Abstract Korovkin theory in modular spaces in the sense of power series method

Tuğba Yurdakadim*

Abstract

In this paper, using the power series method we obtain an abstract Korovkin type approximation theorem for a sequence of positive linear operators defined on modular spaces.

1. Introduction

The classical Korovkin theorem states the uniform convergence in $C[a,b]$, the space of all continuous real valued functions defined on $[a,b]$ of a sequence of positive linear operators by proving the convergence only on three test functions $\{1, x, x^2\}$. This result lets us to say the convergence with minimum calculations. In fact the key moment of Korovkin's theorem is the proof of Weierstrass's theorem given by Bernstein with the use of Bernstein polynomials [11]. There are also trigonometric versions of this theorem with the test functions $\{1, \cos x, \sin x\}$ and abstract Korovkin type results have also been studied [14, 18]. Later on Korovkin type theorems have been extended in various directions with different aims such as finding other subsets satisfying the same property $\{1, x, x^2\}$, establishing the same results in other function spaces, abstract Banach spaces [1, 2, 3]. Recently some versions of Korovkin type theorems have been proved in modular spaces which include as particular cases L^p, Orlicz and Musielak-Orlicz spaces [9, 21]. Also more general convergences, for example convergences generated by summability methods, statistical, filter convergence, have also been used for the sequence of operators in order to correct the lack of convergence since it is well known that they provide a nonconvergent sequence to converge [6, 10, 12, 15, 16, 17, 22, 23].

In this paper, we give abstract Korovkin theorem in modular spaces in the sense of power series method which includes both Abel and Borel methods. We also give an example which satisfies our theorems.

*Department of Mathematics, Hitit University, Çorum, Turkey
Email : tugbayurdakadim@hotmail.com

2. Notation and definitions

Let us begin with recalling some basic definitions and notation used in the paper.

Let (p_j) be real sequence with $p_0 > 0$ and $p_1, p_2, p_3, \ldots \geq 0$, and such that the corresponding power series $p(t) := \sum_{j=0}^{\infty} p_j t^j$ has radius of convergence R with $0 < R \leq \infty$. If, for all $t \in (0, R)$,

$$\lim_{t \to R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} x_j p_j t^j = L$$

then we say that $x = (x_j)$ is convergent in the sense of power series method [19, 24]. Note that the power series method is regular if and only if

$$\lim_{t \to R^-} \frac{p_j t^j}{p(t)} = 0, \ \ for \ each \ j \in \mathbb{N}$$

holds [13]. Throughout the paper we assume that power series method is regular.

Let G be a locally compact Hausdorff topological space, endowed with a uniform structure $\mathcal{U} \subset 2^{G \times G}$ which generates the topology of G, see [20]. Let $\mathbb{B}$ be the σ-algebra of all Borel subsets of G, and $\mu : \mathbb{B} \to \mathbb{R}$ be a positive σ-finite regular measure. We denote by $L^0(G)$ the space of all real-valued μ-measurable functions on G with identification up to sets of measure μ zero, by $C_b(G)$ the space of all continuous real valued and bounded functions on G, and by $C_c(G)$ the subspace of $C_b(G)$ of all functions with compact support on G. A functional $\varrho : L^0(G) \to [0, \infty]$ is a modular on $L^0(G)$ provided that the following conditions hold:

(i) $\varrho[f] = 0$ if and only if $f = 0$ μ-a.e on G,

(ii) $\varrho[-f] = \varrho[f]$ for every $f \in L^0(G)$,

(iii) $\varrho[\alpha f + \beta g] \leq \varrho[f] + \varrho[g]$ for every $f, g \in L^0(G)$ and for any $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

A modular ϱ is said to be Q-quasi convex if there exists a constant $Q \geq 1$ such that the inequality

$$\varrho[\alpha f + \beta g] \leq Q \alpha \varrho[Qf] + Q \beta \varrho[Qg]$$

holds for every $f, g \in L^0(G)$, $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. In particular if $Q = 1$, then ϱ is called convex.

A modular ϱ is said to be Q-quasi semiconvex if there exists a constant $Q \geq 1$ such that the inequality

$$\varrho[af] \leq Q a \varrho[Qf]$$

holds for every $f \in L^0(G)$, $f \geq 0$ and $a \in (0, 1]$. It is clear that every Q-quasi convex modular is Q-quasi semiconvex. A modular ϱ is said to be monotone if $\varrho[f] \leq \varrho[g]$ for all $f, g \in L^0(G)$ with $|f| \leq |g|$.

