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RESULTS ON TSALLIS ENTROPY OF ORDER STATISTICS AND RECORD VALUES

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Abstract: The Tsallis entropy is a generalization of type α of the Shannon entropy (Tsallis, 1988) that is a non-additive entropy unlike the Shannon entropy and some of other generalized entropy, such as Renyi entropy that introduced by Renyi (1961). In this paper, we study the Tsallis entropy based on order statistics and record values. We show that the parent distributions can be determined uniquely by the equality of Tsallis entropy of order statistics or record values. Also, we characterize symmetric distributions based on Tsallis entropy of order statistics and record values. Finally, we prove that the Tsallis information between order statistics and parent random variable, and Tsallis information between record values and parent random variable are distribution free. The results are useful in modeling problems and testing statistical hypotheses.

Key words: Characterization; Hazard rate function; Muntz-Szasz theorem; Series (parallel) system; Symmetric distributions; Tsallis information.
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1. Introduction

Let $X_1, X_2, \ldots, X_n$ be iid observations, each with the cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$. The order statistics of the sample is defined by the smallest to the largest, denoted as $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$. Order statistics have been used in a wide range of problems like detection of outliers, characterization of probability distributions, testing strength of materials, robust statistical estimation and entropy estimation, goodness-of-fit tests and also in reliability theory it is well known that $X_{i:n}$ represents the lifetime of a $(n-i+1)$-out-of-n system. Particulary, $X_{1:n}$ and $X_{n:n}$ give the lifetimes of the series and the parallel systems, respectively (see Arnold et al., 1992 and David and Nagaraja, 2003, for more details).

Suppose that $\{X_i\}_{i\geq 1}$, be a sequence of iid continuous random variables from the cdf $F(x)$ and the pdf $f(x)$. An observation X_j will be called an upper record value if its value exceeds that of all previous observation i.e. $X_j > X_i$, for every $i < j$. An analogous definition can be given for lower record values. The times at which upper record values appear are given by the random variable T_j which are called record times and are defined by $T_1 = 1$ with probability 1, and for $j \geq 2$, $T_j = \min\{i : X_i > X_{T_{j-1}}\}$ and T_0 is defined 0. The sequence of upper record values can thus be defined by $U_j = X_{T_j}$, $j = 1, 2, 3, \cdots$.

Records can be used in a wide range of problem, incluging seimology, sporting and athletic events, meteorological analysis, industrial stress testing, hydrology, oil and minig surveys, characterization of probability distribution and in reliability theory (see Arnold et al., 1998, for more details). Also, record values are used in shock models and minimal repair systems, such that, if X_n denotes the lifetime of the component and if n minimal repairs are allowed, then the survival function of the X_n is the same as that of the $(n+1)^{th}$ upper record value (see Shaked and Shanthikumar, 1994 and Kamps, 1994). Records can be viewed as order statistics from a sample whoes size is determined

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by the value and the order of occurrence of the observations. Also, one of the important applications of order statistics is to construct median filters for image and signal processing. Considering importance of Tsallis entropy, order statistics and record values, we try to extend the concept of Tsallis entropy using order statistics and record values which can be further used in image or signal processing.
The concept of entropy is important for studies in many areas such as physics, probability and statistics, communication theory and economics. An early definition of a measure of the entropy is the Shannon entropy (Shannon, 1948). In information theory, the Shannon information is described as a measure of the uncertainty of a source. The definition of this measure is

$$H(X) = -\int_{-\infty}^{+\infty} f(x)\log f(x)dx = -E[\log f(X)], \tag{1.1}$$

where X is a random variable having an absolutely continuous cdf $F(x)$ with pdf $f(x)$. The measure 1.1 is an additive entropy because, for any two independent random variables X and Y

$$H(X,Y) = H(X) + H(Y).$$

One main drawback of $H(X)$ is that for some probability distribution, it may be negative and then it is no longer an uncertainty measure. This drawback is removed in the generalized entropies. One of this generalized entropies is the Tsallis entropy, was first introduced by Havrda and Charvat (1967) in the context of cybernetics theory. Then, Tsallis (1988) exploited its non-extensive features and placed it in a physical setting. This measure is defined as

$$S_\alpha(X) = \frac{1}{\alpha-1}\left[1 - \int_{-\infty}^{+\infty} f^\alpha(x)dx\right],\ \alpha \neq 1,\ \alpha > 0, \tag{1.2}$$

where X is a random variable having an absolutely continuous cdf $F(x)$ with pdf $f(x)$. As $\alpha \longrightarrow 1$ in 1.2, it reduces to $H(X)$ given in 1.1 (Tsallis, 1988).
By changing variable $u = F(x)$ in 1.2, we have

$$S_\alpha(X) = \frac{1}{\alpha-1}\left[1 - \int_0^1 f^{\alpha-1}(F_X^{-1}(u))du\right]. \tag{1.3}$$

Also, for a non-negative random variabel X, we can conclude that

$$S_\alpha(X) = \frac{1}{\alpha-1}\left[1 - \frac{1}{\alpha}E_{f_{X,\alpha}}[r_X^{\alpha-1}(X)]\right], \tag{1.4}$$

where $f_{X,\alpha}(x) = \frac{-d\bar{F}^\alpha(x)}{dx} = \alpha\bar{F}^{\alpha-1}(x)f(x)$; $\alpha > 1$, $\bar{F}(x) = 1 - F(x)$ and $r_X(t) = \frac{f(t)}{\bar{F}(t)}$ is the hazard rate function of X. Moreover, the Tsallis entropy is a non-additive entropy as for any two independent random variables X and Y

$$S_\alpha(X,Y) = S_\alpha(X) + S_\alpha(Y) + (1-\alpha)S_\alpha(X)S_\alpha(Y).$$

