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Lyapunov-type inequality for a Riemann-Liouville fractional differential boundary value problem

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Abstract

The aim of this paper is to present a Lyapunov-type inequality for a Riemann-Liouville fractional differential equation of order $2 < \alpha \leq 3$ subject to mixed boundary conditions.

Keywords: Lyapunov's inequality, Riemann-Liouville derivative, Caputo fractional derivative, mixed boundary conditions.

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1. Introduction

In this paper, we present a Lyapunov's inequality for the following boundary value problem:

$$(1.1)\qquad \begin{cases} ({}_aD^{\alpha}u)(t) + q(t)u(t) = 0, \quad a < t < b, \quad 2 < \alpha \leq 3, \\ u(a) = u'(a) = u'(b) = 0, \end{cases}$$

where a and b are consecutive zeros of the solution u. As $u = 0$ is a trivial solution, only non-negative solutions are taken in consideration.
We prove that problem (1.1) has a non-trivial solution for $\alpha \in (2, 3]$ provided that the real and continuous function q satisfies

$$(1.2)\qquad \int_a^b |q(t)|\, dt > \frac{\Gamma(\alpha)}{(b-a)^{(\alpha-1)}} \left(\frac{\alpha-1}{\alpha-2}\right)^{\alpha-2}.$$

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Before we prove this result, let us dwell upon some references.
For the problem

$$\begin{cases} u''(t) + q(t)u(t) = 0, & a < t < b \\ u(a) = u(b) = 0, \end{cases}$$

where a and b are consecutive zeros of u and the function $q \in C([a,b];\mathbb{R})$. Lyapunov [7] proved a necessary condition of existence of non-trivial solutions is that

$$\int_a^b |q(t)|\,dt > \frac{4}{b-a}. \tag{1.3}$$

After this result, similar type inequalities have been obtained for other kind of differential equations and boundary conditions see [3], [8].
Concerning differential equation with fractional derivative's in [2], Ferreira derived Lyapunov's inequality for the problem

$$\begin{cases} ({}_aD^\alpha u)(t) + q(t)u(t) = 0, & a < t < b,\ 1 < \alpha \leq 2, \\ u(a) = u(b) = 0, \end{cases} \tag{1.4}$$

where $q \in C([a,b],\mathbb{R})$, a and b are consecutive zeros of u, and ${}_aD^\alpha$ is the Riemann-Liouville fractional derivative of order $\alpha > 0$ defined for an absolute continuous function on $[a,b]$ by

$$({}_aD^\alpha f)(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^\alpha f(s)\,ds$$

where $n \in \mathbb{N}, n < \alpha \leq n+1$ (For more details of fractional derivatives see [6]). His inequality reads

$$\int_a^b |q(t)|\,dt > \Gamma(\alpha)\left(\frac{4}{b-a}\right)^{\alpha-1} = \Gamma(\alpha)\left(\frac{2^{2(\alpha-1)}}{(b-a)^{(\alpha-1)}}\right), \tag{1.5}$$

which in the particular case $\alpha = 2$ corresponds to Lyapunov's classical inequality (1).

Then, Ferreira [3] and Jleli and Samet [5] dealt with fractional differential boundary value problems with Caputo's derivative which is defined for a function $f \in AC^n[a,t]$ by

$$({}_a^C D^\alpha f)(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^\alpha f^{(n)}(s)\,ds.$$

For the boundary value problem

$$\begin{cases} ({}_a^C D^\alpha u)(t) + q(t)u(t) = 0, & a < t < b,\ 1 < \alpha \leq 2, \\ u(a) = u(b) = 0, \end{cases} \tag{1.6}$$

where $q \in C([a,b];\mathbb{R})$ and a and b are consecutive zeros of u, Ferreira [2] proved that if (1.6) has a nontrivial solution, then the following necessary condition is satisfied

$$\int_a^b |q(t)|\,dt > \frac{\Gamma(\alpha)\alpha^\alpha}{[(\alpha-1)(b-a)]^{\alpha-1}}. \tag{1.7}$$

In [5], Jleli and Samet considered the equation (1.6) subject to either

$$u'(a) = 0,\ u(b) = 0, \tag{1.8}$$

or

$$u(a) = 0,\ u'(b) = 0. \tag{1.9}$$

They showed that the associated non trivial solution exists if

$$\int_a^b (b-s)^{\alpha-2}|q(t)|\,dt \geq \Gamma(\alpha) \tag{1.10}$$

is satisfied.
However, in the case of (1.9), the corresponding nontrivial solution exists if:

$$(1.11)\qquad \int_a^b (b-s)^{\alpha-2}|q(t)|\,dt \;\geq\; \frac{\Gamma(\alpha)}{\max\{\alpha-1, 2-\alpha\}\,(b-a)}.$$

