**IConTES 2018: International Conference on Technology, Engineering and Science**


# Infinitely Remote Singularities of Special Differential Dynamic Systems

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**Abstract**: The work is devoted to the results of a fundamental study on the arithmetical plane of a broad special family of differential dynamic systems having polynomial right parts. Let those polynomials be a cubic and a square reciprocal forms. A task of a whole investigation was to find out all topologically different phase portraits in a Poincare circle and indicate close to coefficient criteria of them. To achieve this goal a Poincare method of the central and the orthogonal consecutive displays (or mappings) has been used. As a rezult more than 250 topologically different phase portraits in a total have been constructed. Every portrait we depict with a special table called a descriptive phase portrait. Each line of such a special table corresponds to one invariant cell of the phase portrait and describes its boundary, a source of its phase flow and a sink of it. All finite and infinitely remote singularities of dynamic systems under consideration were fully investigated. Namely infinitely remote singularities are discussed in the present article.

**Keywords:** Dynamic systems, Phase portraits, Phase flows, Poincare sphere, Poincare circle, Singular points, Separatrices, Trajectories

## Introduction

A dynamic system works as a mathematical model of a phenomenon or a process, for which fluctuations and other statistical events we don't consider. A dynamic system can be characterized with its initial state and a law transforming it into a different state. A *phase space* of a dynamic system is a totality of all admissible states of it. Its naturally to distinguish dynamic systems with the discrete and with the continuous time. For dynamic systems with the discrete time (the *cascades*) a system's behavior is described with a sequence states. For dynamic systems with continuous time (the *flows*) a state of a system is defined for each moment of time on a real or an imaginary axis. Cascades and flows are the main subject of study in a symbolic and topological dynamics.

Dynamic systems of the both kinds can be usually described with an autonomous system of differential equations, defined in a certain domain and satisfying in it the conditions of the Cauchy theorem of existence and uniqueness of solutions of the differential equations.Singular points of differential equations correspond to equilibrium positions of dynamic systems, and periodical solutions of differential equations correspond to closed phase curves of dynamic systems.

The mostly important problem of the theory of dynamic systems is a study of curves, defined by differential equations. This process includes splitting of a phase space into trajectories and study of a limit behavior of those: finding out and classification of equilibrium positions, revealing of attracting and repulsive manifolds.

The important notions of a theory of dynamic systems are a notion of a stability of equilibrium states, i.e. an ability of a system to remain near an equilibrium state (or on a given manifold) for an arbitrary long period of time under considerably small changes of initial data, as well as a notion of a roughness of a system (i.e. saving of system`s properties under small changes of a model itself). A rough dynamic system is a system which preserve its qualitative character of motion under a satisfactory small change of parameters.

Jules H. Poincare has shown, that any normal autonomous second-order differential system with polynomial right parts in principle allows its full qualitative investigation on an extended arithmetical plane $\bar{R}^2_{x,y}$ [1]. Further

investigators have successfully studied some of such systems, for example quadratic dynamic systems [2], systems containing nonzero linear terms, homogeneous cubic systems, as well as systems with nonlinear homogeneous terms of the odd degrees (3, 5, 7) [3], which have a center or a focus in a singular point O (0,0) [4], and some other particular kinds.

Here we consider a special family of dynamic systems on a real plane *x, y*

$$\frac{dx}{dt} = X(x,\ y), \quad \frac{dy}{dt} = Y\ (x,\ y), \tag{1}$$

such as *X (x, y), Y (x, y)* are reciprocal forms of *x* and *y*, *X* be a cubic, and *Y* be a square form, and *X (0,1) > 0, Y (0, 1) > 0*. Our task is to depict in a Poincare circle all kinds of possible for the Eq.(1) - systems phase portraits, and indicate close to coefficient criteria of each portrait appearance. We use a Poincare method of consecutive mappings: firstly the central mapping of a plane *x, y* (from a center (0, 0, 1) of a sphere $\sum$), augmented with a line at infinity (i.e. $\bar{R}^2_{x,y}$ plane) on a sphere $\sum : X^2 + Y^2 + Z^2 = 1$ with identified diametrically opposite points, and secondly the orthogonal mapping of a lower enclosed semi sphere of a sphere $\sum$ to a circle $\bar{\Omega}: x^2 + y^2 \leq 1$ with identified diametrically opposite points of its boundary $\Gamma$.

The circle $\bar{\Omega}$ and the sphere $\sum$ are called the Poincare circle and the Poincare sphere correspondingly [1].

