İSTATİSTİK: JOURNAL OF THE TURKISH STATISTICAL ASSOCIATION İSTATİSTİK
Vol. 9, No. 2, July 2016, pp. 56–66
ISSN 1300-4077 | 16 | 2 | 56 | 66

THE PREDICTION OF THE TWO PARAMETER RIDGE ESTIMATOR

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Abstract: The prediction of a regression model can be adversely affected by multicollinearity. Although biased estimation procedures have been proposed as an alternative to least squares, there has been little analysis of the predictive performance of the resulting equations. Therefore, we discuss the predictive performance of the Two Parameter Ridge (2PR) estimator compared to ordinary least squares, principal components and ridge regression estimators. Also, the theoretical results are illustrated by a numerical example and a region is established where the 2PR estimator is uniformly superior to the other estimators.

Key words: Biased estimation; Two Parameter Ridge estimator; Multicollinearity; Prediction Mean Square Error
History: Submitted: 16 June 2016; Revised: 20 July 2016; Accepted: 26 July 2016

1. Introduction

The multiple linear regression model is

$$y = X\beta + \epsilon, \tag{1.1}$$

where y is an $n\times 1$ vector of responses, X is an $n\times p$ full column rank matrix of non-stochastic predetermined regressors, β is a $p\times 1$ vector of unknown parameters, and ϵ is an $n\times 1$ vector of i.i.d. $(0,\sigma^2)$ random errors.

The Ordinary Least Squares (OLS) estimator of β in model (1.1) is

$$\widehat{\beta} = (X^{'}X)^{-1}X^{'}y. \tag{1.2}$$

This estimator is widely used technique for estimating the linear regression models. The main reason for focusing on the OLS estimator is because it is unbiased and has the minimum variance among all linear unbiased estimators.

One of the oldest technique used to combat collinearity between regressors is the principal components regression (PCR) (Massy [7]) which is given by

$$\widehat{\beta}_r = U_r(U_r^{'}X^{'}XU_r)^{-1}U_r^{'}X^{'}y = U_r\Lambda^{-1}U_r^{'}X^{'}y, \tag{1.3}$$

where $U = [u_1,\ u_2,\ ...,\ u_p]$ is an orthogonal matrix such that $U^{'}X^{'}XU = \Lambda$ (i.e., U is the $p\times p$ matrix of eigenvectors of $X^{'}X$). $\Lambda = diag\ (\lambda_1,\ \lambda_2,\ ...,\ \lambda_p)$ is the matrix of eigenvalues of $X^{'}X$ and the eigenvalues are in descending order. U_r contains the remaining r eigenvectors of after deleting the last $p-r$ columns of U and $\Lambda_r = diag\ (\lambda_1,\ \lambda_2,\ ...,\ \lambda_r)$.

One of the most popular estimator dealing with multicollinearity is the ordinary ridge regression (ORR) estimator proposed by Hoerl and Kennard [2] and is defined as

$$\widehat{\beta}_k = (X^{'}X + kI)^{-1}X^{'}y, \tag{1.4}$$

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where $k \geq 0$ is the biasing parameter. The purpose of adding the constant k to the diagonal elements of the $X^{'}X$ matrix is the reduction of the sensitivity of the OLS estimator when $X^{'}X$ is not close to the identity matrix.

The two parameter ridge (2PR) estimator of β (Lipovetsky and Conklin [3]) is given by

$$\widehat{\beta}_{2PR} = \widehat{\beta}_{2PR}(q,k) = q(X^{'}X + kI)^{-1}X^{'}y, \qquad k \geq 0 \tag{1.5}$$

where q and k are the biasing parameters. 2PR estimator differs from ORR estimator only in the parameter q. Also, the value of q reaches its maximum when

$$q = \frac{y^{'}X^{'}(X^{'}X + kI)^{-1}Xy}{y^{\prime}X^{\prime}(X^{\prime}X + kI)^{-1}X^{\prime}X(X^{\prime}X + kI)^{-1}Xy}. \tag{1.6}$$

The 2PR estimator is a general estimator which includes the OLS estimator and ORR estimator as special cases:

$$\widehat{\beta}_{2PR}(1,0) = (X^{'}X)^{-1}X^{'}y, \text{ is the OLS estimator}, \tag{1.7}$$

$$\widehat{\beta}_{2PR}(1,k) = (X^{'}X + kI)^{-1}X^{'}y, \text{ is the ORR estimator}. \tag{1.8}$$

Lipovetsky and Conklin [3] found that the 2PR estimator always outperforms the ORR estimator by better approximation and has good properties of orthogonality between residuals and predicted values of the dependent variable. The results that they found are very convenient for the analysis and interpretation of the regression such that the numerical runs proved that this technique works very well. Lipovetsky [4] improved the two parameter model and investigated various characteristics of the 2PR estimator. Then, Toker and Kaçıranlar [9] compared the 2PR estimator to the OLS and the ORR estimators according to the matrix mean square error (MMSE) criterion. Also, other properties of the 2RP estimator have been discussed in the literature (see for instance, Lipovetsky [5] and Li and Yang [6]).

