \in V(G)$. The r-dynamic chromatic number of a graph G is the minimum k such that G has an r-dynamic coloring with k colors. In this paper, we obtain the r-dynamic chromatic number of middle, total, central and line graph of Bistar graph.

1. Introduction

In this paper all graphs are loopless and connected. All undefined symbols and concepts may be looked up from [1]. The r-dynamic chromatic number was first introduced by Montgomery [12]. An r-dynamic coloring of a graph G is a map c from $V(G)$ to the set of colors such that (i) if $uv \in E(G)$, then $c(u) \neq c(v)$, and (ii) for each vertex $v \in V(G), |c(N(v))| \geq min\{r, d(v)\}$, where $N(v)$ denotes the set of vertices adjacent to v, $d(v)$ its degree and r is a positive integer. The first condition characterizes proper colorings, the adjacency condition and second condition is double-adjacency condition. The r-dynamic chromatic number of a graph G, written $\chi_r(G)$, is the minimum k such that G has an r-dynamic proper k-coloring. The 1-dynamic chromatic number of a graph G is equal to its chromatic number. The 2-dynamic chromatic number of a graph has been studied under the name dynamic chromatic number in [2, 3, 4, 6, 9]. There are many upper bounds and lower bounds for $\chi_d(G)$ in terms of graph parameters. For example, for a graph G with $\Delta(G) \geq 3$, Lai et al. [9] proved that $\chi_d(G) \leq \Delta(G) + 1$. An upper bound for the dynamic chromatic number of a d-regular graph G in terms of $\chi(G)$ and the independence number of G, $\alpha(G)$, was introduced in [7]. In fact, it was proved that $\chi_d(G) \leq \chi(G) + 2log_2\alpha(G) + 3$. Taherkhani gave in [13] an upper bound for

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$\chi_2(G)$ in terms of the chromatic number, the maximum degree Δ and the minimum degree δ. i.e.,

$$\chi_2(G) - \chi(G) \leq \left\lceil (\Delta e)/\delta log \left(2e\left(\Delta^2+1\right)\right)\right\rceil$$

Li et al. proved in [11] that the computational complexity of $\chi_d(G)$ for a 3-regular graph is an NP-complete problem. Furthermore, Liu and Zhou [10] showed that to determine whether there exists a $3-$dynamic coloring, for a claw free graph with the maximum degree 3, is NP-complete.

In this paper, we study $\chi_r(G)$, we find the $r-$dynamic chromatic number of the middle, total, central and line graphs of the Bistar graph.

2. Preliminaries

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The middle graph [14] of G, denoted by $M(G)$ is defined as follows. The vertex set of $M(G)$ is $V(G)\cup E(G)$. Two vertices x, y of $M(G)$ are adjacent in $M(G)$ in case one of the following holds: (i) x, y are in $E(G)$ and x, y are adjacent in G. (ii) x is in $V(G)$, y is in $E(G)$, and x, y are incident in G.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The total graph [14] of G, denoted by $T(G)$ is defined in the following way. The vertex set of $T(G)$ is $V(G)\cup E(G)$. Two vertices x, y of $T(G)$ are adjacent in $T(G)$ in case one of the following holds: (i) x, y are in $V(G)$ and x is adjacent to y in G. (ii) x, y are in $E(G)$ and x, y are adjacent in G. (iii) x is in $V(G), y$ is in $E(G)$, and x, y are incident in G.

The central graph [15] $C(G)$ of a graph G is obtained from G by adding an extra vertex on each edge of G, and then joining each pair of vertices of the original graph which were previously non-adjacent.

The line graph [8] of G denoted by $L(G)$ is the graph whose vertex set is the edge set of G. Two vertices of $L(G)$ are adjacent whenever the corresponding edges of G are adjacent.

The Bistar graph [5] $B_{m,n}$ is defined as the graph obtained from K_2 by joining m pendant edges to one end and n pendant edges to the other end of K_2. Let

$$V\left(B_{m,n}\right) = \{u, v\} \cup \{u_i : 1 \leq i \leq m\} \cup \{v_i : 1 \leq i \leq n\}$$

and

$$E\left(B_{m,n}\right) = \{e_i : 1 \leq i \leq m+n+1\},$$

where $e_i = uu_i \ (1 \leq i \leq m)\,,\ e_{m+1} = uv,\ e_{m+1+i} = vv_i \ (1 \leq i \leq n)\,.$

Theorem 2.1. *Let $m, n \geq 2$, $m \leq n$, the $r-$dynamic chromatic number of the line graph of a Bistar graph is*

$$\chi_r(L(B_{m,n})) = \begin{cases} n+1, & 1 \leq r \leq \Delta - m \\ r+1, & \Delta - m + 1 \leq r \leq \Delta \end{cases}$$

Proof. Let $V\left(L\left(B_{m,n}\right)\right)=\{e_1,e_2,\ldots,e_{m+n+1}\}$. Note that $deg(e_i)=m \ \ (1\leq i\leq m)$, $deg(e_{m+1})=m+n$, $deg(e_{m+1+i})=n \ \ (1\leq i\leq n)$.