## Basic Definitions and Notation

$\varphi(t,p),\ p = (x,y)$ —a fixed point := a solution (a motion) of an Eq.(1) – system with initial data $(0,p)$.
$L_p : \varphi = \varphi(t,p), t \in I_{max,\ .}$ a trajectory of a motion $\varphi(t,p)$.
$L_p^{+(-)} :=$ +(-)- a semi trajectory of a trajectory $L_p$.

*O*-curve of a system := its semi trajectory $L^s_p$ ($p \neq O$, $s \in \{+,-\}$ ), adjoining to a point *O* under a condition that $st \to +\infty$.

$O^{+(-)}$- curve of a system := its *O*-curve $L_p^{+(-)}$.
$O_{+(-)}$-curve of a system := its *O*-curve, adjoining to a point *O* from a domain $x > 0$ $(x < 0)$.

*TO*-curve of a system := its *O*-curve, which, being supplemented by a point *O,* touches some ray in it.

A nodal bundle of *NO*-curves of a system := an open continuous family of its *TO*-curves $L^s_{p,}$ where $s \in \{+,-\}$ is a fixed index, $p \in \varLambda$, $\varLambda$ - a simple open arc, $L^s_p \cap \varLambda = \{p\}$.

A saddle bundle of *SO*-curves of a system, a separatrix of the point *O*:= a fixed *TO* -curve, which is not included into some bundle of *NO*-curves of a system.

*E, H, P* - *O*-sectors of a system: an elliptical, a hyperbolic, a parabolic ones.

A topological type (T-type) of a singular point *O* of a system := a word $A_O$ consisting of letters *N,S* (a word $B_O$ consisting of letters *E,H,P*), which describes a circular order of bundles *N, S* of its *O*-curves (of its *O*-sectors *E, H, P* ) when traversing the point *O* in the «+»-direction, i. e. counterclockwise, starting with some of them.

$P(u) := X(1,u) \equiv p_0 + p_1 u + p_2 u^2 + p_3 u^3,$
$Q(u) := Y(1,u) \equiv a + bu + cu^2.$

**Note 1.** For every Eq.(1) - system:

1) T-type of a singular point *O* in its form $B_O$ is easy to construct using its T-type in the form $A_O$ , and backwards (we need to know the both forms, see below the Corollary 1);
2) real roots of a polynomial $P(u)$ (polynomial $Q(u)$) are in fact angular coefficients of isoclines of infinity (isoclines of a zero);

3) when we write out the real roots of the system`s polynomials $P(u)$, $Q(u)$, separately or all together, we always number the roots of each one of them in an ascending order.

## Topological Type (T – type) of a Singular Point O (0, 0)

In order to find all $O$-curves and to split their totality into the bundles $N$, $S$, let us use the method of exceptional directions of a system in the point O [1]. According to this source, the equation of exceptional directions for the point O of the Eq.(1) - system has the form

$$xY(x,y) \equiv x(ax^2 + bxy + cy^2) = 0.$$

For it the follows cases are possible:

1) when $d \equiv b^2 - 4ac > 0$ this equation defines simle straight lines $x = 0$ and
$y = q_i x$, $i = 1,2$, $q_1 < q_2$,
2) when $d = 0$ this equation defines the straight line $x = 0$ and the double straight line
$y = qx, q = -\frac{b}{2c}$,
3) when $d < 0$ – only a straight line $x = 0$.

The follows Theorem 1 takes place for them [5] .

**Theorem 1.** Words $A_O$ and $B_O$ which define a topological type (T-type) of a singular point O(0, 0) of the Eq.(1) - system:

1) in the case of $d > 0$ depending on signs of values $P(q_i)$, $i = 1,2$, have forms, indicated in a Table 1,
2) in the case of $d = 0$ depending on signs of values $q$ and $P(q)$ - forms, indicated in a Table 2,
3) in the case of $d < 0$ a form: $A_O = S_0 S^0$, $B_O = HH$:

Table 1. T-type of a singular point in the case of $d > 0$ ($r = \overline{1,6}$)

| $r$ | $P(q_1)$ | $P(q_2)$ | $A_O$ | $B_O$ |
|---|---|---|---|---|
| 1, 4 | + | + | $S_0 S_+^1 N_+^2 S^0 N_-^1 S_-^2 = S_0 S_+^1 N S_-^2$ | $PH^2$ |
| 2 | − | − | $S_0 N_+^1 S_+^2 S^0 S_-^1 N_-^2 = N S_+^2 S^0 S_+^1$ | $PH^2$ |
| 3, 6 | − | + | $S_0 N_+^1 N_+^2 S^0 S_-^1 S_-^2$ | $PEPH^3$ |
| 5 | + | − | $S_0 S_+^1 S_+^2 S^0 N_-^1 N_-^2$ | $H^3 PEP$ |

