İSTATİSTİK: JOURNAL OF THE TURKISH STATISTICAL ASSOCIATION

İSTATİSTİK

Vol. 7, No. 2, July 2014, pp. 43–54
ISSN 1300-4077 | 14 | 2 | 43 | 54

PREDICTIVE ESTIMATION OF FINITE POPULATION MEAN USING GENERALISED FAMILY OF ESTIMATORS

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Abstract: Using predictive estimation procedure, an attempt has been made to develop some generalised family of estimators for the finite population mean $\bar{J}$ of survey variable J in the presence of known auxiliary variable K. The proposed class consists of mainly two different types of estimators namely, Ratio and Exponential ratio-product type estimator. Theoretical conditions under which the proposed classes are less biased and more efficient then usual unbiased, ratio estimator and estimator due to [2], [12] and [13] have been obtained . It is also evaluated that at optimum values of unknown scalars the mean square error (MSE) of suggested classes tends to the MSE of regression estimator. Finally, these theoretical findings are illustrated by a numerical example.

Key words: Auxiliary variable asymptotic bias; asymptotic variance; prediction approach.
History: Submitted: 5 August 2014; Revised: 23 December 2014; Accepted: 30 December 2014

1. Introduction

Let J be the study variable taking real value J_i for all i(1≤ i ≤)defined over finite population ψ=1,2,...,N of size N.We wish to estimate the population mean $\bar{J} = \sum_{i=1}^{N} J_i$.
on the basis of observed values of J on the units in a sample which is an ordered subset of finite population ψ.Let S denotes the collection of all possible samples from ψ. For given s∈S, let $\gamma(s)$ denotes its effective sample size and $\widetilde{s}$ denote the set of all units of which are not in s.
Thus for any given s∈S , we have

$$\bar{J} = \frac{\gamma(s)}{N}\bar{J}_s + \frac{N-\gamma(\widetilde{s})}{N}\bar{J}_{\widetilde{s}}. \qquad (1.1)$$

where, $\bar{J}_s = \frac{1}{\gamma(s)}\sum_{i\in s}\bar{J}_i$ and $\bar{J}_{\widetilde{s}} = \frac{1}{N-\gamma(s)}\sum_{i\in \widetilde{s}}\bar{J}_i$

The representation of $\bar{J}$ above in (1.1), the sample mean $\bar{J}_{\widetilde{s}}$, being based on the units in the sample whose j values have been observed, is known. Therefore, the statisticians should attempt a prediction of the mean $\bar{J}_{\widetilde{s}}$ of the unobserved units of the population on the basis of observed units in s. For any given s∈S using SRSWOR and effective sample size $\gamma(s) = n$ and $\bar{J}_s = \bar{\bar{j}}$, we can write (1.1) as

$$\bar{J} = \frac{\gamma(s)}{N}\bar{\bar{j}}_s + \frac{N-\gamma(\widetilde{s})}{N}\bar{J}_{\widetilde{s}}. \qquad (1.2)$$

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From (1.2), an estimator of population mean $\bar{J}$ can be written as:

$$t = \frac{\gamma(s)}{N}\bar{j} + \frac{N-\gamma(\widetilde{s})}{N}T. \quad (1.3)$$

where T is considered as a predictor of $\bar{J}_{\widetilde{s}}$.

The literature describes a great variety of techniques using auxiliary information by means of ratio, product and regression methods for estimating population mean $\bar{J}$ of study variable j. The main aim of survey statistician is to reduce the errors either by means of suitable sampling techniques or by formulating efficient estimators of parameters, see [14]. The main approach of prediction is that a model is specified for the population values and used to predict the non-sampled values of S. Prediction theory is came into play in late nineties, when [13] has shown that if the usual product estimator is used as the predictor for the mean of the unobserved units of the population, the resulting estimator of the mean of the whole population is different from the usual product estimator. Further [12] have proposed exponential estimator under prediction approach. [1] and [9] provided some predictive estimators for population variance.
Moving along this direction, we intend in this paper to show how the problem of estimating the unknown population mean $\bar{J}$ of study variable j. Motivated by [5] and [7], we have defined two different classes of estimators which are less biased and encompasses many that are present in literature. Further different possible families of estimators are developed using proposed classes. The asymptotic bias and MSEs of the classes are obtained and it is shown how to establish if an estimator belongs to the class and whether it is optimum.
The remaining part of the work is organized as follows: Section 2 introduces the notation and a brief review of some existing estimators for the mean of the study variable. In section 3.1, 3.2 and 3.3, three new classes of estimators under prediction approach are suggested and the expressions for the asymptotic bias and the mean square error are obtained. Section 4 addresses the problem of theoretical efficiency comparisons, while in section 5, an empirical study is carried out to demonstrate the superiority of suggested estimators over others in literature. Section 6 concludes the paper with final remarks.