From the years 2000 on, an increasingly wide spectrum of natural, artificial and social complex systems have been identified which confirm the predictions and consequences that are derived from this non-additive entropy, such as non-extensive statistical mechanics (Tsallis, 2009), which generalizes the Boltzmann-Gibbs theory. In a recent conference (Recent Innovations in Info-Metrics An Interdisciplinary Perspective 2014, American University, Washington DC), Tsallis presented a classification of physical systems according to their complexities and identified the systems where additive entropy (Shannon entropy) is applicable and where it is not, so a non-addative measures

of uncertainty (Tsallis entropy) is needed. Also, in physics, Tsallis entropy is used to describe a number of non-extensive systems (Hamity and Barraco, 1996), image processing (Yu et al., 2009) and signal processing (Tong et al., 2002). Properties of the Tsallis entropy have been investigated by several authors including Nanda and Paul (2006), Zhang (2007), Wilk and Woldarczyk (2008) and Kumar and Taneja (2011).
Several authors have studied the subject of characterization of distribution function $F(x)$ based on entropies of order statistics and record values. Raqab and Awad (2000, 2001) obtained a characterization of the generalization pareto distribution based on Shannon entropy of k-record statistics. Baratpour et al. (2007, 2008) obtained several characterization based on Shannon entropy and Renyi entropy of order statistics and record values. Fashandi and Ahmadi (2012) have derived characterization result for the symmetric distributions based on Renyi entropy of order statistics, k-record statistics and the FGM family of bivariate distributions. Thapliyal and Taneja (2013) established a characterization based on past entropy of order statistics. Gupta et al. (2014) achieved some characterization results based on dynamic entropy of order statictics. In this paper, we extend the Tsallis entropy based on order statistics and record values and obtain some similar results. We also study the Tsallis information measure in order statistics and record values and its properties. The paper is organized as follows: In section 2, we express Tsallis entropy of i^{th} order statistics. Also, we obtain some characterization results based on Tsallis entropy for order statistics. Section 3 is devoted of Tsallis entropy of j^{th} upper record values and characterization results for the record values. Characterization of symmetric distributions by Tsallis entropy of k-records is explicitly given. Finally, in section 4, we consider Tsallis information of order statistics and record values.

2. Tsallis entropy of order statistics

Tsallis entropy associated with the i^{th} order statistics $X_{i:n}$ is given by

$$S_\alpha(X_{i:n}) = \frac{1}{\alpha-1}\left[1-\int_{-\infty}^{+\infty} f_{i:n}^{\alpha}(x)dx\right], \tag{2.1}$$

where $\alpha \neq 1, \alpha > 0$ and $f_{i:n}(x)$ is pdf of i^{th} order statistics, for $i = 1, 2, ..., n$, that given by

$$f_{i:n}(x) = \frac{1}{B(i, n-i+1)} F^{i-1}(x)\bar{F}^{n-i}(x) f(x), \tag{2.2}$$

where

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx, \quad a>0,\ b>0,$$

is beta function. Note that for $n = 1$, 2.1 reduces to 1.2.
Next, we have the following Lemma.

Lemma 1. *Let $X_1, X_2, \ldots, X_n$ be a random sample with size n from continuous cdf $F(x)$ and pdf $f(x)$. Let $X_{i:n}$ denotes the i^{th} order statistics. Then the Tsallis entropy of $X_{i:n}$ can be expressed as*

$$S_\alpha(X_{i:n}) = \frac{1}{\alpha-1}\left[1-C_i E[f^{\alpha-1}(F^{-1}(Z_i))]\right], \tag{2.3}$$

where

$$C_i = \frac{B\left(\alpha(i-1)+1, \alpha(n-i)+1\right)}{B^\alpha(i, n-i+1)},$$

and Z_i has beta distribution with parameters $\alpha(i-1)+1$ and $\alpha(n-i)+1$.

Proof. By 2.1 and 2.2, and by changing variable $z = F(x)$, we have

$$S_\alpha(X_{i:n}) = \frac{1}{\alpha-1}\left[1-\int_0^1 \frac{1}{B^\alpha(i,n-i+1)} z^{\alpha(i-1)}(1-z)^{\alpha(n-i)} f^{\alpha-1}(F^{-1}(z))dz\right]$$

$$
\begin{aligned}
&= \frac{1}{\alpha-1}\left[1-\frac{1}{B^{\alpha}(i,n-i+1)}B\left(\alpha(i-1)+1,\alpha(n-i)+1\right)\right.\\
&\quad \left.\times\int_0^1 \frac{1}{B\left(\alpha(i-1)+1,\alpha(n-i)+1\right)} z^{\alpha(i-1)}(1-z)^{\alpha(n-i)} f^{\alpha-1}(F^{-1}(z))dz\right]\\
&= \frac{1}{\alpha-1}\left[1-\frac{1}{B^{\alpha}(i,n-i+1)}B\left(\alpha(i-1)+1,\alpha(n-i)+1\right)\right.\\
&\quad \left.\times E[f^{\alpha-1}(F^{-1}(Z_i))]\right]. \qquad (2.4)
\end{aligned}
$$

Thus, the proof is complete.

EXAMPLE 1. Suppose that X is a random variable having the exponential distribution with mean $\frac{1}{\lambda}$. Here, $f^{\alpha-1}(F^{-1}(t)) = \lambda^{\alpha-1}(1-t)^{\alpha-1}$. Then,

$$E[f^{\alpha-1}(F^{-1}(Z_i))] = \lambda^{\alpha-1}\frac{B\left(\alpha(i-1)+1,\alpha(n-i+1)\right)}{B\left(\alpha(i-1)+1,\alpha(n-i)+1\right)}.$$

So, by 2.1, we have

$$S_\alpha(X_{i:n}) = \frac{1}{\alpha-1}\left[1-\frac{\lambda^{\alpha-1}B\left(\alpha(i-1)+1,\alpha(n-i+1)\right)}{B^{\alpha}(i,n-i+1)}\right],$$

and for the sample minimum, $i=1$, we find

$$S_\alpha(X_{1:n}) = \frac{1}{\alpha-1}\left[1-\frac{(n\lambda)^{\alpha-1}}{\alpha}\right].$$

Also, by replacing n by 1, we conclude that

$$S_\alpha(X) = \frac{1}{\alpha-1}\left[1-\frac{\lambda^{\alpha-1}}{\alpha}\right].$$

Thus, in this case, we find the fact that the sample minimum has an exponential distribution with parameter $n\lambda$.