Table 2. T-type of the singular point O(0, 0) in the case of d = 0

| $q$ | $P(q)$ | $A_O$ | $B_O$ |
|---|---|---|---|
| + | + | $S_0 S_+ N_+ S^0$ | $H^2 P$ |
| − | − | $S_0 N_+ S_+ S^0$ | $PH^2$ |
| + | − | $S_0 S^0 S_- N_-$ | $H^2 P$ |
| − | + | $S_0 S^0 N_- S_-$ | $PH^2$ |
| 0 | + | $S_0 S_+ N S_-$ | $H^2 P$ |
| 0 | − | $N S_+ S^0 S_-$ | $PH^2$ |

**Note 2**. Let`s clarify the meaning of new symbols introduced in the Theorem 1.

A symbol $S_0$ (a symbol $S^0$) means a bundle $S$, adjoining to the point $O(0, 0)$ from the domain $x > 0$ along a semi axis $x = 0, y < 0$, when $t \to +\infty$ (along a semi axis $x = 0$, $y > 0$, when $t \to -\infty$).

A lower sign index «+» or «-» of every bundle $N$ or $S$, different from $S_0$ and $S^0$, indicates wheather the bundle consists of $O_+$-curves or of $O_-$-curves. Upper index 1 or 2 of every such a bundle indicates wheather its $O$-curves adjoining to the point $O$ along a straight line $y = q_1 x$ or along a straight line $y = q_2 x$.

In the Table 2, lines No. 5, 6, a bundle $N$ doesn`t have a lower sign index, because it contains both $O_+$ -curves and $O_-$ -curves simultaneously.

**Corollary 1.** From the Theorem 1 it follows, that Eq.(1) – systems do not have limit cycles on the $R^2{}_{x;y}$ plane.

Indeed, such a cycle could surround a singular point O (0,0) of a Eq.(1) - system, and then the Poincare index of this singular point must be equal to 1 [1]. But Bendixon`s formula for the index of an isolated singular point of a smooth dynamic system is the follows:

$$I(O) = 1 + \frac{e-h}{2},$$

where $e$ $(h)$ is a number of elliptical (hyperbolic) $O$-sectors of the system. This formula and our Theorem 1 give: for the singular point O (0, 0) of every Eq.(1) – system Poincare index $I(O) = 0$.

**Corollary 2.** For the singular point O (0, 0) of the Eq.(1) - system 11 (eleven) different topological types (T-types) are possible, and from the analysis of these 11 its T-types follows:

For every Eq.(1) – system the singular point O(0, 0) has not more than four separatrices (actually 2, 3 or 4 ones).

## Infinitely Remote Singular Points (IR-points)

Now it`s time to discuss a behavior of trajectories of the Eq.(1) – systems in a neighborhood of infinity. For the investigation of this question we use a method of the Poincare consecutive transformations, or mappings [1].

The first Poincare transformation

$$x = \frac{1}{z}, \quad y = \frac{u}{z} \quad (u = \frac{y}{x}, \quad z = \frac{1}{x})$$

anambiguosly maps a phase plane $R^2{}_{x,y}$ of the Eq.(1) - system onto a Poincare sphere $\sum$ : $x^2 + y^2 + z^2 = 1$ (where $z = -Z$ [1]) with the diametrically opposite points identified, which is considered without it`s equator $E$, and an infinitely remote straight line of a plane $\overline{R^2_{x,y}}$ the first Poincare transformation maps onto the equator $E$ of the sphere $\sum$, which diametrically opposite points are also considered to be identified.

The Eq.(1) - system this mapping transforms into a system, which in the Poincare coordinates $u, z$ after a time change $dt = -z^2 d\tau$ looks like

$$\frac{du}{d\tau} = P(u)u - Q(u)z, \quad \frac{dz}{d\tau} = P(u)z,$$

where $P(u) :\equiv X(1, u)$ and $Q(u) :\equiv Y(1, u)$ are reciprocal polynomials. This new system is determined on the whole sphere $\sum$, including its equator, and on the whole $(u, z) -$ **plane**
$\alpha^*$, **which is tangent to a sphere** $\sum$ **at a point** $C = (1,0,0)$. We shall study it namely on a plane $\overline{R^2_{u,z}}$, and received results we`ll project onto a closed circle $\overline{\Omega}$, sequentially mapping firstly a plane $R^2{}_{u,z}$ onto the sphere $\sum$ from its center, and secondly its lower semi sphere $\overline{H}$ onto the Poincare circle $\overline{\Omega}$, i. e. onto a closed unit circle of a plane $R^2{}_{x,y}$ through the orthogonal mapping.