2. Reviewing estimators of the population mean

Let K being the auxiliary variable highly positively correlated with study variable J and K_i being the value of K on the i^{th} unit $1 \leq i \leq \psi$ of the population ψ , let $\bar{K} = \frac{1}{N}\sum_{i=1}^{N}$ and further, for k_i being the value of k on the unit of the sample s, let $\bar{k} = \frac{1}{n}\sum_{i \in S} k_i$
[13] has shown that if we adapt the prediction approach as defined earlier in section 1, use of.

$\bar{j} = \frac{1}{n}\sum_{i \in \widetilde{s}} j_i$, (mean per unit estimator)

$\bar{j}_{lr} = [\bar{j} + b(\bar{K}_{\widetilde{s}} - k)]$, (the regression estimator)

$\bar{j}_{\widetilde{R}} = \bar{K}_{\widetilde{s}}\frac{\bar{j}}{\bar{k}}$. (the usual ratio estimator)

for predicting the mean $\bar{J}_{\widetilde{s}}$ of the unobserved units of the population result in the corresponding customary:

$\bar{j} = \frac{1}{n}\sum_{i \in s} j_i$, (mean per unit estimator)

$\bar{j}_{lr} = [\bar{j} + b(\bar{K} - k)]$, (the regression estimator)

$\bar{j}_R = \bar{K}\frac{\bar{j}}{\bar{k}}$. (the usual ratio estimator)

Of the population mean $\bar{J}$, where b is the regression coefficient estimated from the sample s and $\bar{K} = \frac{1}{N}\sum_{i\in S} k_i$.
Note: Here t and T stands for $[T = \bar{j}, t = \bar{j}; T = \bar{j}_{\tilde{R}}, t = \bar{j}_R; T = \bar{j}, t = \bar{j}_{\tilde{lr}}, t = \bar{j}_{lr}]$.

However, if the product estimator $\bar{j}_P = \bar{j}\frac{\bar{k}}{\bar{K}_{\tilde{s}}}$ is used with such an approach, the resulting estimator of population mean $\bar{J}$ is not the usual product estimator $\bar{j}_P = \bar{j}\frac{\bar{k}}{\bar{K}_{\tilde{s}}}$.
Here if we take $T = \bar{j}_P$,then $t_P = \frac{n\bar{K} + (N-2n)\bar{k}}{N\bar{K} - n\bar{k}}$.
Up to the first order of approximation the bias (B) and MSE of the estimators $\bar{j}_R, \bar{j}_P$and t_p are given as:

$$B(\bar{j}_R) = \theta\bar{J}C_k^2(1 - C) \tag{2.1}$$

$$B(\bar{j}_P) = \theta\bar{J}C_k^2 C \tag{2.2}$$

$$B(t_p) = \theta\bar{J}C_k^2\left[C + \frac{f}{(1-f)}\right] \tag{2.3}$$

and

$$MSE(\bar{j}_R) = \theta\bar{J}^2\left[C_j^2 + C_k^2(1 - 2C)\right] \tag{2.4}$$

$$MSE(\bar{j}_P) = MSE(t_p) = \theta\bar{J}^2\left[C_j^2 + C_k^2(1 + 2C)\right] \tag{2.5}$$

Following [2],[12] have suggested exponential-ratio and exponential-product type estimator using prediction approach given as

$$t_{re} = \left[\frac{n}{N}\bar{j} + \left(\frac{N-n}{N}\right)\bar{j}exp\left(\frac{\bar{K}_{\tilde{s}} - \bar{k}}{\bar{K}_{\tilde{s}} + \bar{k}}\right)\right] = \left[\frac{n}{N}\bar{j} + \left(\frac{N-n}{N}\right)\bar{j}exp\left(\frac{N(\bar{K} - \bar{k})}{N(\bar{K} - \bar{k}) - 2n\bar{k}}\right)\right] \tag{2.6}$$

and

$$t_{Pr} = \left[\frac{n}{N}\bar{j} + \left(\frac{N-n}{N}\right)\bar{j}exp\left(\frac{\bar{k} - \bar{K}_{\tilde{s}}}{\bar{k} + \bar{K}_{\tilde{s}}}\right)\right] = \left[\frac{n}{N}\bar{j} + \left(\frac{N-n}{N}\right)\bar{j}exp\left(\frac{N(\bar{k} - \bar{K})}{N\bar{K} + (N - 2n)\bar{k}}\right)\right] \tag{2.7}$$