For the case of the sample maximum, $i=n$, we get

$$S_\alpha(X_{n:n}) = \frac{1}{\alpha-1}\left[1-n^{\alpha}\lambda^{\alpha-1}B(\alpha(n-1)+1,\alpha)\right].$$

The Tsallis entropy of order statistics 2.3 for $i=1$ and $i=n$, that are, the lifetimes of the series and the paraller systems, respectively, for several well-known distributions are provided in Table 1.

2.1. Characterizations based on order statistics We know that the m^{th} order statistics in a sample of size n represents the life length of a $(n-m+1)$-out-of-n system. In this subsection, we show that the parent distribution can be characterized by Tsallis entropy of $X_{m:n}$ and $X_{n:m+n-1}$. First, we recall the following lemma, due to Aliprantis and Burkinshaw (1981).

LEMMA 2. *If η is a continuous function on $[0,1]$ such that $\int_0^1 x^n\eta(x)dx = 0$, for $n \geq 0$, then $\eta(x) = 0$ for all $x \in [0,1]$.*

Now, we use Lemma 2 in the proof of the following characterization theorem.

Table 1. Tsallis entropy of first and last order statistics for some common distributions

Density function	$S_\alpha(X_{1:n})$	$S_\alpha(X_{n:n})$
Uniform distribution $f(x)=\frac{1}{b-a},\quad a<x<b$	$\frac{1}{\alpha-1}[1-(\frac{n}{b-a})^\alpha \frac{b-a}{\alpha(n-1)+1}]$	$\frac{1}{\alpha-1}[1-(\frac{n}{b-a})^\alpha \frac{b-a}{\alpha(n-1)+1}]$
Beta distribution $f(x)=\frac{x^{\theta-1}}{B(\theta,1)},\quad 0<x<1$	$\frac{1}{\alpha-1}[1-n^\alpha\theta^{\alpha-1}B(\frac{(\alpha-1)(\theta-1)}{\theta},\alpha(n-1)+1)]$	$\frac{1}{\alpha-1}[1-n^\alpha\theta^{\alpha-1}B(\alpha n+\frac{1-\alpha}{\theta},1)]$
Exponential distribution $f(x)=\theta e^{-\theta x},\quad \theta>0$	$\frac{1}{\alpha-1}-\frac{(n\lambda)^{\alpha-1}}{\alpha(\alpha-1)}$	$\frac{1}{\alpha-1}[1-n^\alpha\lambda^{\alpha-1}B(\alpha(n-1)+1,\alpha)]$
Pareto distribution $f(x)=\lambda\beta^\lambda x^{-\lambda+1}$ $x\geq\beta>0,\quad \lambda>0$	$\frac{1}{\alpha-1}[1-\frac{(n\lambda)^\alpha}{\beta^{\alpha-1}(n\alpha\lambda+\alpha-1)}]$	$\frac{1}{\alpha-1}[1-\frac{(n\lambda)^\alpha}{\lambda\beta^{\alpha-1}}B(\alpha(n-1)+1,\alpha+\frac{\alpha-1}{\lambda})]$
Finite Range distribution $f(x)=\frac{a}{b}(1-\frac{x}{b})^{a-1}$ $a>1,\quad 0\leq x\leq b$	$\frac{1}{\alpha-1}[1-\frac{(na)^\alpha}{(\alpha na+1-\alpha)b^{\alpha-1}}]$	$\frac{1}{\alpha-1}[1-\frac{(na)^\alpha}{ab^{\alpha-1}}B(\alpha(n-1)+1,\alpha+\frac{1-\alpha}{a})]$

Theorem 1. *Let X and Y be two random variable with absolutely continuous cdfs $F(x)$ and $G(y)$ and pdfs $f(x)$ and $g(y)$, respectively, then for a fixed m $(1\leq m\leq n)$ and for a change location c,*

$$X\stackrel{d}{=}Y+c\Longleftrightarrow S_\alpha(X_{m:n})=S_\alpha(Y_{m:n}),\ \forall n\geq 1,$$

where $\stackrel{d}{=}$ stands for equality in distribution.

Proof. The necessity is clear, hence it remains to prove the sufficiency part only.
By 2.1 and 2.2, we have

$$S_\alpha(X_{m:n})=\frac{1}{\alpha-1}\Bigg[1-\int_{-\infty}^{+\infty}\frac{1}{B^\alpha(m,n-m+1)}F^{\alpha(m-1)}(x)\\ \bar{F}^{\alpha(n-m)}(x)f^\alpha(x)dx\Bigg].$$