We now consider some subspaces of $L^0(G)$ by means of a modular ϱ as follows

$$L^{\varrho}(G) := \{f \in L^0(G) : \lim_{\lambda \to 0^+} \varrho[\lambda f] = 0\}$$

and

$$E^{\varrho}(G) := \{f \in L^{\varrho}(G) : \ \varrho[\lambda f] < \infty \ for \ all \ \lambda > 0\}$$

are called the modular space generated by ϱ and the space of the finite elements of $L^{\varrho}(G)$, respectively. Observe that if ϱ is Q-quasi semiconvex then the space

$$\{f \in L^0(G) : \ \varrho[\lambda f] < \infty \ for \ some \ \lambda > 0\}$$

coincides with $L^{\varrho}(G)$. The notions about modulars have been introduced and widely discussed in [4, 5, 7, 8, 9].

Now we define the convergences in the sense of power series method in modular spaces. Let $\{f_j\}$ be a function sequence whose terms belong to $L^\varrho(G)$. Then, $\{f_j\}$ is modularly convergent to a function $f \in L^\varrho(G)$ in the sense of power series method if and only if

$$\lim_{t \to R^-} \frac{1}{p(t)} \sum_{j=0}^\infty p_j t^j \varrho[\lambda_0(f_j - f)] = 0 \ for \ some \ \lambda_0 > 0.$$

Also, $\{f_j\}$ is strongly convergent to a function $f \in L^\varrho(G)$ in the sense of power series method if and only if

$$\lim_{t \to R^-} \frac{1}{p(t)} \sum_{j=0}^\infty p_j t^j \varrho[\lambda(f_j - f)] = 0 \ for \ every \ \lambda > 0.$$

Recall that $\{f_j\}$ is modularly convergent to a function $f \in L^\varrho(G)$ if and only if

$$\lim_{j \to \infty} \varrho[\lambda_0(f_j - f)] = 0 \ for \ some \ \lambda_0 > 0,$$

also $\{f_j\}$ is strongly convergent to a function $f \in L^\varrho(G)$ if and only if

$$\lim_{j \to \infty} \varrho[\lambda(f_j - f)] = 0 \ for \ every \ \lambda > 0.$$

If there exists a constant $M > 0$ such that

$$\varrho[2u] \le M \varrho[u]$$

holds for all $u \ge 0$ then it is said to be that ϱ satisfies the Δ_2-condition.

Now we can give the following:

2.1. Theorem. *Let $L^\varrho(G)$ be a modular space. Δ_2-condition is sufficient in order that strong convergence in the sense of power series method and modular convergence in the sense of power series method be equivalent in $L^\varrho(G)$.*

Proof. Obviously, strong convergence in the sense of power series method of $\{f_j\}$ to f is equivalent to the condition $\lim_{t \to R^-} \frac{1}{p(t)} \sum_{j=0}^\infty p_j t^j \varrho[2^N \lambda(f_j - f)] = 0$, for some $\lambda > 0$ and all $N = 1, 2, \dots$. Let $\{f_j\}$ be modularly convergent to f in the sense of power series method. Then there exists a $\lambda > 0$ such that $\lim_{t \to R^-} \frac{1}{p(t)} \sum_{j=0}^\infty p_j t^j \varrho[\lambda(f_j - f)] = 0$. Δ_2-condition implies by induction that $\varrho[2^N \lambda(f_j - f)] \le M^N \varrho[\lambda(f_j - f)]$. Therefore we get

$$\lim_{t \to R^-} \frac{1}{p(t)} \sum_{j=0}^\infty p_j t^j \varrho[2^N \lambda(f_j - f)] = 0.$$

This completes the proof. $\qquad \square$

A modular ϱ is said to be finite if χ_A, the characteristic function associated with A, belongs to $L^\varrho(G)$ whenever $A \in \mathbb{B}$ with $\mu(A) < \infty$. A modular ϱ is said to be strongly finite if χ_A belongs to $E^\varrho(G)$ for all $A \in \mathbb{B}$ with $\mu(A) < \infty$ and a modular ϱ is said to be absolutely continuous if there is a positive constant a with the property: for all $f \in L^0(G)$ with $\varrho[f] < \infty$,

- for each $\varepsilon > 0$ there exists a set $A \in \mathbb{B}$ with $\mu(A) < \infty$ and $\varrho[af\chi_{G \setminus A}] \le \varepsilon$,
- for every $\varepsilon > 0$ there is a $\delta > 0$ with $\varrho[af\chi_B]$ for every $B \in \mathbb{B}$ with $\mu(B) < \delta$.