Up to the first order of approximation the bias (B) and MSE of the estimators t_{re} and t_{Pr} are given as:

$$B(t_{re}) = \frac{\theta}{8}\bar{J}C_k^2\left[3 - 4(C + f)\right] \tag{2.8}$$

$$B(t_p) = \frac{\theta}{8}\bar{J}C_k^2\left[4C - \frac{1}{(1-f)}\right] \tag{2.9}$$

and

$$MSE(t_{re}) = \theta\bar{J}^2\left[C_j^2 + \frac{Ck^2}{4}(1 - 4C)\right] \tag{2.10}$$

$$MSE(t_{Pr}) = \theta\bar{J}^2\left[C_j^2 + \frac{Ck^2}{4}(1 + 4C)\right] \tag{2.11}$$

where,
$C_j^2 = \frac{S_j^2}{\bar{J}^2}$,$C_k^2 = \frac{S_k^2}{\bar{K}^2}$, $C = \rho\frac{C_j}{C_k}$,$S_j^2 = \frac{1}{(N-1)}\sum_{i=1}^{N}(j_i - \bar{J})^2$, $S_k^2 = \frac{1}{(N-1)}\sum_{i=1}^{N}(k_i - \bar{K})^2$,$S_{jk} = \frac{1}{(N-1)}\sum_{i=1}^{N}(j_i - \bar{J})(k_i - \bar{K})$,$\theta = \frac{1}{(1-f)}$ and $f = \frac{n}{N}$
In the following, we will focus only on solutions ascribable to a general class of estimators that can be more efficient than the existing estimators.

3. Generalised class of estimators

3.1. Predictive estimation of population means using exponential-type estimators:

A general class of exponential-ratio type estimators for population mean $\bar{J}$ has been proposed by [5] given as

$$t_{\sqrt{1}} = \bar{j} exp\left[\frac{\sqrt{\bar{K}} - \sqrt{\bar{k}}}{\sqrt{\bar{K}} + \sqrt{\bar{k}}}\right] \tag{3.1}$$

$$t_{\sqrt{2}} = \bar{j} exp\left[\frac{\sqrt{\bar{k}} - \sqrt{\bar{K}}}{\sqrt{\bar{k}} + \sqrt{\bar{K}}}\right] \tag{3.2}$$

In case, when information on auxiliary variable k is available and we intends to use $t_{\sqrt{1}}$ and $t_{\sqrt{2}}$ on for T under prediction approach, then if j is positively correlated with k, a modified form of exponential-ratio type estimator under prediction approach is given as

$$\eta_{\sqrt{1}} = \left[\frac{n}{N}\bar{j} + \left(\frac{N-n}{N}\right)\bar{j} exp \frac{\sqrt{\bar{K}_s} - \sqrt{\bar{k}}}{\sqrt{\bar{K}_s} + \sqrt{\bar{k}}}\right] \tag{3.3}$$

and if the auxiliary variable k is negatively correlated with the study variable j, then a modified form of exponential-product type estimator under prediction approach is given as

$$\eta_{\sqrt{2}} = \left[\frac{n}{N}\bar{j} + \left(\frac{N-n}{N}\right)\bar{j} exp \frac{\sqrt{\bar{k}} - \sqrt{\bar{K}_s}}{\sqrt{\bar{k}} + \sqrt{\bar{K}_s}}\right] \tag{3.4}$$

Large sample properties of classes are obtained according to simple random sampling without replacement (SRSWOR). In so doing, we define:
$\delta_0 \left(\frac{\bar{j}-\bar{J}}{\bar{J}}\right)$ and $\delta_1 = \left(\frac{\bar{k}-\bar{K}}{\bar{K}}\right)$
Up to the first order of approximation, $E(\epsilon_i) = 0 \forall (i = 0, 1)$

and $E(\epsilon_0^2) = \theta C_j^2$,$E(\epsilon_1^2) = \theta C_k^2$ and $E(\epsilon_0 \epsilon_1) = \theta C C_k^2$

Expressing (3.3) in terms of δ_i's, we have

$$\eta_{\sqrt{1}} = \bar{J}(1+\delta_0)\left[f + (1-f) exp\left(\frac{\sqrt{(1-\phi\delta_1)\bar{K}} - \sqrt{(1+\delta_1)\bar{K}}}{\sqrt{(1-\phi\delta_1)\bar{K}} + \sqrt{(1+\delta_1)\bar{K}}}\right)\right]$$

Assuming $|\phi = \frac{n}{(N-n)}| < 1$, so that $\sqrt{(1-\phi\delta_1)}$ and $\sqrt{(1-\delta_1)}$ is expendable functions, hence we have

$$\eta_{\sqrt{1}} = \bar{J}\left[1 + \delta_0 - \frac{\delta_1}{4} - \frac{\delta_0\delta_1}{4} + \frac{\delta_1^2(3-4f)}{32(1-f)}\right]$$

$$\eta_{\sqrt{1}} - \bar{J} = \bar{J}\left[+\delta_0 - \frac{\delta_1}{4} - \frac{\delta_0\delta_1}{4} + \frac{\delta_1^2(3-4f)}{32(1-f)}\right] \tag{3.5}$$

Taking expectation on both sides of (3.5), we get the bias of $\eta_{\sqrt{1}}$ up to the first order of approximation as:

$$B(\eta_{\sqrt{1}}) = \bar{J}\theta C_k^2\left[\frac{(3-4f)}{32(1-f)} - \frac{C}{4}\right] \tag{3.6}$$