Now, let $S_\alpha(X_{m:n})=S_\alpha(Y_{m:n})$, for all $n\geq m$. Substituting $u=F(x)$ on the above equation and also using $u=G(y)$ for $S_\alpha(Y_{m:n})$ and taking $n-m=j$, we find

$$\int_0^1 u^{\alpha(m-1)}(1-u)^{\alpha j}\left(f^\alpha(F^{-1}(u))-g^\alpha(G^{-1}(u))\right)du=0,$$

for all $j\geq 0$. Using the transformation $v=(1-u)^\alpha$, we get

$$\int_0^1\frac{(1-v^{\frac{1}{\alpha}})^{\alpha(m-1)}}{\alpha v^{\alpha-\frac{1}{\alpha}}}(f^\alpha(F^{-1}(1-v^{\frac{1}{\alpha}}))-g^\alpha(G^{-1}(1-v^{\frac{1}{\alpha}})))v^j dv=0,$$

for all $j\geq 0$. So, from Lemma 2, we can conclude that $f(F^{-1}(1-v^{\frac{1}{\alpha}}))=g(G^{-1}(1-v^{\frac{1}{\alpha}}))$ for all $v\in(0,1)$. By taking $1-v^{\frac{1}{\alpha}}=u$, we have $f(F^{-1}(u))=g(G^{-1}(u))$ for all $u\in(0,1)$. Thus, $\frac{d}{du}F^{-1}(u)=\frac{d}{du}G^{-1}(u)$ for all $u\in(0,1)$. It then follows that $F^{-1}(u)=G^{-1}(u)+c$ for all $u\in(0,1)$, where c is a constant. This means $F(x)$ and $G(y)$ belong to the same family of distribution, but for a location shift. Thus, the proof is completed. Thapliyal et al. (2015) proved the above theorem by the hazard rate function.
By taking $m=1$ in the Theorem 1, we have the following corollary, that characterizes the lifetime of the series system.

Corollary 1. *Under the assumption of Theorem 1, we have*

$$X\stackrel{d}{=}Y+c\Longleftrightarrow S_\alpha(X_{1:n})=S_\alpha(Y_{1:n}),\quad \forall n\geq 1.$$

THEOREM 2. *Under the assumption of Theorem 1, for a fixed m and for a change location c, we have*

$$X \overset{d}{=} Y + c \Longleftrightarrow S_\alpha(X_{n:m+n-1}) = S_\alpha(Y_{n:m+n-1}), \quad \forall n \geq 1.$$

PROOF. The necessity is clear, hence it remains to prove the sufficiency part. By using 2.1 and 2.2, we have

$$S_\alpha(X_{n:m+n-1}) = \frac{1}{\alpha-1}\left[1 - \int_{-\infty}^{+\infty} \frac{1}{B^\alpha(n,m)} F^{\alpha(n-1)}(x) \bar{F}^{\alpha(m-1)}(x) f^\alpha(x) dx\right]. \tag{2.5}$$

Now, let $S_\alpha(X_{n:m+n-1}) = S_\alpha(Y_{n:m+n-1})$, for all $n \geq 1$ and fixed m. By using $u = F(x)$ in 2.5 and $u = G(y)$ for $S_\alpha(Y_{n:m+n-1})$, we have

$$\int_0^1 u^{\alpha(n-1)}(1-u)^{\alpha(m-1)}\left(f^\alpha(F^{-1}(u)) - g^\alpha(G^{-1}(u))\right) du = 0,$$

for all $n \geq 1$. Substituting $v = (1-u)^\alpha$, we get

$$\int_0^1 \frac{(1-v^{\frac{1}{\alpha}})^{\alpha(n-1)}}{\alpha v^{\alpha - \frac{1}{\alpha}}} (f^\alpha(F^{-1}(1-v^{\frac{1}{\alpha}})) - g^\alpha(G^{-1}(1-v^{\frac{1}{\alpha}})))v^{m-1} dv = 0,$$

for all $n \geq 1$. So, from Lemma 2, we can conclude that $f(F^{-1}(1-v^{\frac{1}{\alpha}})) = g(G^{-1}(1-v^{\frac{1}{\alpha}}))$ for all $v \in (0,1)$. The rest of the proof is similar to the proof of Theorem 1. Thus, the proof is complete. By taking $m = 1$ in the Theorem 2, we have the following corollary, that characterizes the lifetime of a parallel system.

COROLLARY 2. *Under the assumptions of Theorem 2, we have*

$$X \overset{d}{=} Y + c \Longleftrightarrow S_\alpha(X_{n:n}) = S_\alpha(Y_{n:n}), \quad \forall n \geq 1.$$

The previous results can be used in the modeling problems and testing statistical hypotheses. For example, testing $H_0 : F(x) = G(y) + c$ against all alternatives is equivalent to testing

$$H_0 : S_\alpha(X_{1:n}) = S_\alpha(Y_{1:n})[S_\alpha(X_{n:n}) = S_\alpha(Y_{n:n})], \quad \forall n \geq 1.$$

Baratpour et al. (2007) obtained similar properties based on Shannon entropy of order statistics.

2.2. Characterizations of symmetric distributions based on order statistics We know that the class of symmetric distributions is broad and includes several well-know distributions. In this subsection, we show that symmetric distributions can be characterized by Tsallis entropy of $X_{i:n}$. We use the following lemma in the proof of the Theorem 3.

LEMMA 3. *Let X be a continuous random variable with cdf $F(x)$ and pdf $f(x)$ with support S_X. Then, the identity*

$$f(F^{-1}(u)) = f(F^{-1}(1-u)), \quad \textit{for almost all} \quad u \in (0,1), \tag{2.6}$$

implies that there exists a constant c such that $F(c-x) = 1 - F(c+x)$ for all $x \in S_X$.

PROOF. See Fashandi and Ahmadi (2012).