By definition of the line graph, the vertices $\{e_i:(1\leq i\leq m+1)\}$induce a clique of order K_{m+1}in $L(B_{m,n})$. Also the vertices $\{e_i:(m+1\leq i\leq m+n+1)\}$induce a clique of order K_{n+1} in $L(B_{m,n})$. Thus, $\chi_r(L(B_{m,n}))\geq n+1$, for any r.

Case 1: $1\leq r\leq \Delta-m$

Consider the color function $c:V\left(L\left(B_{m,n}\right)\right)\to\{c_1,c_2,\ldots,c_{n+1}\}$ defined by $c(e_{i+m})=c_i$, $(1\leq i\leq n+1)$and $c(e_i)=c_{i+1}$, $(1\leq i\leq m)$.

It is clear that c is a r dynamic coloring and hence $\chi_r(L(B_{m,n}))\leq n+1$, $(1\leq r\leq\Delta-m)$.

Case 2: $\Delta-m+1\leq r\leq\Delta$

Consider the color function $c:V\left(L\left(B_{m,n}\right)\right)\to\{c_1,c_2,\ldots,c_{r+1}\}$ defined by $c(e_{i+m})=c_i$, $(1\leq i\leq n+1)$. In order to maintain r-adjacency condition we need at least $r-n$ new colors to color the remaining vertices. Color the vertices $\{e_i:(1\leq i\leq m)\}$ consecutively with the colors $c_{n+2},\ldots,c_{r+1},c_i$. Hence, $\chi_r(L(B_{m,n}))\leq r+1$.

It is clear that c is a r dynamic coloring and hence

$$\chi_r(L(B_{m,n}))=\begin{cases}n+1, & 1\leq r\leq\Delta-m\\ r+1, & \Delta-m+1\leq r\leq\Delta\end{cases}$$

□

Theorem 2.2. *Let $m,n\geq 2$, $m\leq n$, the r-dynamic chromatic number of the middle graph of a Bistar graph is*

$$\chi_r(M(B_{m,n}))=\begin{cases}n+2, & 1\leq r\leq n+1\\ r+1, & n+2\leq r\leq\Delta\end{cases}$$

Proof. Let $V\left(M\left(B_{m,n}\right)\right)=\{u,v,g\}\cup\{u_i,e_i:(1\leq i\leq m)\}\cup\{v_i,f_i:(1\leq i\leq n)\}$, where e_i is the vertex corresponding to the edge uu_i, $(1\leq i\leq m)$, f_i is the vertex corresponding to the edge vv_i, $(1\leq i\leq n)$ and g is the vertex corresponding to the edge uv of $B_{m,n}$.

Note that $deg(e_i)=m+1$, $deg(f_i)=n+1$, $deg(g)=m+n+2$, $deg(u_i)=deg(v_i)=1$, $deg(u)=m+1$, $deg(v)=n+1$.

By definition of the Middle graph, the vertices $\{g,\ v,\ f_i:(1\leq i\leq n)\}$ induce a clique of order K_{n+2} in $M(B_{m,n})$. Thus, $\chi_r(M(B_{m,n}))\geq n+2$, for any r.

Case 1: $1\leq r\leq n+1$

Consider the color function $c:V\left(M\left(B_{m,n}\right)\right)\to\{c_1,c_2,\ldots,c_{n+2}\}$ defined by $c(g)=c_1$, $c(v)=c(u)=c_2$, $c(f_i)=c_{i+2}$, $(1\leq i\leq n)$, $c(e_i)=c_{i+2}$, $(1\leq i\leq m)$, $c(v_i)=c_1$, $(1\leq i\leq n)$ and for $(1\leq i\leq m)$

$$c(u_i)=\begin{cases}c_{n+2}, & m<n\\ c_1, & m=n\end{cases}$$

It is clear that c is a r dynamic coloring and hence $\chi_r(M(B_{m,n}))\leq n+2$.