Squaring both sides of (3.5) and neglecting the terms having power greater than two, we have

$$(\eta_{\sqrt{1}} - \bar{J})^2 = \bar{J}^2\left[\delta_0^2 + \frac{\delta_1^2}{16} - \frac{\delta_0\delta_1}{2}\right] \tag{3.7}$$

Thus after applying the expectation on both sides of (3.7), we get

$$MSE(\eta_{\sqrt{1}}) = \theta \bar{J}^2 \left[C_j^2 + \frac{C_k^2}{16} - \frac{CC_k^2}{2} \right] \tag{3.8}$$

Expressing (3.4) in terms of δ_i's , we have

$$\eta_{\sqrt{2}} = \bar{J}(1+\delta_0)\left[f + (1-f)exp\left(\frac{\sqrt{(1+\delta_1)\bar{K}} - \sqrt{(1-\phi\delta_1)\bar{K}}}{\sqrt{(1+\delta_1)\bar{K}} + \sqrt{(1-\phi_1)\bar{K}}} \right) \right]$$

Assuming $|\phi = \frac{n}{(N-n)}| < 1$, so that $\sqrt{(1-\phi\delta_1)}$ and $\sqrt{(1-\delta_1)}$ is expendable functions, hence we have

$$\eta_{\sqrt{2}} = \bar{J}\left[1 + \delta_0 + \frac{\delta_1}{4} + \frac{\delta_0\delta_1}{4} - \frac{\delta_1^2(1-4f)}{32(1-f)} \right]$$

$$\eta_{\sqrt{2}} - \bar{J} = \bar{J}\left[\delta_0 + \frac{\delta_1}{4} + \frac{\delta_0\delta_1}{4} - \frac{\delta_1^2(1-4f)}{32(1-f)} \right] \tag{3.9}$$

Taking expectation on both sides of equation (3.9), we get the bias of $\eta_{\sqrt{2}}$ to the first order of approximation as:

$$B(\eta_{\sqrt{2}}) = \bar{J}\theta C_k^2 \left[\frac{C}{4} - \frac{1-4f}{32(1-f)} \right] \tag{3.10}$$

Squaring both sides of (3.9) and neglecting the terms having power greater than two, we have

$$(\eta_{\sqrt{2}} - \bar{J})^2 = \bar{J}^2 \left[\delta_0^2 + \frac{\delta_1^2}{16} + \frac{\delta_0\delta_1}{2} \right] \tag{3.11}$$

Thus after applying the expectation on both sides of (3.11), we get

$$MSE(\eta_{\sqrt{2}}) = \theta \bar{J}^2 \left[C_j^2 + \frac{C_k^2}{16} + \frac{CC_k^2}{2} \right] \tag{3.12}$$

3.2. A generalised exponential type estimator under prediction approach:

Following [5], we have suggested the generalised estimator under prediction approach given as

$$\eta^d_{/\alpha,h,a} = \left[\frac{n}{N}\bar{j} + \left(\frac{N-n}{N} \right) \bar{j} exp\left(\alpha \left\{ 1 - \frac{a\bar{k}^{\frac{1}{h}}}{\bar{K}_s^{\frac{1}{h}} + (a-1)\bar{k}^{\frac{1}{h}}} \right\} \right) \right] \tag{3.13}$$

where two real constants ($-\infty < \alpha < \infty$), and h ($h > 0$) are assumed to be known, and a ($a \neq 0$) is assumed to estimate such that $\eta^d_{/\alpha,h,a}$ is optimal and MSE of $\eta^d_{/\alpha,h,a}$ is minimum. Expressing (3.13) in terms of δ_i's , we have

$$\eta^d_{/\alpha,h,a} = \bar{J}(1+\delta_0)\left[\frac{n}{N}\bar{j} + \left(\frac{N-n}{N} \right) exp\alpha \left\{ \frac{\left(N\bar{K} - \frac{n(1+\delta_1)\bar{K}}{N-n} \right)^{\frac{1}{h}} - \left((1+\delta_1)\bar{K} \right)^{\frac{1}{h}}}{\left(N\bar{K} - \frac{n(1+\delta_1)\bar{K}}{N-n} \right)^{\frac{1}{h}} + (a-1)\left((1+\delta_1)\bar{K} \right)^{\frac{1}{h}}} \right\} \right] \tag{3.14}$$

Rearranging (3.14), we have

$$\eta^d_{/\alpha,h,a} = \bar{J}(1+\delta_0)\left[f + (1-f)exp\alpha \left\{ \frac{(1-\phi\delta_1)^{\frac{1}{h}} - ((1+\delta_1))^{\frac{1}{h}}}{((1-\phi\delta_1))^{\frac{1}{h}} + (a-1)\left((1+\delta_1) \right)^{\frac{1}{h}}} \right\} \right]$$