Theorem 3. *Let $X_1, X_2, \cdots, X_n$ be a random sample with size n from continuous cdf $F(x)$ and pdf $f(x)$. Then, $F(x)$ is symmetric if and only if for a fixed $m \geq 1$, $S_\alpha(X_{m:n}) = S_\alpha(X_{n-m+1:n})$, for all $n \geq m$.*

Proof. The necessity is clear, because if $F(x)$ is a symmetric distribution function about μ (without loss of generality take $\mu = 0$), then $X_{m:n} \stackrel{d}{=} -X_{n-m+1:n}$. Therefore, the Tsallis entropy of this two statistics are equal.
Now, for proving the sufficiency part, let $S_\alpha(X_{m:n}) = S_\alpha(X_{n-m+1:n})$. By using 2.3, we have

$$\frac{1}{\alpha-1}\left[1 - C_m E_{g_m}[f^{\alpha-1}(F^{-1}(Z_m))]\right] - \frac{1}{\alpha-1}\left[1 - C_{n-m+1} E_{g_{n-m+1}}[f^{\alpha-1}(F^{-1}(Z_{n-m+1}))]\right] = 0, \tag{2.7}$$

where for $i = m$, we said that $C_m = \frac{B(\alpha(m-1)+1, \alpha(n-m)+1)}{B^\alpha(m, n-m+1)}$, and Z_m has the beta distribution with parameters $(\alpha(m-1)+1)$ and $(\alpha(n-m)+1)$. By using 2.7 and noting that $C_m = C_{n-m+1}$ and $Z_{n-m+1} \stackrel{d}{=} 1 - Z_m$, we can conclude that

$$E_{g_{n-m+1}}\left[f^{\alpha-1}(F^{-1}(1-Z_m))\right] - E_{g_m}\left[f^{\alpha-1}(F^{-1}(Z_m))\right] = 0.$$

Thus, we find

$$\int_0^1 (1-z)^{\alpha(m-1)}\left[f^{\alpha-1}(F^{-1}(1-z)) - f^{\alpha-1}(F^{-1}(z))\right] z^{\alpha(n-m)} dz = 0. \tag{2.8}$$

Now, by taking $z^\alpha = u$ and $n - m = k$ in 2.8, we get

$$\int_0^1 (1-u^{\frac{1}{\alpha}})^{\alpha(m-1)} u^{-\frac{k}{\alpha}}\left[f^{\alpha-1}(F^{-1}(1-u^{\frac{1}{\alpha}})) - f^{\alpha-1}(F^{-1}(u^{\frac{1}{\alpha}}))\right] u^k du = 0, \tag{2.9}$$

for all $k \geq 0$.
Using Lemma 2, we conclude that

$$f(F^{-1}(1-u^{\frac{1}{\alpha}})) = f(F^{-1}(u^{\frac{1}{\alpha}})), \tag{2.10}$$

for all $u \in (0,1)$. By taking $v = 1 - u^{\frac{1}{\alpha}}$ in 2.10, we have $f(F^{-1}(v)) = f(F^{-1}(1-v))$, for all $v \in (0,1)$. Then, by using Lemma 3, this means $F(x)$ is symmetric. Thus, the proof is completed. Fashandi and Ahmadi (2012) obtained similar properties based on Renyi entropy of order statistics.

3. Tsallis entropy of record values Tsllis entropy of the j^{th} upper record value, U_j, is given by

$$S_\alpha(U_j) = \frac{1}{\alpha-1}\left[1 - \int_{-\infty}^{+\infty} f_{U_j}^\alpha(x)dx\right], \tag{3.1}$$

where $\alpha \neq 1$, $\alpha > 0$ and $f_{U_j}(x)$ is the pdf of j^{th} record value, for $j = 1, 2, \cdots, n$, that given by

$$f_{U_j}(x) = \frac{\{-\log(\bar{F}(x))\}^{j-1}}{\Gamma(j)} f(x), \ -\infty < x < +\infty, \tag{3.2}$$

where $\Gamma(j)$, the complete gamma function, is defined as

$$\Gamma(j) = \int_0^\infty t^{\alpha-1} e^{-t} dt, \quad x > 0, \ \alpha > 0.$$

Note that for $j = 1$, (3.1) reduces to (1.2).
Next, we prove the following lemma.

LEMMA 4. *Let $X_1, X_2, \cdots, X_n$ be a random sample with size n from continuous cdf $F(x)$ and pdf $f(x)$. Let U_j denotes the j^{th} upper record. Then, the Tsallis entropy of U_j can be expressed as*

$$S_\alpha(U_j) = \frac{1}{\alpha-1}\left[1 - \frac{\Gamma(\alpha(j-1)+1)}{\Gamma^\alpha(j)} E\left[f^{\alpha-1}(F^{-1}(1-e^{-V_j}))\right]\right], \tag{3.3}$$

where V_j has gamma distribution with parameters $\alpha(j-1)+1$ and 1.

PROOF. By (3.1) and (3.2), and by substituting $u = -\log(\bar{F}(x))$, we have

$$S_\alpha(U_j) = \frac{1}{\alpha-1}\left[1 - \frac{1}{\Gamma^\alpha(j)} \int_0^\infty u^{\alpha(j-1)} e^{-u} f^{\alpha-1}(F^{-1}(1-e^{-u}))du\right],$$

where $F^{-1}(x)$ is the inverse function of $F(x)$. It can be rewritten as

$$S_\alpha(U_j) = \frac{1}{\alpha-1}\left[1 - \frac{\Gamma(\alpha(j-1)+1)}{\Gamma^\alpha(j)} E\left[f^{\alpha-1}(F^{-1}(1-e^{-V_j}))\right]\right], \tag{3.4}$$

where $\alpha \neq 1$, $\alpha > 0$. Thus, the proof is complete.

LEMMA 5. *Under the assumptions of Lemma 4, if L_j denotes the j^{th} lower record, then the Tsallis entropy of L_j can be expressed as*

$$S_\alpha(L_j) = \frac{1}{\alpha-1}\left[1 - \frac{\Gamma(\alpha(j-1)+1)}{\Gamma^\alpha(j)} E\left[f^{\alpha-1}(F^{-1}(e^{-V_j}))\right]\right],$$

where V_j is as Lemma 4.

PROOF. The proof is similar to that of Lemma 4.