Case 2: $n+2\leq r\leq\Delta$

Consider the color function $c : V(M(B_{m,n})) \to \{c_1, c_2, \dots, c_{r+1}\}$ defined by $c(g) = c_1$, $c(v) = c_2$, $c(f_i) = c_{i+2}$, $(1 \le i \le n)$, $c(u_i) = c_2$, $(1 \le i \le m)$.

In order to maintain r-adjacency condition we need at least $r-n-1$ new colors to color the remaining vertices. $c(v_i) = c(u) = c_{n+3}$. For $(1 \le i \le m)$, $c(e_i) = c_{i+2}$, if $c(u) = c_{r+1}$, otherwise color the vertices $\{e_i : (1 \le i \le m)\}$ consecutively with the colors $c_{n+4}, \dots, c_{r+1}, c_{i+2}$. Hence, $\chi_r(M(B_{m,n})) \le r+1$.
It is clear that c is a r dynamic coloring and hence

$$\chi_r(M(B_{m,n})) = \begin{cases} n+2, & 1 \le r \le n+1 \\ r+1, & n+2 \le r \le \Delta \end{cases}$$

□

Theorem 2.3. *Let $m, n \ge 2$, $m \le n$, the r-dynamic chromatic number of the total graph of a Bistar graph is*

$$\chi_r(T(B_{m,n})) = \begin{cases} n+2, & 1 \le r \le n+1 \\ r+1, & n+2 \le r \le \Delta \end{cases}$$

Proof. Let $V(T(B_{m,n})) = \{u, v, g\} \cup \{u_i, e_i : (1 \le i \le m)\} \cup \{v_i, f_i : (1 \le i \le n)\}$, where e_i is the vertex corresponding to the edge uu_i, $(1 \le i \le m)$, f_i is the vertex corresponding to the edge vv_i, $(1 \le i \le n)$ and g is the vertex corresponding to the edge uv of $B_{m,n}$.
Note that $deg(e_i) = m+2$, $deg(f_i) = n+2$, $deg(g) = m+n+2$, $deg(u_i) = deg(v_i) = 2$, $deg(u) = 2m+2$, $deg(v) = 2n+2$.

By definition of the Total graph, the vertices $\{g,\ v,\ f_i : (1 \le i \le n)\}$ induce a clique of order K_{n+2} in $M(B_{m,n})$. Thus, $\chi_r(T(B_{m,n})) \ge n+2$, for any r.
Case 1: $1 \le r \le n+1$
Consider the color function $c : V(T(B_{m,n})) \to \{c_1, c_2, \dots, c_{n+2}\}$ defined by $c(g) = c(v_i) = c_1$, $c(v) = c_2$, $c(f_i) = c_{i+2}$, $(1 \le i \le n)$, $c(e_i) = c_{i+1}$, $(1 \le i \le m)$, $c(u) = c_{m+2}$ and

$$c(u_i) = \begin{cases} c_{n+1}, n \text{ is odd} \\ c_{n+2}, n \text{ is even} \end{cases}$$

It is clear that c is a r dynamic coloring and hence $\chi_r(T(B_{m,n})) \le n+2$.
Case 2: $n+2 \le r \le \Delta$
Consider the color function $c : V(T(B_{m,n})) \to \{c_1, c_2, \dots, c_{r+1}\}$ defined by $c(g) = c_1$, $c(v) = c_2$, $c(f_i) = c_{i+2}$, $(1 \le i \le n)$.

In order to maintain r-adjacency condition we need atleast $r-n-1$ new colors to color the remaining vertices. Color the vertices $\{u, v_i : (1 \le i \le n)\}$ consecutively with the colors $c_{n+3}, \dots, c_{r+1}$ and for $(1 \le i \le m)$ $c(e_i) = c_{i+1}$, if $c(u) = c_{r+1}$, otherwise color the vertices $\{e_i : (1 \le i \le m)\}$ consecutively with the colors $c_{n+4}, \dots, c_{r+1}, c_{i+1}$. For $(1 \le i \le m)$ assign to the vertex u_i one of the allowed colors - such color exists, because $deg(u_i) = 2$.

Hence, $\chi_r(T(B_{m,n})) \le r+1$.