Regression models are widely used in prediction. The predictive performance of a regression model can be adversely affected by multicollinearity. Although biased estimation procedures have been proposed as an alternative to least squares, there has been little analysis of the predictive performance of the resulting equations. Friedman and Montgomery [1] focused on evaluating the predictive performance at a particular observation of the ORR estimator compared to the OLS and the PCR estimators in terms of the prediction mean square error (PMSE) criterion. Then, Özbey and Kaçıranlar [8] adopted a similar approach for comparing Liu estimator with OLS, PCR and ORR estimators.

As a consequence, it appears reasonable to evaluate the predictive performance of the 2PR estimator in which the ORR estimator is a special case of it. In this paper, the predictive performance of the 2PR estimator compared to OLS, PCR and ORR estimators will be discussed according to the PMSE criterion. A numerical example will be given to demonstrate the theoretical results.

2. Evaluations of the prediction Mean Squared Errors

Following Özbey and Kaçıranlar [8], we will recall PMSEs developed by Friedman and Montgomery [1] for OLS, PCR and ORR estimators and then obtain the PMSE of the 2PR estimator.

The PMSE is a measure of the closeness of a predictor to the response being predicted

$$PMSE = E(y_0 - \widehat{y}_0)^2. \tag{2.1}$$

Let J represents the PMSE. J is the sum of the variance (V) and the squared bias(B) :

$$J = V + B. \tag{2.2}$$

If y_0 is the value to be predicted, and $\widehat{y}_0$ is the prediction of that value, then the variance and the bias of the prediction error are

$$V(y_0 - \widehat{y}_0) = V(y_0) + V(\widehat{y}_0) \tag{2.3}$$

and

$$Bias = E(y_0 - \widehat{y}_0). \tag{2.4}$$

For convenience, the orthogonal form of model (1.1)

$$y = Z\alpha + \epsilon \tag{2.5}$$

will be used. Here $Z = XU$ and $\alpha = U^{'}\beta$. Then the OLS estimator of α in model (2.5) is

$$\widehat{\alpha} = (Z^{'}Z)^{-1}Z^{'}y = \Lambda^{-1}Z^{'}y. \tag{2.6}$$

If z_0 is the orthonormalized point at which the prediction y_0 is made, the PMSE of the OLS estimator is

$$J_{OLS} = \sigma^2\left(1 + \sum_{i=1}^{p}\frac{z_{oi}^2}{\lambda_i}\right) \tag{2.7}$$

Note that, since the OLS estimator is unbiased, its PMSE is equal to its prediction variance

$$J_{OLS} = J_{OLS}. \tag{2.8}$$

The PCR estimator of α in model (2.5) is

$$\widehat{\alpha}_r = (Z_r^{'}Z_r)^{-1}Z_r^{'}y = \Lambda_r^{-1}Z_r^{'}y, \tag{2.9}$$

where $Z_r = XU_r$. The PMSE of the PCR estimator is

$$J_r = \sigma^2\left(1 + \sum_{i=1}^{r}\frac{z_{oi}^2}{\lambda_i}\right) + \left(\sum_{i=r+1}^{p} z_{oi}\alpha_i^2\right)^2. \tag{2.10}$$

The ORR estimator of α in model (2.5) is

$$\widehat{\alpha}_k = (Z^{'}Z + kI)^{-1}Z^{'}y = (\Lambda + kI)^{-1}Z^{'}y, \quad k \geq 0 \tag{2.11}$$

and the PMSE of the ORR estimator is

$$J_k = \sigma^2\left(1 + \sum_{i=1}^{p}\frac{z_{oi}^2\lambda_i}{(\lambda_i + k)^2}\right) + k^2\left(\sum_{i=1}^{p}\frac{z_{oi}\alpha_i}{\lambda_i + k}\right)^2 \tag{2.12}$$

(see also, Friedman and Montgomery [1]).