Recall that if a modular ϱ is monotone and finite, then we have $C(G) \subset L^\varrho(G)$ [7]. In a similar manner, if ϱ is monotone and strongly finite, then $C(G) \subset E^\varrho(G)$.

3. Modular Korovkin theorem by power series method

Let ϱ be monotone and finite modular on $L^0(G)$. Assume that D is a set satisfying $C_b(G) \subset D \subset L^0(G)$. Assume further that $T := \{T_j\}$ is a sequence of positive linear operators from D into $L^0(G)$ for which there exists a subset $X_T \subset D \bigcap L^\varrho(G)$ with $C_b(G) \subset X_T$ such that the inequality

$$(3.1) \qquad \limsup_{t \to R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \varrho[\lambda(T_j h)] \leq P\varrho(\lambda h)$$

holds for every $h \in X_T$, $\lambda > 0$ and for an absolute positive constant P. Then it is said that ϱ satisfies the property modified $(\varrho) - (*)$.

Set $e_0(t) \equiv 1$ for all $t \in G$, let e_i, $i = 1, 2, 3, ..., m$, and a_i, $i = 0, 1, 2, 3, ..., m$, be functions in $C_b(G)$. Put

$$(3.2) \qquad P_s(t) = \sum_{i=0}^{m} a_i(s) e_i(t), \ s, t \in G,$$

and suppose that $P_s(t)$, $s, t \in G$ satisfies the following properties:

- (P1) $P_s(s) = 0$, for all $s \in G$,
- (P2) for every neighbourhood $U \in \mathcal{U}$ there is a positive real number η with $P_s(t) \geq \eta$ whenever $s, t \in G$, $(s, t) \notin U$.

Some examples of P_s for which (P1) and (P2) are satisfied have been given in [6].

To give our main theorem, we first give the following result.

3.1. Theorem. *Let ϱ be a strongly finite, monotone and Q-quasi semiconvex modular. Assume that e_i and a_i, $i = 0, 1, ..., m$ satisfy properties (P1) and (P2). Let T_j, $j \in \mathbb{N}$, be a sequence of positive linear operators satisfying property modified $(\varrho) - (*)$. If*

$$\lim_{t \to R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \varrho[\lambda(T_j e_i - e_i)] = 0,$$

for some $\lambda > 0$ and $i = 0, 1, ..., m$, then for every $f \in C_c(G)$

$$\lim_{t \to R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \varrho[\gamma(T_j f - f)] = 0,$$

for some $\gamma > 0$. Moreover if

$$\lim_{t \to R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \varrho[\lambda(T_j e_i - e_i)] = 0,$$

for every $\lambda > 0$ and $i = 0, 1, ..., m$, then for every $f \in C_c(G)$

$$\lim_{t \to R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \varrho[\lambda(T_j f - f)] = 0$$

for every $\lambda > 0$.

Proof. Let $f \in C_c(G)$. Then f is uniformly continuous and bounded on G since G is endowed with $\mathcal{U}$ uniformity. Let $\varepsilon > 0$. Without losing anything from generality we can choose $0 < \varepsilon \leq 1$. From the uniform continuity of f, there exists a set $U \in \mathcal{U}$ such that

$$|f(s) - f(t)| \leq \varepsilon, \quad s, t \in G, \ (s, t) \in U.$$

For every $s, t \in G$ let $P_s(t)$ be as in (3.2) and let $\eta > 0$ satisfy condition $(P2)$. If $M = \sup\limits_{t \in G} |f(t)|$, for $s, t \in G$, $(s, t) \notin U$ we have

$$|f(s) - f(t)| \leq 2M \leq \frac{2M}{\eta} P_s(t).$$

For every $s, t \in G$, we obtain

$$|f(s) - f(t)| \leq 2M \leq \varepsilon + \frac{2M}{\eta} P_s(t).$$

Therefore for every $s, t \in G$, we get

$$(3.3) \qquad -\varepsilon - \frac{2M}{\eta} P_s(t) \leq f(s) - f(t) \leq \varepsilon + \frac{2M}{\eta} P_s(t).$$