EXAMPLE 2. Suppose that X is a random variable having the exponential distribution with mean $\frac{1}{\lambda}$. Here, $f^{\alpha-1}(F^{-1}(1-e^{-t})) = \lambda^{\alpha-1}e^{-t(\alpha-1)}$. Then, we have

$$E\left[f^{\alpha-1}(F^{-1}(1-e^{-V_j}))\right] = \frac{\lambda^{\alpha-1}}{\alpha^{\alpha(j-1)+1}}.$$

By Lemma 4, we find

$$S_\alpha(U_j) = \frac{1}{\alpha-1}\left[1 - \frac{\lambda^{\alpha-1}\Gamma(\alpha(j-1)+1)}{\Gamma^\alpha(j)\alpha^{\alpha(j-1)+1}}\right]. \tag{3.5}$$

EXAMPLE 3. Suppose that X is a random variable having the weibull distribution with pdf $f(x) = abx^{b-1}\exp\{-ax^b\}$, $a, b > 0$, $x > 0$. Here, $f^{\alpha-1}(F^{-1}(1-e^{-t})) = (ba^{\frac{1}{b}})^{\alpha-1}t^{\frac{(\alpha-1)(b-1)}{b}}e^{-t(\alpha-1)}$. Then, we have

$$E\left[f^{\alpha-1}(F^{-1}(1-e^{-V_j}))\right] = \left[\frac{(ba^{\frac{1}{b}})^{\alpha-1}\Gamma(\alpha j - \frac{\alpha+1}{b})}{\Gamma(\alpha(j-1)+1)\alpha^{(\alpha j - \frac{\alpha+1}{b})}}\right].$$

From Lemma 4, we obtain

$$S_\alpha(U_j) = \frac{1}{\alpha-1}\left[1 - \frac{(ba^{\frac{1}{b}})^{\alpha-1}\Gamma(\alpha j - \frac{\alpha+1}{b})}{\Gamma^\alpha(j)\alpha^{(\alpha j - \frac{\alpha+1}{b})}}\right]. \tag{3.6}$$

REMARK 1. For $b = 1$, (3.6) reduces to (3.5). and by taking $b = 2$ in (3.6), we have

$$S_\alpha(U_j) = \frac{1}{\alpha-1}\left[1 - \frac{(2\sqrt{a})^{\alpha-1}\Gamma(\alpha j - \frac{\alpha+1}{2})}{\Gamma^\alpha(j)\alpha^{(\alpha j - \frac{\alpha+1}{2})}}\right],$$

that is the Tsallis entropy of the j^{th} record value from a Rayleigh distribution with parameter $a > 0$.

3.1. Characterizations based on record values In this subsection, we show that the parent distributions can be uniquely specified up to a location change by the equality of Tsallis entropy of record values. First, we recall the following lemma due to Goffman and Pedrick (1965).

Lemma 6. *A complete orthonormal system for the space $L_2(0,\infty)$ is given by the sequence of Laguerre functions*

$$\Phi_n(x) = \frac{1}{n!} e^{-\frac{x}{2}} L_n(x), \quad n \geq 0,$$

where $L_n(x)$ is the Laguerre polynomial, defined as the sum of coefficients of e^{-x} in the nth derivative of $x^n e^{-x}$, that is

$$L_n(x) = e^x \frac{d^n}{dx^n}(x^n e^{-x}) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} n(n-1)\cdots(k+1)x^k.$$

The meaning of the completeness of Laguerre functions in $L_2(0,\infty)$ is that if $f \in L_2(0,\infty)$ and

$$\int_0^\infty f(x) e^{-\frac{x}{2}} L_n(x) dx = 0, \quad \forall n \geq 0,$$

then f is zero a.e. .

Now, we use the this lemma in the proof of the following characterization theorem.

Theorem 4. *Let X and Y be two random variables with absolutely cdfs $F(x)$ and $G(y)$ and pdfs $f(x)$ and $g(y)$, respectively. Let $E(f^2(X)) < \infty$ and $E(g^2(X)) < \infty$, then for a change location c,*

$$X \stackrel{d}{=} Y + c \Longleftrightarrow S_\alpha(U_j^X) = S_\alpha(U_j^Y), \quad \forall n \geq 1,$$

where U_j^X and U_j^Y are the j^{th} upper records of X and Y, respectively.

Proof. The necessity is clear, hence it remains to prove the sufficiency part only. By (3.1) and (3.2) and substituting $u = (-\log \bar{F}(x))^\alpha$, we have

$$S_\alpha(U_j^X) = \frac{1}{\alpha-1}\left[1 - \frac{1}{\alpha\Gamma^\alpha(j)} \int_0^\infty u^{\frac{1}{\alpha}-1} u^{j-1} e^{-u^{\frac{1}{\alpha}}} \left[f^{\alpha-1}(F^{-1}(1-e^{-u^{\frac{1}{\alpha}}}))\right] du\right].$$

Similarly using $u = (-\log \bar{G}(y))^\alpha$, we find

$$S_\alpha(U_j^Y) = \frac{1}{\alpha-1}\left[1 - \frac{1}{\alpha\Gamma^\alpha(j)} \int_0^\infty u^{\frac{1}{\alpha}-1} u^{j-1} e^{-u^{\frac{1}{\alpha}}} \left[g^{\alpha-1}(G^{-1}(1-e^{-u^{\frac{1}{\alpha}}}))\right] du\right].$$