The 2PR estimator of α in model (2.5) is

$$\widehat{\alpha}_{2PR} = q\ (Z^{'}Z + kI)^{-1}Z^{'}y = q\ (\Lambda + kI)^{-1}Z^{'}y, \quad k \geq 0, \quad q > 0. \tag{2.13}$$

The variance of the prediction error of the 2PR estimator is

$$\begin{aligned} V_{2PR}(y_0-\widehat{y}_0) &= V(y_0)+V_{2PR}(\widehat{y}_0) \\ &= \sigma^2+V(z_0^{'}\widehat{\alpha}_{2PR}) \\ &= \sigma^2\left(1+q^2\sum_{i=1}^{p}\frac{z_{oi}^2\lambda_i}{(\lambda_i+k)^2}\right). \end{aligned} \tag{2.14}$$

The bias of the prediction error of the 2PR estimator is

$$\begin{aligned} Bias_{2PR} &= E(y_0-\widehat{y}_0)=z_0^{'}\alpha-z_0^{'}E(\widehat{\alpha}_{2PR}) \\ &= \sum_{i=1}^{p}\frac{z_{oi}\alpha_i[(1-q)\lambda_i+k]}{\lambda_i+k} \end{aligned} \tag{2.15}$$

so, the squared bias is

$$B_{2PR}=Bias_{2PR}^2=\left(\sum_{i=1}^{p}\frac{z_{oi}\alpha_i[(1-q)\lambda_i+k]}{\lambda_i+k}\right)^2. \tag{2.16}$$

By summing up the variance and the squared bias of the 2PR estimator we obtain

$$\begin{aligned} J_{2PR} &= V_{2PR}+B_{2PR} \\ &= \sigma^2\left(1+q^2\sum_{i=1}^{p}\frac{z_{oi}^2\lambda_i}{(\lambda_i+k)^2}\right)+\left(\sum_{i=1}^{p}\frac{z_{oi}\alpha_i[(1-q)\lambda_i+k]}{\lambda_i+k}\right)^2. \end{aligned} \tag{2.17}$$

Therefore, as a special case, if $q=1$ in Equation (2.17), then $J_{2PR}=J_k$.

3. Comparisons of the prediction Mean Squared Errors in two dimensional space

Following Friedman and Montgomery [1] and Özbey and Kaçıranlar [8], we will discuss the predictive performance of the 2PR estimator such that our inferences will be based on the subspace of the observation to be predicted (i.e., the ratio $\frac{z_{02}^2}{z_{01}^2}$) and α_1^2 will be set to zero because non-zero values of α_1^2 increase only the intercept values for J_k and J_{2PR} but leave the curves for J_{OLS} and J_r unchanged. So, comparisons of J_{2PR} with J_{OLS}, J_r and J_k will be made and stated in the following three theorems.

Theorem 1. *a. If* $\alpha_2^2>\frac{\sigma^2\left((\lambda_2+k)^2-q^2\lambda_2^2\right)}{\lambda_2((1-q)\lambda_2+k)^2}$*, then*
- $J_{2PR}<J_{OLS}$*, for* $(\lambda_1+k)^2<q^2\lambda_1^2$*,*
- $J_{2PR}<J_{OLS}\Leftrightarrow\frac{z_{02}^2}{z_{01}^2}<f_1(\alpha_2^2)$*, for* $(\lambda_1+k)^2>q^2\lambda_1^2$*.*

b. If $\alpha_2^2<\frac{\sigma^2\left((\lambda_2+k)^2-q^2\lambda_2^2\right)}{\lambda_2((1-q)\lambda_2+k)^2}$*, then*
- $J_{2PR}<J_{OLS}$*, for* $(\lambda_1+k)^2>q^2\lambda_1^2$*,*
- $J_{2PR}<J_{OLS}\Leftrightarrow\frac{z_{02}^2}{z_{01}^2}<f_1(\alpha_2^2)$*, for* $(\lambda_1+k)^2<q^2\lambda_1^2$*.*

c. As a special case, if $q=1$, we get the results of Friedman and Montgomery [1] which are corrected by Özbey and Kaçıranlar [8].

Where

$$f_1(\alpha_2^2) = \frac{\sigma^2\left(\frac{1}{\lambda_1} - \frac{q^2\lambda_1^2}{(\lambda_1+k)^2}\right)}{\left(\frac{\sigma^2 q^2\lambda_2}{(\lambda_2+k)^2} + \frac{((1-q)\lambda_2+k)^2\alpha_2^2}{(\lambda_2+k)^2} - \frac{\sigma^2}{\lambda_2}\right)}. \tag{3.1}$$

Proof. See Appendix 1.