Since T_j is a linear positive operator, using (3.3) for each $j \in \mathbb{N}$ and every $s \in G$ we have

$$-\varepsilon(T_j e_0)(s) - \frac{2M}{\eta}(T_j P_s)(s) \leq f(s)(T_j e_0)(s) - (T_j f)(s)$$

$$\leq \varepsilon(T_j e_0)(s) + \frac{2M}{\eta}(T_j P_s)(s)$$

and hence

$$|(T_j f)(s) - f(s)| \leq |(T_j f)(s) - f(s)(T_j e_0)(s)| + |f(s)(T_j e_0)(s) - f(s)|$$
$$(3.4) \qquad \leq \varepsilon(T_j e_0)(s) + \frac{2M}{\eta}(T_j P_s)(s) + M|(T_j e_0)(s) - e_0(s)|.$$

Let $\gamma > 0$. By applying the modular ϱ, from the above inequality, we have

$$(3.5) \qquad \varrho[\gamma(T_j f - f)] \leq \varrho[3\gamma\varepsilon(T_j e_0)] + \varrho[3\gamma M(T_j e_0 - e_0)] + \varrho[6\gamma\frac{M}{\eta}(T_j P_{(.)})(.)]$$

$$= J_1 + J_2 + J_3, \; for \; each \; j \in \mathbb{N}.$$

So to prove the theorem it is sufficient to show that there exists a positive real number γ such that $\lim\limits_{t \to R^-} \frac{1}{p(t)} \sum\limits_{j=0}^{\infty} p_j t^j \varrho[\gamma(T_j f - f)] = 0$. From hypothesis there exists a $\lambda > 0$ such that for each $i = 0, 1, ..., m$

$$\lim\limits_{t \to R^-} \frac{1}{p(t)} \sum\limits_{j=0}^{\infty} p_j t^j \varrho[\lambda(T_j e_i - e_i)] = 0.$$

For each $i = 0, 1, ..., m$ and $s \in G$, choose $N > 0$ and $\gamma > 0$ such that $|a_i(s)| \leq N$ and $\max\{3\gamma M, \; 6\gamma\frac{M}{\eta}(m+1)N\} \leq \lambda$. By considering the property (P1), for each $j \in \mathbb{N}$ we get

$$J_3 = \varrho[6\gamma\frac{M}{\eta}(T_j P_{(.)})(.)] = \varrho[6\gamma\frac{M}{\eta}(T_j P_{(.)})(.) - P_{(.)}(.)]$$

$$(3.6) \qquad \leq \sum\limits_{i=0}^{m} \varrho[6\gamma\frac{M}{\eta}(m+1)N(T_j e_i - e_i)]$$

$$\leq \sum\limits_{i=0}^{m} \varrho[\lambda(T_j e_i - e_i)].$$

Hence we obtain

$$\lim\limits_{t \to R^-} \frac{1}{p(t)} \sum\limits_{j=0}^{\infty} p_j t^j J_3 = 0.$$

Moreover from choosing λ and γ, it is clear that $\lim_{t \to R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j J_2 = 0$. Since ϱ is Q-quasi semiconvex and $0 < \varepsilon \le 1$, we have

$$(3.7) \qquad \varrho[3\gamma \varepsilon e_0] \le Q\varepsilon \varrho[3\gamma Q e_0].$$

If the property modified $(\varrho) - (*)$ is considered in (3.5), (3.7), we get

$$(3.8) \qquad 0 \le \limsup_{t \to R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \varrho[\gamma(T_j f - f)] \le \limsup_{t \to R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \varrho[3\gamma \varepsilon(T_j e_0)]$$
$$\le P\varrho[3\gamma \varepsilon e_0] \le PQ\varepsilon \varrho[3\gamma Q e_0]$$

by applying the limit superior. Since ε is arbitrary positive real number and ϱ is strongly finite from (3.8) we have

$$\limsup_{t \to R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \varrho[\gamma(T_j f - f)] = 0,$$

and hence $\{T_j f\}$ is modularly convergent to f in the sense of power series method in $L^\varrho(G)$ by using the properties of limit in the sense of power series method and limit superior. The second part can be proved similarly to the first one. $\qquad \square$

Now we can give our main theorem.

