Çankaya University Journal of Science and Engineering
Volume 15, No. 2 (2018) 049–053
Date Received: 16/09/2018
Date Accepted: 03/10/2018

A prey-predator model with a refuge for prey

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Abstract: In this paper, we see a prey-predator mathematical model with a refuge for prey. Dynamical behaviours such as boundary and stability are considered.

Keywords: Prey-predator system, stability, equilibrium point, refuge.

1. Introduction

Mathematical modeling has been an interesting and useful subject for research work in studying of ecosystems. Mathematical modeling of ecosystems was initiated in 1924 by Lotka [6] and in 1931 by Volterra [8]. They introduced three relations as predator-prey interaction, competitive system and cooperative or mutualist system. Since then, many interesting situations in these ecosystems were investigated by researches with helpful mathematical models.

Lakshminarayan and Apparao investigated a prey-predator model with a cover linearly varying with the prey population and an alternative food for the predator [4]. Disease in prey population and its probable affects on predator were considered in [for example 9]. Also, some investigations with prey-predator subject with basis on delay were considered [for example 5,7]. We can see stability of more than one predator (prey) in [2,3].

In this paper , we consider a prey-predator interaction; meaning, one of species is served as food for the other and is destroyed. The population is divided into two sections. One section is considered as group of prey and the other is group of predator.

We assume $N(t)$ and $P(t)$ denote the number of population of prey and the number of population of predator, respectively. In the absence of predation, prey population grows logistically with carrying capacity K. In the absence of prey, the predator suffers natural mortality with positive constant coefficient c. A positive coefficient for per capita growth rate of prey is denoted by r. b and q are two positive constant for predation and conversion rate, respectively.

ISSN 2564 - 7954

2. The model

In this section, at first we see a mathematical modeling for prey and predator population . In the following, by considering boundary we can see the model is acceptable. We have this assumption that the prey population is served for the predator population by $\frac{NP}{aP+N}$, $a > 0$.

In this model, we have refuge; meaning, some prey are saved by a coefficient $m \in [0,1)$ or the population of prey who are exposed in danger is $(1-m)N(t)$. Then, in the presence of refuge, the model is as the following:

$$\begin{cases} \dot{N(t)} = rN(1-\frac{N}{K}) - \frac{b(1-m)NP}{aP+(1-m)N} \\ \\ \dot{P(t)} = -cP + \frac{qb(1-m)NP}{aP+(1-m)N} \end{cases} \tag{1}$$

By choosing $p := \frac{aP}{K}$ and $n := \frac{N}{K}$, the system (1) can be the form

$$\begin{cases} \dot{n(t)} = rn(1-n) - \frac{b(1-m)np}{ap+a(1-m)n} \\ \\ \dot{p(t)} = -cp + \frac{qb(1-m)np}{p+(1-m)n} \end{cases} \tag{2}$$

with initial data $p(0) \geq 0$, $n(0) \geq 0$.

Case (a), Now we see if the population of prey is less than the capacity of environment, then $p(t)$ and $n(t)$ are bounded.
Since the population of prey is less than the capacity of environment, we have $n(t) \leq 1$.
Also $\dot{p(t)} = -cp(t) + \frac{qb(1-m)n(t)p(t)}{p(t)+(1-m)n(t)}$ or

$$\dot{p(t)} + cp(t) = \frac{qb(1-m)n(t)p(t)}{p(t)+(1-m)n(t)} \leq qb(1-m)\frac{p(t)}{p(t)+(1-m)n(t)} \leq qb(1-m)$$

Then we get,
$\limsup\ p(t) \leq \frac{1}{c}qb(1-m)$.
$t \to \infty$
Case (b), if $N(t) > K$ for some t, in a period of time, then by (2), $\dot{n(t)} < 0$ or N is decreasing. We can see in this period of times $n(t) < n(t_0)$, for some t_0.

Now, $p(t)$ is bounded, because

$$\limsup_{t \to \infty} p(t) \leq \frac{1}{c} qb(1-m)n(t_0) \text{ in the period of times.}$$

Then by cases (a) and (b) we get:

Theorem. $p(t)$ is bounded where is a good evaluation for natural population of predator

3. Stability

In this section, we want to consider the stability of fixed points of the model. Clearly, we see $(0,0), (0,1)$ and $(n, \frac{arnq(1-n)}{c}), \forall n \in (0,1]$ are fixed points of the system (2).
It is very important to be careful that $n \in (0,1)$ if and only if $N < K$ or $(n, \frac{arnq(1-n)}{c})$ can be equilibrium if the population of prey is less than the capacity of environment, K.

Theorem. Fixed point $(1,0)$ is globally asymptotically stable if the product of predation rate and conversion rate is less than coefficient of mortality of predator.

Proof.
By choosing $g := \frac{b(1-m)}{a}$ and $q^{'} = aq$, system (2) is formed as the following:

$$\begin{cases} \dot{n(t)} = rn(1-n) - \frac{gnp}{p+(1-m)n} \\ \\ \dot{p(t)} = -cp + \frac{q^{'} gnp}{p+(1-m)n} \end{cases} \qquad (3)$$

The Jacobian matrix in $(1,0)$ is

$$\begin{bmatrix} -r & \frac{-g}{1-m} \\ 0 & -c + \frac{q^{'} g}{1-m} \end{bmatrix}$$

. Since the eigenvalues of the matrix are $-r$ and $-c + \frac{q^{'} g}{1-m}$, we have $(1,0)$ is globally asymptotically stable if $qb < c$ or the product of predation rate and conversion rate is less than coefficient of mortality of predator.

Remark. (1) Clearly, $(0,0)$ is globally asymptotically stable.

(2) $(n, \frac{rnq^{'}(1-n)}{c})$ can be asymptotically stable for $n \in (0,1)$; it is depended on $r, g, h, q^{'}, m, c$. These quantities are given by ecosystem.

The Jacobian matrix of system (3) at equilibrium point $(n, \frac{rnq^{'}(1-n)}{c})$, for all $n \in (0,1)$ is

$$\begin{bmatrix} r-2rn-\frac{gh^2(1-n)^2}{(h(1-n)+(1-m))^2} & \frac{-g(1-m)}{(h(1-n)+(1-m))^2} \\ \frac{q^{'}gh^2(1-n)^2}{(h(1-n)+(1-m))^2} & -c+\frac{q^{'}g(1-m)}{(h(1-n)+(1-m))^2} \end{bmatrix}$$

where, $h := \frac{rq^{'}}{c}$ is a constant.
The characteristic polynomial of the above matrix is $\lambda^2 + a_1\lambda + a_0 = 0$, where

$a_1 = 2rn - r + c + \frac{gh^2(1-n)^2 - gq^{'}(1-m)}{(h(1-n)+(1-m))^2}$ and

$$a_0 = (r - 2rn - \frac{gh^2(1-n)^2}{(h(1-n)+(1-m))^2})(-c + \frac{q^{'}g(1-m)}{(h(1-n)+(1-m))^2}) + \frac{q^{'}g^2h^2(1-n)^2(1-m)}{(h(1-n)+(1-m))^2}$$

for some $n \in (0,1)$ where $a_1, a_0 > 0$, we have $(n, \frac{rnq^{'}(1-n)}{c})$ is asymptotically stable [by 1].
It is very clear that, for this aim the quantities of $r, g, h, q^{'}, m, c$ are important.

4. Conclusions

In this paper, we considered a mathematical model for a prey-predator ecosystem. In this ecosystem, some prey can be saved from danger of the predator. We studied some notes about this model. In the next search, we see a disease in prey and its affects on predator in this ecosystem.

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