If for two cdfs $F(x)$ and $G(y)$, these measures coincide, we conclude that

$$\int_0^\infty e^{-u^{\frac{1}{\alpha}}} u^{\frac{1}{\alpha}-1} \left[g^{\alpha-1}(G^{-1}(1-e^{-u^{\frac{1}{\alpha}}})) - f^{\alpha-1}(F^{-1}(1-e^{-u^{\frac{1}{\alpha}}}))\right] u^{j-1} du = 0, \tag{3.7}$$

for all $j \geq 1$. Thus, by (3.7), we obtain

$$\int_0^\infty e^{\frac{u}{2}-u^{\frac{1}{\alpha}}} u^{\frac{1}{\alpha}-1} \left[g^{\alpha-1}(G^{-1}(1-e^{-u^{\frac{1}{\alpha}}})) - f^{\alpha-1}(F^{-1}(1-e^{-u^{\frac{1}{\alpha}}}))\right] e^{-\frac{u}{2}} L_j(u) du = 0, \tag{3.8}$$

for all $j \geq 1$, where $L_j(u)$ is Laguerre polynomial given in Lemma 6. Using Lemma 6, in (3.8) and after some simplifications, we can conclude that

$$f\left(F^{-1}(1-e^{-u^{\frac{1}{\alpha}}})\right) = g\left(G^{-1}(1-e^{u^{\frac{1}{\alpha}}})\right), \quad \forall u \in (0,1).$$

By taking $1 - e^{-u^{\frac{1}{\alpha}}} = v$, we have $f(F^{-1}(v)) = g(G^{-1}(v))$ for all $v \in (0,1)$. The rest of the proof is similar to the proof of Theorem 1. Thus, the proof is complete. We have the following remark for lower records

Remark 2. Under the assumptions of Theorem 4, we have

$$X \overset{d}{=} Y + c \Longleftrightarrow S_\alpha(L_j^X) = S_\alpha(L_j^Y), \quad \forall n \geq 1,$$

where L_j^X and L_j^Y are the j^{th} lower records of X and Y, respectively.

Remark 3. By letting $\alpha \to 1$, the result of this subsection are seen to hold for Shannon entropy, as it was shown directly by Baratpour et al. (2007).

3.2. Characterization of symmetric distributions based on k-records Suppose that $\{X_i\}_{i\geq 1}$, be a sequence of iid continuous random variables from the cdf $F(x)$ and pdf $f(x)$ with support S_X. An upper k-record process is defined in terms of the k^{th} largest observation in a partial sample. Here, for the continuous case, we consider the formal definition given in Arnord et al. (1998): for a fixed k, the sequence of upper k-record values are defined as $R_{j,k}^U = X_{T_{j,k}^U - k+1:T_{j,k}}$, $j \geq 1$, where $T_{1,k}^U = k$ and for $j \geq 2$, $T_{j,k}^U = \min\{j : j > T_{j-1,k}^U, X_j > X_{T_{j-1,k}^U - k+1:T_{j-1,k}^U}\}$. It is clear that the first k-records is the first smallest observation in a finit sequence $X_1, X_2, \cdots, X_k$, i.e. $R_{1,k}^U = X_{1:k}$. In the literature, $\{T_{j,k}^U, j \geq k\}$ is said to be the k-record times sequence. In reliability theory the j^{th} upper k-record value can be regarded as the life length of a k-out-of-$T_{j,k}^U$ system. An analogous definition can be given for lower k-record values. The sequence of lower k-record values are defined as $R_{j,k}^L$. We recall that $R_{j,k}^U$ is identical in distribution with the j^{th} usual upper record ($k = 1$) from cdf $G(x) = 1 - (1 - F(x))^k$. Hence, the marginal pdf of $R_{j,k}^U$ is given by

$$f_{R_{j,k}^U}(x) = \frac{k^j}{\Gamma(j)} \big[-\log(\bar{F}(x)) \big]^{j-1} (\bar{F}(x))^{k-1} f(x), \quad x \in S_X. \tag{3.9}$$

Replacing $F(x)$ by $\bar{F}(x)$ in (3.9) the pdf of $R_{j,k}^L$ is deduced. In this subsection we show that symmetric distributions can be characterized by Tsallis entropy of k-record values. We use the following theorems and the Lemma 3, in the proof of the characterization Theorem 7.

Theorem 5 **(Higgins, 2004, pp. 95-96)**. *The set $\{x^{\lambda_1}, x^{\lambda_2}, \cdots : 1 \leq \lambda_1 < \lambda_2 < \cdots\}$ forms a complete sequence in $L^2(0,1)$ if and only if*

$$\sum_{j=1}^{+\infty} \lambda_j^{-1} = +\infty, \quad \textit{where } 1 \leq \lambda_1 < \lambda_2 < \cdots. \tag{3.10}$$

This theorem is well-known as the Muntz-Szasz Theorem.

Theorem 6 **(Hwang and Lin, 1984)**. *Let $f(x)$ be an absolutely continuous function on (a,b) with $f(a)f(b) \geq 0$, and let its derivative satisfies $f'(x) \neq 0$ a.e. on (a,b). Then, under the assumption (3.10), the sequence $\{f^{\lambda_j}(x), j \geq 1\}$ is complete on (a,b) if and only if the function $f(x)$ is monotone on (a,b).*

Theorem 7. *Let $\{X_i\}_{i\geq 1}$ be a sequence of iid continuous random variables from cdf $F(x)$ and pdf $f(x)$. Then, $F(x)$ is symmetric if and only if for a fixed k, $S_\alpha(R_{\lambda_j,k}^U) = S_\alpha(R_{\lambda_j,k}^L)$, such that*

$$\sum_{j=1}^{+\infty} \lambda_j^{-1} = +\infty,$$

where $1 \leq \lambda_1 < \lambda_2 < \cdots$.