3.2. Theorem. *Let ϱ be a strongly finite, monotone, absolutely continuous and Q-quasi semiconvex modular on $L^0(G)$. Let T_j, $j \in \mathbb{N}$, be a sequence of positive linear operators satisfying property modified $(\varrho) - (*)$. If*

$$\lim_{t \to R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \varrho[\lambda(T_j e_i - e_i)] = 0$$

for every $\lambda > 0$ and $i = 0, 1, ..., m$, then for every $f \in L^\varrho(G) \cap D$ with $f - C_b(G) \subset X_T$,

$$\lim_{t \to R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \varrho[\gamma(T_j f - f)] = 0,$$

for some $\gamma > 0$ where X_T and D are as before.

Proof. Let $f \in L^\varrho(G) \cap D$ such that $f - C_b(G) \subset X_T$. From Proposition 3.2 of [6] there exist a $\lambda > 0$ and a sequence $\{f_m\}$ in $C_c(G)$ such that $\varrho[3\lambda f] < \infty$ and $\lim_m \varrho[3\lambda(f_m - f)] = 0$. Take arbitrary fixed $\varepsilon > 0$ and choose a positive integer $\overline{m}$ such that

$$(3.9) \qquad \varrho[3\lambda(f_{\overline{m}} - f)] \le \varepsilon.$$

For each $j \in \mathbb{N}$, we have

$$(3.10) \qquad \varrho[\lambda(T_j f - f)] \le \varrho[3\lambda(T_j f - T_j f_{\overline{m}})] + \varrho[3\lambda(T_j f_{\overline{m}} - T_j f_{\overline{m}})] + \varrho[3\lambda(f_{\overline{m}} - f)].$$

Using the similar technique in the previous theorem, we obtain

$$0 = \lim_{t \to R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \varrho[3\lambda(T_j f_{\overline{m}} - f_{\overline{m}})] = \limsup_{t \to R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \varrho[3\lambda(T_j f_{\overline{m}} - f_{\overline{m}})].$$

From the property modified $(\varrho) - (*)$, there exists an $P > 0$ such that

$$(3.11) \qquad \lim_{t \to R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \varrho[3\lambda(T_j f - T_j f_{\overline{m}})] \le P\varrho[3\lambda(f - f_{\overline{m}})] \le P\varepsilon.$$

From (3.9)-(3.11) and subadditivity of the operator lim sup we have

$$(3.12) \quad 0 \le \limsup_{t \to R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \varrho[\lambda(T_j f - f)] \le \varepsilon(P+1).$$

From (3.12) and arbitrariness of ε we get

$$\limsup_{t \to R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \varrho[\lambda(T_j f - f)] = 0.$$

Thus one can get

$$\lim_{t \to R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \varrho[\lambda(T_j f - f)] = 0$$

and this completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4. Concluding remarks

Take $G = [0,1]$ and let $\varphi : [0,\infty) \to [0,\infty)$ be a continuous function for which the following conditions hold:

- φ is convex,
- $\varphi(0) = 0, \varphi(u) > 0 \ for \ u > 0$ and $\lim_{u \to +\infty} \varphi(u) = \infty$.

Here, consider the functional ϱ^φ on $L^0(G)$ defined by

$$\varrho^\varphi[f] := \int_0^1 \varphi(|f(x)|)dx, \ for \ f \in L^0(G).$$

In this case, ϱ^φ is a convex modular on $L^0(G)$ (see [7]). Consider the Orlicz space generated by φ as follows

$$L_\varphi^\varrho(G) := \{f \in L^0(G) : \varrho^\varphi[\lambda f] < \infty \ for \ some \ \lambda > 0\}.$$

Then, consider the following classical Bernstein-Kantorovich operator $\mathbb{U} := \{U_j\}$ on the space $L_\varphi^\varrho(G)$ (see [7]) which is defined by

$$U_j(f;x) := \sum_{k=0}^{j} \binom{j}{k} x^k (1-x)^{j-k}(j+1) \int_{\frac{k}{j+1}}^{\frac{k+1}{j+1}} f(t)dt; \ x \in G.$$