Assuming $|\phi = \frac{n}{(N-n)}| < 1$, so that $(1-\phi\delta_1)^{\frac{1}{h}}$ and $((1+\delta_1))^{\frac{1}{h}}$ is expendable functions, hence we have

$$\eta^d_{/\alpha,h,a} = \bar{J}(1+\delta_0)\left[f + (1-f)exp\left\{\frac{-\alpha\delta_1}{ah(1-f)}\left(1+\frac{\delta_1}{h}-\frac{\delta_1}{ah(1-f)}\right)^{-1}\right\}\right]$$

$$\eta^d_{/\alpha,h,a} = \bar{J}\left[1+\delta_0-\frac{\alpha\delta_1}{ah}-\frac{\alpha\delta_0\delta_1}{ah}+\frac{\alpha\delta_1^2}{ah^2}-\frac{\alpha\delta_1^2}{a^2h^2(1-f)}+\frac{\alpha^2\delta_1^2}{a^2h^2(1-f)}\right]$$

$$\eta^d_{/\alpha,h,a} - \bar{J} = \bar{J}\left[\delta_0-\frac{\alpha\delta_1}{ah}-\frac{\alpha\delta_0\delta_1}{ah}+\frac{\alpha\delta_1^2}{ah^2}-\frac{\alpha\delta_1^2}{a^2h^2(1-f)}+\frac{\alpha^2\delta_1^2}{a^2h^2(1-f)}\right] \tag{3.15}$$

Taking expectation on both sides of (3.15), we get the bias of $\eta^d_{/\alpha,h,a}$ to the first order of approximation as:

$$B(\eta^d_{/\alpha,h,a}) = \bar{J}\theta C_k^2\left[\frac{\alpha}{ah^2}-\frac{\alpha}{a^2h^2(1-f)}+\frac{\alpha^2}{a^2h^2(1-f)}-\frac{\alpha C}{ah}\right] \tag{3.16}$$

Squaring both sides of (3.15) and neglecting the terms having power greater than two, we have

$$(\eta^d_{/\alpha,h,a} - \bar{J})^2 = \bar{J}^2\left[\delta_0^2+\frac{\alpha^2\delta_1^2}{a^2h^2}-\frac{2\alpha\delta_0\delta_1}{ah}\right] \tag{3.17}$$

Thus after applying the expectation on both sides of (3.17), we get

$$MSE(\eta^d_{/\alpha,h,a}) = \bar{J}^2\theta\left[C_j^2+\frac{\alpha^2C_k^2}{a^2h^2}-\frac{2\alpha CC_k^2}{ah}\right] \tag{3.18}$$

Partially differentiating equation (3.18) w.r.t. 'a' and equating it to zero, we have
$a_{opt} = \frac{\alpha}{Ch}$
Now, the substitution of $a_{opt} = \frac{\alpha}{Ch}$ and for known α and h, the asymptotically optimal estimator $\eta^d_{/\alpha,h,a}$ is given as

$$\eta^d_{/\alpha,h,a} = \left[\frac{n}{N}\bar{j}+\left(\frac{N-n}{N}\right)\bar{j}exp\left(\alpha\left\{1-\frac{a\bar{k}^{\frac{1}{h}}}{Ch\bar{K}_s^{\frac{1}{h}}+(\alpha-Ch)\bar{k}^{\frac{1}{h}}}\right\}\right)\right] \tag{3.19}$$

The minimum mean square error of $\eta^d_{/\alpha,h,a}$ under the optimum condition is given as,

$$MSE_{min}(\eta^d_{/\alpha,h,a}) = \bar{J}^2\theta C_j^2\left[1-\rho^2\right] \tag{3.20}$$

where, $C = \rho\frac{C_j}{C_k}$.

Note: Here it is noted that minimum MSE of $\eta^d_{/\alpha,h,a}$ is equal to the approximate MSE of linear regression estimator $t_{lr} = \bar{j}+\hat{b}(\bar{K}-\bar{k})$, where b is regression correlation coefficient. Thus estimator $\eta^d_{/\alpha,h,a}$ can be used in practice as an alternative to the linear regression estimator t_{lr} .
In certain situations when information on α and h is not possible to gather from known population data, in such situations we simply replace α and h by their consistent estimates $\hat{\alpha}$ and $\hat{h}$ based on relatively large sample. Below Table.1 determines some proposed classes in literature using suggested class $\eta^d_{/\alpha,h,a}$ for different known possible values of α, h and a.

Note: 1. In Table 1. $\eta^1_{/1,1,1}$,$\eta^2_{/-1,1,1}$, $\eta^3_{/1,2,1}$,$\eta^4_{/-1,2,1}$,$\eta^5_{/1,1,2}$ and $\eta^6_{/-1,1,2}$ are [2] type estimator and, $\eta^7_{/1,2,2}$ and $\eta^8_{/-1,2,2}$ are respectively indicates $t_{\sqrt{1}}$ and $t_{\sqrt{2}}$.