Theorem 2. *a. If* $\alpha_2^2 < \frac{\sigma^2 q^2\lambda_2}{(\lambda_2+k)^2-((1-q)\lambda_2+k)^2}$*, then*
- $J_{2PR} < J_r$*, for* $(\lambda_1+k)^2 < q^2\lambda_1^2$*,*
- $J_{2PR} < J_r \Leftrightarrow \frac{z_{02}^2}{z_{01}^2} < f_2(\alpha_2^2)$*, for* $(\lambda_1+k)^2 > q^2\lambda_1^2$*.*

b. If $\alpha_2^2 > \frac{\sigma^2 q^2\lambda_2}{(\lambda_2+k)^2-((1-q)\lambda_2+k)^2}$*, then*
- $J_{2PR} < J_r$*, for* $(\lambda_1+k)^2 > q^2\lambda_1^2$*,*
- $J_{2PR} < J_r \Leftrightarrow \frac{z_{02}^2}{z_{01}^2} < f_2(\alpha_2^2)$*, for* $(\lambda_1+k)^2 < q^2\lambda_1^2$*.*

c. As a special case, if $q=1$*, we get the results of Friedman and Montgomery [1].*
Where

$$f_2(\alpha_2^2) = \frac{\sigma^2\left(\frac{1}{\lambda_1} - \frac{q^2\lambda_1}{(\lambda_1+k)^2}\right)}{\left(\frac{\sigma^2 q^2\lambda_2}{(\lambda_2+k)^2} + \frac{((1-q)\lambda_2+k)^2\alpha_2^2}{(\lambda_2+k)^2} - \alpha_2^2\right)}. \tag{3.2}$$

Proof. See Appendix 2.

Theorem 3. *a. If* $\alpha_2^2 > \frac{\sigma^2\lambda_2(1-q^2)}{((1-q)\lambda_2+k)^2-k^2}$*, then*
- $J_{2PR} < J_k$*, for* $q^2 > 1$*,*
- $J_{2PR} < J_k \Leftrightarrow \frac{z_{02}^2}{z_{01}^2} < f_3(\alpha_2^2)$*, for* $q^2 < 1$*.*

b. If $\alpha_2^2 < \frac{\sigma^2\lambda_2(1-q^2)}{((1-q)\lambda_2+k)^2-k^2}$*, then*
- $J_{2PR} < J_k$*, for* $q^2 < 1$*,*
- $J_{2PR} < J_k \Leftrightarrow \frac{z_{02}^2}{z_{01}^2} < f_3(\alpha_2^2)$*, for* $q^2 > 1$*.*

Where

$$f_3(\alpha_2^2) = \frac{\sigma^2\left(\frac{\lambda_1}{(\lambda_1+k)^2} - \frac{q^2\lambda_1}{(\lambda_1+k)^2}\right)}{\left(\frac{\sigma^2 q^2\lambda_2}{(\lambda_2+k)^2} + \frac{((1-q)\lambda_2+k)^2\alpha_2^2}{(\lambda_2+k)^2} - \frac{\sigma^2\lambda_2}{(\lambda_2+k)^2} - \frac{k^2\alpha_2^2}{(\lambda_2+k)^2}\right)}. \tag{3.3}$$

Proof. See Appendix 3.

4. Numerical examples

In this section, we will illustrate our theoretical results using the example given by Friedman and Montgomery [1] (i.e., $\sigma^2=1,\ k=0.1$, and $r_{12}=0.95$). Also, we choose the biasing parameter $q=1.1$ for the above mentioned estimator.

Firstly, let us consider the predictive performances of the 2PR and the OLS estimators. From (3.1), we get

$$f_1(\alpha_2^2) = \frac{0.04863}{17.3111 - 0.40111\alpha_2^2}, \tag{4.1}$$

which is a hyperbola with a vertical asymptote at

$$\alpha_2^2 \cong 43.15789. \tag{4.2}$$

Figure 1 illustrates this situation. For values of α_2^2 larger than 43.15789, the 2PR estimator is uniformly superior to the OLS estimator. For smaller values of α_2^2, there is a trade-off between these two estimators. If the value of the ratio $\frac{z_{02}^2}{z_{01}^2}$ is smaller than the value of $f_1(\alpha_2^2)$, then the 2PR estimator is superior to the OLS estimator, otherwise the OLS estimator is superior to the 2PR estimator.