Observe that the operators U_j map the Orlicz space $L_\varphi^\varrho(G)$ into itself. Moreover, it is also known that the property $\limsup_{j \to \infty} \varrho[\lambda(U_j h)] \le M\varrho[\lambda h]$ is satisfied with the choice of $X_U := L_\varphi^\varrho(G)$ and for every function $f \in L_\varphi^\varrho(G)$ such that $f - g \in X_U$ for every $g \in C_b(G)$, $\{U_j f\}$ is modularly convergent to f. Using the operators $\{U_j f\}$ define the sequence of positive linear operators $V := V_j$ on $L_\varphi^\varrho(G)$ as follows:

$$(4.1) \qquad V_j(f;x) = (1 + s_j)U_j(f;x), \ for \ f \in L_\varphi^\varrho(G), \ x \in [0,1] \ and \ j \in \mathbb{N},$$

where $s_j = 1$, j is square and 0 otherwise. Also let $R = 1$, $p(t) = \frac{1}{1-t}$ and for $j \in \mathbb{N}^0$, $p_j = 1$. In this case the power series method coincides with Abel method. Note that

$\{s_j\}$ is convergent to 0 in the sense of power series method. By Lemma 5.1 of [7], for every $h \in X_V := L_\varphi^\varrho(G)$, all $\lambda > 0$ and for an absolute positive constant P, we get

$$\varrho^\varphi[\lambda V_j h] = \varrho^\varphi[\lambda(1 + s_j)U_j h] \leq \varrho^\varphi[2\lambda U_j h] + \varrho^\varphi[2\lambda s_j U_j h]$$
$$= \varrho^\varphi[2\lambda U_j h] + s_j \varrho^\varphi[2\lambda U_j h]$$
$$= (1 + s_j)\varrho^\varphi[2\lambda U_j h] \leq (1 + s_j)P\varrho^\varphi[2\lambda h].$$

Then, we get

$$\limsup_{t \to R^-} \frac{1}{p(t)} \sum_{j=0}^\infty p_j t^j \varrho^\varphi[\lambda V_j h] \leq P\varrho^\varphi[2\lambda h].$$

Now, we show that conditions in the Theorem 3.2 holds. First note that

$$V_j(e_0; x) = 1 + s_j$$

$$V_j(e_1; x) = (1 + s_j)\{\frac{jx}{j+1} + \frac{1}{2(j+1)}\}$$

$$V_j(e_2; x) = (1 + s_j)\{\frac{j(j-1)x^2}{(j+1)^2} + \frac{2jx}{(j+1)^2} + \frac{1}{3(j+1)^2}\}$$

where $e_i(t) = t^i$. So for any $\lambda > 0$, we can see, that

$$\lambda|V_j(e_0; x) - e_0(x)| = \lambda|1 + s_j - 1| = \lambda s_j,$$

which implies

$$\varrho^\varphi[\lambda(V_j(e_0) - e_0)] = \varrho^\varphi[\lambda s_j] = \int_0^1 \varphi(\lambda s_j)dx = \varphi(\lambda s_j) = s_j\varphi(\lambda)$$

because of the definition of $\{s_j\}$. Since $\{s_j\}$ is convergent to 0 in the sense of power series method

$$\lim_{t \to R^-} \frac{1}{p(t)} \sum_{j=0}^\infty p_j t^j \varrho^\varphi[\lambda V_j(e_0) - e_0] = \lim_{t \to R^-} \frac{1}{p(t)} \sum_{j=0}^\infty p_j t^j s_j \varphi(\lambda) = 0,$$

for every $\lambda > 0$. Also

$$\lambda|V_j(e_1; x) - e_1(x)| = \lambda|x(\frac{j}{j+1} + \frac{js_j}{j+1} - 1) + \frac{1}{2(j+1)} + \frac{s_j}{2(j+1)}|$$
$$\leq \lambda\{\frac{3}{2(j+1)} + s_j(\frac{2j+1}{2(j+1)})\},$$

we may write that

$$\varrho^\varphi[\lambda(V_j(e_1) - e_1)] \leq \varrho^\varphi[\lambda\{s_j(\frac{2j+1}{2(j+1)}) + \frac{3}{2(j+1)}\}]$$
$$\leq s_j\varrho^\varphi[\lambda(\frac{2j+1}{j+1})] + \varrho^\varphi[\frac{3\lambda}{j+1}]$$

by the definitions of $\{s_j\}$ and ϱ^φ. Since $\{\frac{2j+1}{j+1}\}$ is convergent, there exists a constant $M > 0$ such that $\{\frac{2j+1}{j+1} \leq M\}$, for every $j \in \mathbb{N}$. Then using the monotonicity of ϱ^φ, we have

$$\varrho^\varphi[\lambda\frac{2j+1}{j+1}] \leq \varrho^\varphi[\lambda M]$$

for any $\lambda > 0$, which implies

$$\varrho^\varphi[\lambda(V_j(e_1) - e_1)] \leq s_j \varrho^\varphi[\lambda M] + \varrho^\varphi[\frac{3\lambda}{j+1}] = s_j\varphi(\lambda M) + \varphi(\frac{3\lambda}{j+1}).$$