TABLE 1. Some members of the family of estimators of $\eta^{d}_{/\alpha,h,a}$

Exponential-ratio type estimator	**Exponential-product type estimator**	**a**	**h**
$g=1$	$g=-1$		
$\eta^{1}_{/1,1,1}=\bar{j}exp\left[\frac{\bar{K}_s-\bar{k}}{\bar{K}_s}\right]$	$\eta^{2}_{/-1,1,1}=\bar{j}exp\left[\frac{\bar{k}-\bar{K}_s}{\bar{K}_s}\right]$	1	1
$\eta^{3}_{/1,2,1}=\bar{j}exp\left[\frac{\sqrt{\bar{K}_s}-\sqrt{\bar{k}}}{\sqrt{\bar{K}_s}}\right]$	$\eta^{4}_{/-1,2,1}=\bar{j}exp\left[\frac{\sqrt{\bar{k}}-\sqrt{\bar{K}_s}}{\sqrt{\bar{K}_s}}\right]$	1	2
$\eta^{5}_{/1,1,2}=\bar{j}exp\left[\frac{\bar{K}_s-\bar{k}}{\bar{K}_s+\bar{k}}\right]$	$\eta^{6}_{/-1,1,2}=\bar{j}exp\left[\frac{\bar{k}-\bar{K}_s}{\bar{k}+\bar{K}_s}\right]$	2	1
$\eta^{7}_{/1,2,2}=\bar{j}exp\left[\frac{\sqrt{\bar{K}_s}-\sqrt{\bar{k}}}{\sqrt{\bar{K}_s}+\sqrt{\bar{k}}}\right]$	$\eta^{8}_{/-1,2,2}=\bar{j}exp\left[\frac{\sqrt{\bar{k}}-\sqrt{\bar{K}_s}}{\sqrt{\bar{k}}+\sqrt{\bar{K}_s}}\right]$	2	2

For $\alpha=1$, the MSEs of the exponential-ratio type estimators defined in Table 1, could be expressed as-

$$MSE(\eta^{i}_{/\alpha,h,a})=\bar{J}^2\theta\left[C_j^2+\frac{\alpha^2C_k^2}{a^2h^2}-\frac{2\alpha CC_k^2}{ah}\right] \qquad \text{for all i}(\in\text{d})=1,3,5,7.$$

And for $\alpha=-1$, the MSEs of the exponential-product type estimators defined in Table 1, could be expressed as-

$$MSE(\eta^{i}_{/\alpha,h,a})=\bar{J}^2\theta\left[C_j^2+\frac{\alpha^2C_k^2}{a^2h^2}+\frac{2\alpha CC_k^2}{ah}\right] \qquad \text{for all i}(\in\text{d})=2,4,6,8.$$

3.3. Predictive estimation of population mean using Ratio-type estimator

[7] defined a family of estimators for the population mean in simple random sampling as

$$t_{Kh}=\bar{j}\left[\frac{a_1\bar{K}+b_1}{\beta(a_1\bar{k}+b_1)+(1-\beta)(a_1\bar{K}+b_1)}\right]^g \tag{3.21}$$

where $\alpha(\neq 0)$, b_1 are either real numbers or functions of the known parameters of the auxiliary variable k such as the standard deviation S_k , coefficient of variation C_k and the correlation coefficient ρ of the population. Using (3.21), we have suggested such estimator under prediction approach for estimating population mean $\bar{J}$ is given as

$$\eta_q^p=\left[\frac{n}{N}\bar{j}+\left(\frac{N-n}{N}\right)\bar{j}\left[\frac{a_1\bar{K}+b_1}{\beta(a_1\bar{k}+b_1)+(1-\beta)(a_1\bar{K}+b_1)}\right]^g\right] \tag{3.22}$$

Expressing (3.22) in terms of $\delta_i's$, we have

$$\eta_q^p=\bar{J}(1+\delta_0)\left[f+(1-f)\left\{1-\beta+\beta(1+\lambda\delta_1)(1+\lambda\phi\delta_1+\lambda^2\phi^2\delta_1^2)\right\}^{-g}\right] \tag{3.23}$$

where, $\bar{K}_s=\frac{N\bar{K}-n\bar{k}}{(N-n)}$, $\phi=\frac{n}{(N-n)}$ and $\lambda=\frac{a_1\bar{K}}{a_1\bar{K}+b_1}$

Expanding equation (3.23) in terms of $\lambda,\beta,\delta_i's$,f and g , we have

$$\eta_q^p=\bar{J}\left[1+\delta_0-\lambda g\beta\delta_1-\frac{fg\beta\lambda^2\delta_1^2}{(1-f)}-g\beta\lambda\delta_0\delta_1+\frac{g(g+1)\beta^2}{2(1-f)^2}\lambda^2\delta_1^2\right]$$

$$\eta_q^p - \bar{J} = \bar{J}\left[\delta_0 - \lambda g\beta\delta_1 - \frac{fg\beta\lambda^2\delta_1^2}{(1-f)} - g\beta\lambda\delta_0\delta_1 + \frac{g(g+1)\beta^2}{2(1-f)}\lambda^2\delta_1^2\right] \tag{3.24}$$