It was shown in [4] that a non trivial solution corresponding to equation (1.6) where $q \in C([a,b];\mathbb{R})$, a and b are consecutive zeros of u, subject to the boundary conditions

$$(1.12)\qquad u(a) - u'(a) = u(b) + u'(b) = 0,$$

exists if the following necessary condition

$$(1.13)\qquad \int_a^b (b-s)^{\alpha-2}(b-s+\alpha-1)|q(s)|\,ds \geq \frac{(b-a+2)\,\Gamma(\alpha)}{\max\{b-a+1, \frac{2-\alpha}{\alpha-1}\,(b-a)-1\}}$$

is satisfied.

2. Main results

2.1. A Lyapunov-type inequality for problem (1.1)**.** The strategy in getting Lyapunov-type inequality for (1.1) is to re-write the considered problem in its equivalent integral form.
As in [2], the solution can be written in the integral form

$$u(t) = \int_a^t G(t,s)q(s)u(s)\,ds + \int_t^b G(t,s)q(s)u(s)\,ds,$$

where the Green function $G(x,t)$ is defined by

$$(2.1)\qquad \Gamma(\alpha)G(t,s) \;=\; \begin{cases} \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-2}}(b-s)^{\alpha-2} - (t-s)^{\alpha-1}, & a \leq s \leq t, \\ \\ \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-2}}(b-s)^{\alpha-2}, & t \leq s \leq b. \end{cases}$$

$$(2.2)\qquad = \begin{cases} g_1(t,s), & a \leq s \leq t \leq b, \\ g_2(t,s), & a \leq t \leq s \leq b, \end{cases}$$

which in the particular case $a = 0$, $b = 1$ corresponds to that of M. El-Shahed [1].

2.1. Theorem. *The Green function G satisfies:*

(1) $G(t,s) \geq 0$ *for all* $a \leq t, s \leq b$.
(2) $\max_{t\,\in\,[a,b]} G(t,s) = G(b,s), \quad s \in [a,b]$,
(3) *$G(b,s)$ has a unique maximum given by:*

$$\max_{s\,\in\,[a,b]} G(b,s) = \frac{1}{\Gamma(\alpha)}(b-a)^{(\alpha-1)}\left(\frac{\alpha-2}{\alpha-1}\right)^{\alpha-2}.$$

Proof. For the proof of Theorem 2.1, we start with the function $g_1(t,s)$. The function g_1 is non-decreasing. Indeed, to show this fact, we need to make the following observation of Ferreira in [2]:

$$(a + \frac{(s-a)(b-a)}{t-a}) \geq s \text{ is equivalent to } s \geq a;$$

this allows us to write

$$
\begin{aligned}
(t-s)^{\alpha-1} = (t-a+a-s)^{\alpha-1} &= [(t-a)(1+\frac{a-s}{t-a})]^{\alpha-1} \\
&= [(b-a)(1+\frac{a-s}{t-a})]^{\alpha-1}\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}} \\
&= [b-(a+\frac{(s-a)(b-a)}{t-a})]^{\alpha-1}\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}},
\end{aligned}
$$

which is used to show that g_1 is positive and non-decreasing.
Indeed,
For $a \leq s \leq t \leq b$,

$$
\begin{aligned}
g_1(t,s) &:= \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-2}}(b-s)^{\alpha-2} - (t-s)^{\alpha-1} \\
&= \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-2}}(b-s)^{\alpha-2} - [b-(a+\frac{(s-a)(b-a)}{t-a})]^{\alpha-1}\,\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-2}} \\
&\geq \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-2}}(b-s)^{\alpha-2} - (b-s)^{\alpha-2}\,\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-2}} \\
&\geq 0.
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
\frac{\partial g_1}{\partial t}(t,s) &= (\alpha-1)\frac{(t-a)^{\alpha-2}}{(b-a)^{\alpha-2}}(b-s)^{\alpha-2} - (\alpha-1)(t-s)^{\alpha-2} \\
&= (\alpha-1)\frac{(t-a)^{\alpha-2}}{(b-a)^{\alpha-2}}(b-s)^{\alpha-2} - (\alpha-1)[b-(a+\frac{(s-a)(b-a)}{t-a})]^{\alpha-2} \\
&\quad \frac{(t-a)^{\alpha-2}}{(b-a)^{\alpha-2}} \\
&= (\alpha-1)(t-a)^{\alpha-2}\left(\frac{(b-s)^{\alpha-2}}{(b-a)^{\alpha-2}} - [b-(a+\frac{(s-a)(b-a)}{t-a})]^{\alpha-2}\frac{1}{(b-a)^{\alpha-2}}\right) \\
&\geq (\alpha-1)(t-a)^{\alpha-2}\left(\frac{(b-s)^{\alpha-2}}{(b-a)^{\alpha-2}} - (b-s)^{\alpha-2}\frac{1}{(b-a)^{\alpha-2}}\right) \\
&\geq 0.
\end{aligned}
$$