It is clear that c is a r dynamic coloring and hence

$$\chi_r(T(B_{m,n})) = \begin{cases} n+2, & 1 \leq r \leq n+1 \\ r+1, & n+2 \leq r \leq \Delta \end{cases}$$

□

Theorem 2.4. *Let $m, n \geq 2$, $m \leq n$, the r-dynamic chromatic number of the Central graph of a Bistar graph is*

$$\chi_r(C(B_{m,n})) = \begin{cases} m+n, & r=1 \\ m+n+2, & 2 \leq r \leq \Delta - 1 \\ m+2n+3, & r = \Delta \end{cases}$$

Proof. Let $V\left(C\left(B_{m,n}\right)\right) = \{u, v, g\} \cup \{u_i, e_i : (1 \leq i \leq m)\} \cup \{v_i, f_i : (1 \leq i \leq n)\}$, where e_i is the vertex corresponding to the edge uu_i, $(1 \leq i \leq m)$, f_i is the vertex corresponding to the edge vv_i, $(1 \leq i \leq n)$ and g is the vertex corresponding to the edge uv of $B_{m,n}$.

Note that $deg(e_i) = deg(f_i) = deg(g) = 2$, $deg(u_i) = deg(v_i) = deg(u) = deg(v) = m+n+1$.

By definition of the Central graph, the vertices $\{u, v_i : (1 \leq i \leq n)\}$ induce a clique of order K_{n+1} in $C(B_{m,n})$. Moreover the vertices u_i $(1 \leq i \leq m)$ is adjacent to the vertices v_i $(1 \leq i \leq n)$. Thus, $\chi_r(C(B_{m,n})) \geq m+n$, for any r.

Case 1: $r = 1$

Consider the color function $c : V\left(C\left(B_{m,n}\right)\right) \to \{c_1, c_2, \ldots, c_{m+n}\}$ defined by $c(u_i) = c_i$, $(1 \leq i \leq m)$, $c(v_i) = c_{m+i}$, $(1 \leq i \leq n)$, $c(u) = c(f_i) = c_1$, $c(g) = c_2$, and $c(v) = c(e_i) = c_{m+1}$.

It is clear that c is a r dynamic coloring and hence $\chi_r(C(B_{m,n})) \leq m+n$.

Case 2: $2 \leq r \leq \Delta - 1$

Consider the color function $c : V\left(C\left(B_{m,n}\right)\right) \to \{c_1, c_2, \ldots, c_{m+n+2}\}$ defined by $c(g) = c_1$, $c(u_i) = c_i$, $(1 \leq i \leq m)$, $c(u) = c_{m+n+1}$, $c(v) = c_{m+n+2}$, $c(v_i) = c_{m+i}$ $(1 \leq i \leq n)$, $c(f_{n-i}) = c_{m+1+i}$, $(0 \leq i \leq n-1)$, $c(e_i) = c_{m+1-i}$, $(1 \leq i \leq m)$.

It is clear that c is a r dynamic coloring and hence $\chi_r(C(B_{m,n})) \leq m+n+2$.

Hence, $\chi_r(C(B_{m,n})) \leq m+n+2$.

Case 3: $r = \Delta$

Consider the color function $c : V\left(C\left(B_{m,n}\right)\right) \to \{c_1, c_2, \ldots, c_{m+2n+3}\}$ defined by $c(u_i) = c_i$, $c(v_i) = c_{m+i}$, $(1 \leq i \leq n)$, $c(u) = c_{m+n+1}$, $c(v) = c_{m+n+2}$, $c(g) = c_{m+2n+3}$, $c(f_i) = c_{m+n+2+i}$, $c(e_i) = c_{m+n+2+i}$.

It is clear that c is a r dynamic coloring and hence $\chi_r(C(B_{m,n})) \leq m+2n+3$.

Hence, $\chi_r(C(B_{m,n})) \leq m+2n+3$.

It is clear that c is a r dynamic coloring and hence

$$\chi_r(C(B_{m,n})) = \begin{cases} m+n, & r=1 \\ m+n+2, & 2 \leq r \leq \Delta - 1 \\ m+2n+3, & r = \Delta \end{cases}$$

□

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Current address: G.Nandini: Department of Mathematics, SNS College of Technology, Coimbatore - 641 035 Tamil Nadu, India.
E-mail address: nandiniap2006@gmail.com
ORCID Address: http://orcid.org/0000-0002-9368-571X
Current address: M.Venkatachalam: Department of Mathematics, Kongunadu Arts and Science College, Coimbatore - 641 029, Tamil Nadu India
E-mail address: venkatmaths@gmail.com
ORCID Address: http://orcid.org/0000-0001-5051-4104
Current address: S.Gowri: Department of Mathematics, SNS College of Technology, Coimbatore 641035 Tamil Nadu, India.
E-mail address: gowrisathasivam@gmail.com
ORCID Address: http://orcid.org/0000-0001-5849-3213