Taking expectation on both sides of (3.24), we get the bias of η_q^p to the first order of approximation as:

$$B(\eta_q^p) = \bar{J}\theta\left[\frac{g(g+1)\beta^2}{2(1-f)}\lambda^2 C_k^2 - \frac{fg\beta\lambda^2 C_k^2}{(1-f)} - g\beta\lambda C C_k^2\right] \tag{3.25}$$

Squaring both sides of (3.24) and neglecting the terms having power greater than two and then taking expectation , we have

$$MSE(\eta_q^p) = \bar{J}^2\theta\left[C_j^2 + \beta^2\lambda^2 g^2 C_k^2 - 2\beta\lambda g C C_k^2\right] \tag{3.26}$$

Partially differentiating equation (3.26) w.r.t. β and equating it to zero ie:

$$\frac{\partial}{\partial\beta}MSE(\eta_q^p) = 2\beta\lambda^2 g^2 C_k^2 - 2\lambda g C C_k^2 = 0 \tag{3.27}$$

Thus from equation (3.27), the mean square error of η_q^p is minimum under the optimum condition that $\beta_{opt} = \frac{C}{\lambda g}$.

Now the substitution of $\beta_{opt} = \frac{C}{\lambda g}$ and for known a_1 and b_1 , the asymptotically optimal estimator of η_q^p is given as

$$\eta_q^p = \left[\frac{n}{N}\bar{j} + \left(\frac{N-n}{N}\right)\bar{j}\left[\frac{\lambda g(a_1\bar{K}+b_1)}{C(a_1\bar{k}+b_1) + (\lambda g - C)(a_1\bar{K}+b_1)}\right]^g\right] \tag{3.28}$$

The minimum mean square error of η_q^p under the optimum condition is given as,

$$MSE_{min}(\eta_q^p) = \bar{J}^2\theta C_j^2\left[1-\rho^2\right] \tag{3.29}$$

where, $C = \rho\frac{C_j}{C_k}$.
Note: Here it is noted that minimum MSE of η_q^p is also equal to the approximate MSE of linear regression estimator.
In certain situations when information on a_1 and b_1 is not known, we simply replace them by their estimates $\hat{a_1}$ and $\hat{b_1}$.
Below in Table 2 there are some possible family of estimators of η_q^p using different values of g,β ,a_1 and b_1 given as

For $\beta = 1$and g=1 the MSEs of the ratio-type estimator defined in table 2, could be expressed as-
$MSE(\eta_q^i) = \bar{J}^2\theta\left[C_j^2 + \lambda^2 g^2 C_k^2 - 2\lambda g C C_k^2\right]$ for all i(∈d)=1,3,5,7.

And for $\beta = -1$ and g=-1 the MSEs of the product-type estimators defined in Table 2, could be expressed as-
$MSE(\eta_q^i) = \bar{J}^2\theta\left[C_j^2 + \lambda^2 g^2 C_k^2 - 2\lambda g C C_k^2\right]$ for all i(∈d)=2,4,6,8.

TABLE 2. Some well known members of proposed estimator η_q^p

Ratio-type estimator	**Product-type estimator**	β	a_1	b_1
$g=1$	$g=-1$			
$\eta_{qi}^1=\bar{j}\left[\frac{\bar{K}_s}{\bar{k}}\right]$	$\eta_{qi}^2=\bar{j}\left[\frac{\bar{k}}{\bar{K}_s}\right]$	1	1	0
$\eta_{qi}^3=\bar{j}\left[\frac{\bar{K}_s+C_k}{\bar{k}+C_k}\right]$ Estimator due to [15]	$\eta_{qi}^4=\bar{j}\left[\frac{\bar{k}+C_k}{\bar{K}_s+C_k}\right]$ Estimator due to [10]	1	1	C_k
$\eta_{qi}^5=\bar{j}\left[\frac{\bar{K}_s+S_k}{\bar{k}+S_k}\right]$ Estimator due to [16]	$\eta_{qi}^6=\bar{j}\left[\frac{\bar{k}+S_k}{\bar{K}_s+S_k}\right]$ Estimator due to [16]	1	1	S_k
$\eta_{qi}^7=\bar{j}\left[\frac{\bar{K}_s+\rho}{\bar{k}+\rho}\right]$ Estimator due to [17]	$\eta_{qi}^8=\bar{j}\left[\frac{\bar{k}+\rho}{\bar{K}_s+\rho}\right]$ Estimator due to [17]	1	1	ρ

4. Efficiency comparisons

As

$$V(\bar{j})=\theta\bar{J}^2C_j^2 \tag{4.1}$$

(i) From (3.8), (3.12) and (4.1), we have
$V(\bar{j})\geq MSE(\eta_{\sqrt{1}})$

$$\left[\theta\bar{J}^2C_j^2-\theta\bar{J}^2\left\{C_j^2+\frac{C_k^2}{16}-\frac{CC_k^2}{2}\right\}\right]\geq 0$$