For our new system the axis $z = 0$ is invariant (consists of this system`s trajectories). On this axis lie its singular points $O_i$ $(u_i, 0)$, $i = \overline{0, m}$, where $u_i$, $i = \overline{1, m}$ are all real roots of the polynomial $P(u)$, and $u_0 = 0$; the same time may exist $i_0 \in \{1, \dots, m\}$: $u_{i_0} = 0$. Let`s call such points IR-points of the 1-st kind for the Eq.(1) – system.

The second Poincare transformation

$$x = \frac{v}{z}, \qquad y = \frac{1}{z} \qquad \left(v = \frac{x}{y}, \qquad z = \frac{1}{y}\right)$$

also anambiguosly maps a phase plane $R^2_{x,y}$ onto a Poincare sphere $\sum$ with the diametrically opposite points identified, considered without it`s equator, and every Eq.(1) - system transforms into a system, which in the coordinates $\tau, v, z$ looks like:

$$\frac{dv}{d\tau} = -X(v, 1) + Y(v, 1)vz, \quad \frac{dz}{d\tau} = Y(v, 1)z^2.$$

This last system is determined on the whole sphere $\sum$, and on the whole $(v, z) -$ plane $\hat{\alpha}$, which is tangent to a sphere $\sum$ at a point $D = (0, 1, 0)$ [1]. A set $z = 0$ is invariant for this last system. On this set lie its singular points $(v_0, 0)$, where $v_0$ is any real root of the polynomial $X(v, 1) \equiv p_3 + p_2 v + p_1 v^2 + p_0 v^3$. It would be naturally to call such points IR-points of the 2-st kind for the Eq.(1) – system, but each of these points, for which $v_0 \neq 0$, obviously coincides with one of the IR-points of the 1-st kind, namely with the point $\left(\frac{1}{v_0}, 0\right)$, while a number $v_0 = 0$ is not a root of the polynomial $X(x, 1)$, because $X(0, 1) = p_3 \neq 0$ for the Eq.(1) – system. Consequently, correct is the follows

**Corollary 3.** The infinitely remote singular points of any Eq.(1) – system are only IR-points of the 1-st kind.

With the orthogonal projection of a closed lower semi sphere $\bar{H}$ of a Poincare sphere $\sum$ onto a plane $x, y$ its open part $H$ one-to-one maps on an open Poincare circle $\Omega$, while its boundary $E$ (an equator of the Poincare sphere $\sum$) maps on the bondary of the Poincare circle $\Gamma = \partial\Omega$. =>

    1) Own trajectories of any Eq.(1) - system (including its singular point O $(0, 0)$) are displayed into a circle $\Omega$ and fill it.
    2) Such a system`s infinitely remote trajectories (including IR-points) are displayed on a boundary $\Gamma$ of a circle $\Omega$, filling it.
    Following Poincare, we call the first of them trajectories of the Eq.(1) - system in $\Omega$ , and the second we call trajectories of the Eq.(1) - system on $\Gamma$.

As it follows from the abovementioned conclusions, to each IR-point $O_i\ (u_i, 0)$, of the Eq.(1) – system, $i \in \{1, ...,m\}$, correspond two diametrically opposite points situated on the $\Gamma$ circle

$$O_i^{\pm}(u_i, 0): O_i^+\ (O_i^-) \in \Gamma^{+(-)} := \Gamma|_{x>0\ (x<0)}.$$
$\forall\ i\ \in \{1, ...,m\}$ for the point $O_i^+\ (O_i^-)$ we shall introduce the follows notation.

    1) Let a $O_i^+\ (O_i^-)$—curve be a semi trajectory of the Eq.(1) – system in $\Omega$, starting in an ordinary point $p \in \Omega$ and adjacent to a point $O_i^{+(-)}$.
    2) A notation of bandles $N$, $S$, adjacent to a point $O_i^+\ (O_i^-)$ from the circle $\Omega$, let be similar to notation introduced for the point O $(0, 0)$.
    3) A notation of a word $A_i^+\ (A_i^-)$ consisting of letters $N,S$, which is fixing an order of bandles of $O_i^+\ (O_i^-)$-curves at a semi circumvention of the point $O_i^+\ (O_i^-)$ in the circle $\Omega$ in the direction of increasing of $u$.