Proof. The necessity is trivial. For proving sufficiency part, by (3.1) and (3.2), and substituting $u=-\log\bar{F}(x)$, we can conclude that

$$S_\alpha(R^U_{n,k})=\frac{1}{\alpha-1}\left[1-\frac{k^{\alpha n}\Gamma(\alpha(n-1)+1)}{\Gamma^\alpha(n)(\alpha(k-1)+1)^{\alpha(n-1)+1}}E\big[f^{\alpha-1}(F^{-1}(1-e^{-V_n}))\big]\right], \tag{3.11}$$

that is the Tsallis entropy of the n^{th} upper k-record, and V_n has gamma distribution with parameters $\alpha(n-1)+1$ and $\alpha(k-1)+1$. Now, let $S_\alpha(R^U_{n,k})=S_\alpha(R^L_{n,k})$, for a fixed k. By (3.11), we have

$$E\big[f^{\alpha-1}(F^{-1}(e^{-V_n}))\big]-E\big[f^{\alpha-1}(F^{-1}(1-e^{-V_n}))\big]=0. \tag{3.12}$$

So, by (3.12), we find

$$\int_0^1 u^{\alpha(k-1)}(-\log u)^{-\alpha}\left[f^{\alpha-1}(F^{-1}(u))-f^{\alpha-1}(F^{-1}(1-u))\right]\big[(-\log u)^\alpha\big]^n du=0. \tag{3.13}$$

If (3.13) holds for $n=\lambda_j$, such that $\sum_{j=1}^{+\infty}\lambda_j^{-1}=+\infty$ where $1\le\lambda_1<\lambda_2<\cdots$, then by Theorem 6, we can conclude that

$$f(F^{-1}(u))=f(F^{-1}(1-u)),$$

for almost all $u\in(0,1)$. Thus, by using Lemma 3, the proof is completed.

Remark 4. By taking $k=1$ in Theorem 7, similar result given in Theorem 7, holds for usual records.

4. Tsallis information of order statistics and record values

Some goodness-of-fit tests provided based on entropy and information measures. Vasicek (1975) used the sample Shannon entropy estimation to test normality. Ebrahimi et al. (1992) introduced a test for exponentiality based on Kullback-Leibler information. Ebrahimi et al. (2004) studied Kullback-Leibler information measure and its properies for order statistics using Shannon entropy for order statistics. Park (2005) introduced a goodnees-of-fit test for exponentiality based on Kullback-Leibler information of order statistics. Habibi et al. (2007) presented a goodness-of-fit test for exponentiality based on Kullback-Leibler information of records. Abbasnejad and Arghami (2011) proposed Renyi information for order statistics and record values based on Renyi entropy. In this section, we will study the Tsallis information for order statistics and record values.

Tsallis (1998) introduced a generalization of the Kullback-Leibler entropy in the framework of the non-extensive thermodynamics. The Tsallis information between density function $f(x)$ and $g(y)$ is given as

$$T_\alpha(f,g)=\frac{1}{\alpha-1}\left[\int_{-\infty}^{+\infty}\Big(\frac{f(x)}{g(x)}\Big)^{\alpha-1}f(x)dx-1\right]. \tag{4.1}$$

In the case of $\alpha\to1$, the Tsallis information becomes the Kullback-Leibler information between $f(x)$ and $g(y)$ that is defined by

$$D(f,g)=\int_{-\infty}^{+\infty}f(x)\log\frac{f(x)}{g(x)}dx,$$

(see Kullback and Leibler, 1951, for more details).

Lemma 7. *The Tsallis information between the distribution of i^{th} order statistic $f_{i:n}$ and the original distribution f is given by*

$$T_\alpha(f_{i:n},f)=-S_\alpha(W_i),$$

where W_i has beta distribution with parameters i and $n-i+1$.

PROOF. By (4.1), we have

$$T_\alpha(f_{i:n}, f) = \frac{1}{\alpha-1}\left[\int_{-\infty}^{+\infty} (\frac{f_{i:n}(x)}{f(x)})^{\alpha-1} f_{i:n}(x)dx - 1\right].$$

Substituting $u = F(x)$, we can conclude that

$$\begin{aligned} T_\alpha(f_{i:n}, f) &= \frac{1}{\alpha-1}\left[\int_0^1 \left(\frac{\Gamma(n+1)}{\Gamma(i)\Gamma(n-i+1)} u^{i-1}(1-u)^{n-i}\right)^\alpha du - 1\right] \\ &= -S_\alpha(W_i). \end{aligned}$$

Hence, the Tsallis information between the distribution of order statistics and the original distribution is distribution free.

LEMMA 8. *The Tsallis information between the distribution of the j^{th} upper record f_{U_j} and the original distribution f is given by*

$$T_\alpha(f_{U_j}, f) = -D_j S_\alpha(H_j),$$

where $D_j = \alpha^{\alpha(j-1)+1}$ and H_j has gamma distribution with parameters j and 1.

PROOF. From (4.1), we have

$$T_\alpha(f_{U_j}, f) = \frac{1}{\alpha-1}\left[\int_{-\infty}^{\infty} (\frac{f_{U_j}(x)}{f(x)})^{\alpha-1} f_{U_j}(x)du - 1\right].$$

Substituting $u = -\log(\bar{F}(x))$, we conclude that

$$T_\alpha(f_{U_j}, f) = \frac{1}{\alpha-1}\left[\int_0^\infty \frac{u^{\alpha(j-1)}}{\Gamma^\alpha(j)} e^{-u} du - 1\right].$$

Now, by taking $h = \frac{u}{\alpha}$, we can write

$$\begin{aligned} T_\alpha(f_{U_j}, f) &= \frac{1}{\alpha-1}\left[\int_0^\infty \alpha(\frac{(\alpha h)^{j-1}e^{-h}}{\Gamma(j)})^\alpha dh - 1\right] \\ &= \frac{\alpha^{\alpha(j-1)+1}}{\alpha-1}\left[\int_0^\infty (\frac{h^{j-1}e^{-h}}{\Gamma(j)})^\alpha dh - 1\right] \\ &= -\alpha^{\alpha(j-1)+1} S_\alpha(H_j). \end{aligned}$$

Thus, the proof is completed. Hence, the Tsallis information between the distribution of the upper record values and the original distribution is destribution free.

REMARK 5. Similar result as stated in Lemma 8, holds for lower record values.

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