$V(\bar{j})\geq MSE(\eta_{\sqrt{2}})$, if

$$\left[\theta\bar{J}^2C_j^2-\theta\bar{J}^2\left\{C_j^2+\frac{C_k^2}{16}+\frac{CC_k^2}{2}\right\}\right]\geq 0$$

(ii) From (2.4), (3.8), (3.20) and (3.29), we have
$MSE(j_R)\geq MSE(\eta_{\sqrt{1}})$, if

$$\left[\left\{C_j^2+C_k^2(1-2C)\right\}-\left\{C_j^2+\frac{C_k^2}{16}-\frac{CC_k^2}{2}\right\}\right]\geq 0$$

$$MSE(j_R)\geq\left\{MSE_{min}(\eta_{/\alpha,h,a}^d)=MSE_{min}(\eta_q^p)\right\}$$

$$\left[\left\{C_j^2+C_k^2(1-2C)\right\}-\left\{C_j^2\left[1-\rho^2\right]\right\}\right]\geq 0$$

(iii) From (2.10), (3.20) and (3.29), we have

$$MSE(t_{re})\geq\left\{MSE_{min}(\eta_{/\alpha,h,a}^d)=MSE_{min}(\eta_q^p)\right\}$$

$$\left[\left\{C_j^2+\frac{Ck^2}{4}(1-4C)\right\}-\left\{C_j^2\left[1-\rho^2\right]\right\}\right]\geq 0$$

Table 3. Parameters of the populations under study

Source	**N**	**n**	C_j	C_k	**C**	ρ
I. Data set considered is from [3]	256	100	1.42	1.40	0.8996	0.887
II. Data set considered is from [8]	80	20	0.354	0.750	0.4441	0.941
III.Data set considered is from [4]	278	60	1.4451	1.619	0.6435	0.721
IV.Data set considered is from [3]	34	10	1.0123	1.072	0.6449	0.683
V.Data set considered is from [18]	38	6	0.70	0.7493	0.4667	0.4996
VI.Data set considered is from [3]	20	8	0.1445	0.1281	0.7332	0.6500
VII.Data set considered is from [13]	100	30	0.594065	0.59960	0.975906	0.985

5. Empirical Study

In this section, we have considered 7 real data sets to numerically evaluate the performance of suggested and existing estimators

Population I.
j: be the peach production in bushels in an orchard.
k: be the number of peach trees in the orchard.
Population II.
j: Output.
k: Fixed capital.
Population III.
j: the number of agricultural labourers for 1971.
k: the number of agricultural labourers for 1961.
Population IV.
j: be the number of placebo children
k: be the number of paralytic polio cases in the placebo group.
Population V.
j:Steel and Terrie(1960)
k:log of leaf burn in sec.
Population VI.
j:the number of person per block.
k:the number of room per block.

The comparison is performed in terms of the percent relative efficiency (PRE) of the estimator T with respect to $\bar{J}$.

$$PRE(.) = \frac{MSE(\bar{J})}{MSE(.)} * 100$$

The result are displayed in Table 4

Note: Using α, h=2 and optimum values of scalar a, we obtained the min.MSE of $\eta^{d}_{/\alpha,h,a}$. And minimum MSE of η^{p}_{q} is obtained using a_1, b_1 ,g=1 and the optimum value of β

6. Conclusion

For all the populations considered here, our estimators $\eta^{d}_{/\alpha,h,a}$ and η^{p}_{q} outperforms all the other such as usual unbiased, ratio and [12] estimators.

TABLE 4. MSE and PRE values of different estimators with respect to $\bar{J}$.

Estimators	**MSE /(PRE)**						
T	I	II	III	IV	V	VI	VII
$\bar{j}$	3.308	0.167	2.663	1.451	0.5819	0.0348	0.504162
	(100)	(100)	(100)	(100)	(100)	(100)	(100)
$\bar{j}_R$	0.7379	0.2512	1.70	0.977	0.62627	0.022045	0.01531
	(448.39)	(66.58)	(156.39)	(148.50)	(92.9099)	(157.8631)	(3293.113)
$\eta_{\sqrt{1}}$	2.0631	0.0473	1.795	1.028	0.467965	0.026483	0.285649
	(160.385)	(353.05)	(148.3)	(141.23)	(124.34)	(131.4042)	(176.4973)
t_{re}	1.219	0.0214	1.346	0.808	0.437396	0.021585	0.13135
	(271.37)	(781.398)	(197.78)	(179.74)	(133.0315)	(161.2234)	(383.874)
$\eta^d_{/\alpha,h,a}$	0.7056	0.019	1.277	0.7732	0.436657	0.020098	0.015011
	(468.9)	(877.54)	(208.45)	(187.8)	(133.25681)	(173.1546)	(3358.522)
η^p_q	0.7056	0.019	1.277	0.7732	0.436657	0.020098	0.015011
	(468.9)	(877.54)	(208.45)	(187.8)	(133.25681)	(173.1546)	(3358.522)

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