Consequently,

$$\max_{t,s\in[a,b]} g_1(t,s) = \max_{s\in[a,b]} g_1(b,s).$$

In view of (2.1) $-$ (2.2), $g_1(b,s)$ is defined by: $g_1(b,s) = (b-s)^{\alpha-2}(s-a)$. Its derivative with respect to s takes the form

$$\frac{\partial g_1}{\partial s} = (b-s)^{\alpha-3}[s(1-\alpha) + a(\alpha-2) + b].$$

$$\frac{\partial g_1}{\partial s} = 0 \Leftrightarrow s = s_* = \frac{a(\alpha-2)+b}{\alpha-1}.$$

Hence

$$\max_{s\in[a,b]} g_1(b,s_*) = (b-a)^{\alpha-1}\left(\frac{\alpha-2}{\alpha-1}\right).$$

The function g_2 is clearly positive and non decreasing in t, so

$$\max_{t,s\in[a,b]} g_2(t,s) = \max_{s\in[a,b]} g_2(b,s) = g_2(s,s) = \frac{(s-a)^{\alpha}}{(b-a)^{\alpha-2}} =: F(s).$$

The function F is increasing for

$$s \leq s^* = \frac{(\alpha-2)a+(\alpha-1)b}{2\alpha-3};$$

and is decreasing for

$$s \geq s^* = \frac{(\alpha-2)a+(\alpha-1)b}{2\alpha-3}.$$

So

$$\max F(s) = \max g_2(s,s) = g_2(s^*, s^*),$$

where

$$g_2(s^*, s^*) = (b-a)^{\alpha-1}\left(\frac{\alpha-2}{\alpha-1}\right)^{\alpha-2}.$$

Now we need to compare $g_1(b, s_*)$ and $g_2(s^*, s^*)$.

Since $2 \leq \alpha \leq 3$ then $(2\alpha-3)^{\alpha-3} \geq (\alpha-1)^{2\alpha-3}$ and therefore

$$(b-a)^{\alpha-1}\left(\frac{\alpha-2}{\alpha-1}\right)^{\alpha-2} \geq (b-a)^{\alpha-1}\frac{(\alpha-1)^{\alpha-1}(\alpha-2)^{\alpha-2}}{(2\alpha-3)^{2\alpha-3}}.$$

Consequently

$$\max_{s\in[a,b]} G(b,s) = \frac{1}{\Gamma(\alpha)}(b-a)^{(\alpha-1)}\left(\frac{\alpha-2}{\alpha-1}\right)^{\alpha-1}.$$

□

We are now ready to prove the Lyapunov's type-inequality for problem (1.1).

2.2. Theorem. *Let u be a solution satisfying the following boundary value problem*

$$\begin{cases} ({}_aD^\alpha u)(t) + q(t)u(t) = 0, & a < t < b,\ 2 < \alpha \leq 3, \\ u(a) = u'(a) = u'(b) = 0, \end{cases} \tag{2.3}$$

where a and b two consecutive zeros of u. Then (2.3) has a non-trivial solution provided that the real and continuous function q satisfies the condition

$$\int_a^b |q(t)|\, dt > \frac{\Gamma(\alpha)}{(b-a)^{\alpha-1}}\left(\frac{\alpha-1}{\alpha-2}\right)^{\alpha-1}. \tag{2.4}$$

Proof. For the proof of Theorem 2.2, we equip the Banach space $C[a,b]$ with the Chebychev norm $||u|| = \max_{t\in[a,b]} |u(t)|$.
As

$$u(t) := \int_a^b G(t,s)q(s)u(s)\, ds,$$

we have

$$||u|| \leq \int_a^b \max_{t,s\,\in[a,b]} |G(t,s)|\,|q(s)|\, ds\ ||u||.$$

Then since u is a non trivial solution, in view of Theorem 2.1, we get

$$1 \leq \int_a^b \frac{1}{\Gamma(\alpha)}(b-a)^{(\alpha-1)}\left(\frac{\alpha-2}{\alpha-1}\right)^{\alpha-1} |q(s)|\, ds.$$

Using the properties of G, the inequality (2.4) is obtained. □

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