We shall describe a T-type of a point $O_i^+\ (O_i^-)$ with a word $A_i^+\ (A_i^-)$, and a T-type of a point $O_i$ with words $A_i^{\pm}$.
T-types of IR-points $O_0^{\pm}\ (0,0)$ of Eq.(1) – systems are described with the follows theorem.

**Theorem 2.** Let a number $u = 0$ be a multiplicity $k \in \{0, ...,3\}$ root of a polynomial $P(u)$ of the Eq.(1) - system. Then words $A_0^{\pm}$, which determine topologycal types (T-types) of IR-points $O_0^{\pm}\ (0,0)$ of this system, depending on the value of $k$ and a sign of a number $ap_k$ (where $a$ and $p_k$ are coefficients of the system), have the forms indicated in the Table 3 [5].

Table 3. T-types of IR-points $O_0^{\pm}$ $(0,0)$.

| $k$ | $ap_k$ | $A_0^+$ | $A_0^-$ |
|-----|--------|---------|---------|
| 0 | 0 | $N$ | $N$ |
| 0, 2 | + (-) | $N_+(N_-)$ | $N_-(N_+)$ |
| 1, 3 | + (-) | $N_-N_+(\emptyset)$ | $\emptyset(N_-N_+)$ |

**Corollary 4**. IR-points $O_0^{\pm}$ of any Eq. (1) — system do not have separatrices.

 T-types of IR-points   $O_i$ $(u_i, 0) \neq O_o$ $(0,0), i = \overline{1,m}$, of Eq.(1) – systems are described with the following theorem.

**Theorem 3**. Let a real number $u_i$ $(\neq 0)$ be a multiplicity $k_i \in \{1,2,3\}$ root of a polynomial $P(u)$ of the Eq.(1) - system. Then for this system a value $g_i = P^{(ki)}(u_i)Q(u_i) \neq 0$ and words $A_i^{\pm}$, which determine topologycal types (T-types) of IR-points $O_i^{\pm}$ $(u_i, 0)$ of this system, depending on the value of $k_i$ and signs of numbers $u_i$ and $g_{i,}$ have forms indicated in the Table 4 [5].

Table 4. T-types of IR-points $O_i^{\pm}$ $(u_i, 0), i \in \{1,...,m\}$.

| $u_i$ | $k_i$ | $g_i$ | $A_i^+$ | $A_i^-$ |
|-------|-------|-------|---------|---------|
| +(-) | 1, 3 | + | $N_+(N_-)$ | $S_-(S_+)$ |
| +(-) | 1, 3 | − | $S_-(S_+)$ | $N_+(N_-)$ |
| +(-) | 2 | + | $S_-N_+(\emptyset)$ | $\emptyset(N_-S_+)$ |
| +(-) | 2 | − | $\emptyset(N_-S_+)$ | $S_-N_+(\emptyset)$ |

**Corollary 5**. As it can be seen from the Theorems 2 and 3, for the IR-points of the Eq.(1) – systems only finite number (13) of different T-types are possible. And the investigation of these T-types shows, that IR-points of each Eq.(1) - system have only $m$ separatrices: one separatrice for every singular point $O_i$ $(u_i, 0), i = 1, m$.

**Note 3**. In the tables 3 and 4 a lower sign index «+» or «-» of every bundle $N$ or $S$, indicates wheather the bundle adjusts to the point $O_i^+$ (or to the point $O_i^-$) from the side $u > u_i$ or from the side $u < u_i$ of the isocline $u = u_i$.

In the Table 3, line No. 1, a bundle $N$ doesn`t have a lower sign index, because the detailed study of this case shows: it contains $O_i^+$-curves $(O_i^-$-curves) in every domain $|u| > 0$ [5].

## Conclusions

This article is devoted to the original study. The main task was to construct all different in the topological sense phase portraits in a Poincare circle, possible for the dynamic differential systems belonging to a special family of the Eq. (1) – systems, and to its numerical subfamilies. We have constructed all those portraits two ways - in a descriptive as well as in a graphical form. Each table of a descriptive form contains from 5 to 6 lines. Every line describes one invariant cell of the phase portrait in detail – its boundary, a source and a sink of its phase flow. Such a table is called a descriptive phase portrait [6, 7].

During this investigation we have fully studied finite and infinitely remote singular points of systems under consideration.Also the task of this work was to develop, outline and successfully apply some new effective methods of investigation [8-10].

## Recommendations

Nevertheless this is a theoretical work, due to abovementioned new methods it may be useful for applied studies of dynamic systems of the second order with polynomial right parts. The work may be interesting for students, postgraduates and scientific researchers as well.

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