Since φ is continuous, we have $\lim_j \varphi(\frac{3\lambda}{j+1}) = \varphi(\lim_j \frac{3\lambda}{j+1}) = \varphi(0) = 0$. So we get $\varphi(\frac{3\lambda}{j+1})$ is convergent to 0 in the sense of power series method. Using this and by the definition of $\{s_j\}$, we obtain

$$\lim_{t \to R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \varrho^\varphi[\lambda V_j(e_1) - e_1] \leq \lim_{t \to R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j [s_j \varphi(\lambda M) + \varphi(\frac{3\lambda}{j+1})]$$

$$= \varphi(\lambda M) \lim_{t \to R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j s_j$$

$$+ \lim_{t \to R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \varphi(\frac{3\lambda}{j+1})$$

$$= 0$$

Finally, since

$$\lambda |V_j(e_2; x) - e_2(x)|$$

$$= \lambda |x^2 \frac{j(j-1)}{(j+1)^2} + \frac{2jx}{(j+1)^2} + \frac{1}{3(j+1)^2} + s_j \frac{j(j-1)x^2}{(j+1)^2} + s_j \frac{2jx}{(j+1)^2}$$

$$+ s_j \frac{1}{3(j+1)^2} - x^2|$$

$$\leq \lambda \{\frac{15j+4}{3(j+1)^2} + s_j(\frac{3j^2+3j+1}{3(j+1)^2})\}.$$

Since $\{\frac{3j^2+3j+1}{3(j+1)^2}\}$ is convergent, there exists a constant $K > 0$ such that $|\frac{3j^2+3j+1}{3(j+1)^2}| \leq K$, for every $j \in \mathbb{N}$. Then using the monotonicity of ϱ^φ and the definition of $\{s_j\}$, we have

$$\varrho^\varphi[\lambda(V_j(e_2) - e_2)] \leq \varrho^\varphi[2\lambda(\frac{15j+4}{3(j+1)^2})] + \varrho^\varphi[2\lambda s_j(\frac{3j^2+3j+1}{3(j+1)^2})]$$

$$\leq \varrho^\varphi[\lambda(\frac{30j+8}{3(j+1)^2})] + \varrho^\varphi[2\lambda s_j K],$$

which yields

$$\varrho^\varphi[\lambda(V_j(e_2) - e_2)] \leq \varphi[\lambda(\frac{30j+8}{3(j+1)^2})] + s_j\varphi(2\lambda K).$$

Since φ is continuous, we have $\lim_j \varphi(\lambda \frac{30j+8}{3(j+1)^2}) = \varphi(\lambda \lim_j \frac{30j+8}{3(j+1)^2}) = \varphi(0) = 0$. So we get $\varphi(\lambda \frac{30j+8}{3(j+1)^2})$ is convergent to 0 in the sense of power series method. Using this and by the definition of $\{s_j\}$, we obtain

$$\lim_{t \to R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \varrho^\varphi[\lambda V_j(e_2) - e_2] = 0, \ for \ every \ \lambda > 0.$$

So we can say that our sequence $V := \{V_j\}$ satisfies all assumptions of Theorem 3.2. Therefore we conclude that

$$\lim_{t \to R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \varrho^\varphi[\lambda_0 V_j(f) - f] = 0, \ for \ some \ \lambda_0 > 0$$

holds for every $f \in L_\varphi^\varrho(G)$ such that $f - g \in X_V$ for every $g \in C_b(G)$. However since $\{s_j\}$ is not convergent to zero, it is clear that $\{V_j(f)\}$ is not modularly convergent to f. Note that

- in the case of $R = 1$, $p(t) = \frac{1}{1-t}$ and for $j \in \mathbb{N}^0$, $p_j = 1$ the power series method coincides with Abel method which is a sequence-to-function transformation,
- in the case of $R = \infty$, $p(t) = e^t$ and for $j \in \mathbb{N}^0$, $p_j = \frac{1}{j!}$ the power series method coincides with Borel method.

We can therefore give all of the theorems of this paper for Abel and Borel convergences.

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