Secondly, let us consider the predictive performances of the 2PR and the PCR estimators. From (3.2), we get

$$f_2(\alpha_2^2) = \frac{0.04863}{0.59889\alpha_2^2 - 2.6889}, \tag{4.3}$$

which is a hyperbola with a vertical asymptote at

$$\alpha_2^2 \cong 4.489796. \tag{4.4}$$

Figure 2 illustrates this situation. For values of α_2^2 smaller than 4.489796, the 2PR estimator is uniformly superior to the PCR estimator. For larger values of α_2^2 there is a trade-off between these two estimators. If the value of the ratio $\frac{z_{02}^2}{z_{01}^2}$ is smaller than the value of $f_2(\alpha_2^2)$, then the 2PR estimator is superior to the PCR estimator, otherwise the PCR estimator is superior to the 2PR estimator.

Therefore, for the previous parts of this example we get the same results of Friedman and Montgomery [1] if $q = 1$. That means, the ORR estimator is just a special case of the 2PR estimator.

Finally, let us consider the predictive performances of the 2PR and the ORR estimators. From (3.3), we get

$$f_3(\alpha_2^2) = \frac{0.09744}{0.04333\alpha_2^2 - 0.4667}, \tag{4.5}$$

which is a hyperbola with a vertical asymptote at

$$\alpha_2^2 \cong 10.71429. \tag{4.6}$$

Figure 3 illustrates this situation. For values of α_2^2 smaller than 10.71429, the 2PR estimator is uniformly superior to the ORR estimator. For larger values of α_2^2 there is a trade-off between these two estimators. If the value of the ratio $\frac{z_{02}^2}{z_{01}^2}$ is smaller than the value of $f_3(\alpha_2^2)$, then the 2PR estimator is superior to the ORR estimator, otherwise the ORR estimator is superior to the 2PR estimator.

5. Conclusions

We investigate the predictive performance of the 2PR estimator compared to the OLS, the PCR and the ORR estimators. The comparisons of these estimators are in terms of the PMSE criterion at a specific point in two-dimensional regressor variable spaces. In this context, the PMSE of the 2PR estimator is developed and three theorems are given. The theoretical consequences are illustrated by a numerical example, and regions are assigned for the superiority of the given estimators. For some values of α_2^2 , there are trade-offs between the relative effectiveness of the estimators. The OLS estimator is effective only when the value of α_2^2 is small compared to the 2PR estimator. The effectiveness of these techniques is also affected by the location of the prediction point. Hence, the choice of the estimator may depend on the location of the point to be predicted. In the numerical example, a region was established where the 2PR estimator is uniformly superior to the mentioned estimators above. This implies that it is theoretically possible to determine such a region. Finally, we get the theoretical and empirical results of Friedman and Montgomery [1] if $q = 1$, as a special case.

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Appendix 1

If the 2PR estimator is superior to the OLS estimator in terms of the PMSE criterion, we have $J_{2PR} <$

J_{OLS}. That is,

$$\sigma^2 + \sigma^2 q^2 \left[\frac{z_{01}^2 \lambda_1}{(\lambda_1 + k)^2} + \frac{z_{02}^2 \lambda_2}{(\lambda_2 + k)^2} \right] + \frac{((1-q)\lambda_2 + k)^2 z_{02}^2 \alpha_2^2}{(\lambda_2 + k)^2} < \sigma^2 + \sigma^2 \left(\frac{z_{01}^2}{\lambda_1} + \frac{z_{02}^2}{\lambda_2} \right) \tag{5.1}$$

Rearranging this inequality we will obtain

$$z_{02}^2 \left(\frac{\sigma^2 q^2 \lambda_2}{(\lambda_2 + k)^2} + \frac{((1-q)\lambda_2 + k)^2 \alpha_2^2}{(\lambda_2 + k)^2} - \frac{\sigma^2}{\lambda_2} \right) < z_{01}^2 \sigma^2 \left(\frac{1}{\lambda_1} - \frac{q^2 \lambda_1}{(\lambda_1 + k)^2} \right). \tag{5.2}$$

If both

$$\frac{\sigma^2 q^2 \lambda_2}{(\lambda_2 + k)^2} + \frac{((1-q)\lambda_2 + k)^2 \alpha_2^2}{(\lambda_2 + k)^2} - \frac{\sigma^2}{\lambda_2} \tag{5.3}$$

and

$$\sigma^2\left(\frac{1}{\lambda_1}-\frac{q^2\lambda_1}{(\lambda_1+k)^2}\right) \tag{5.4}$$

have the same signs, the condition for the superiority of the 2PR estimator over the OLS estimator is

$$\frac{z_{02}^2}{z_{01}^2}<f_1(\alpha_2^2). \tag{5.5}$$

If (5.3) and (5.4) have opposite signs, the condition for the superiority of the 2PR estimator over the OLS estimator is

$$\frac{z_{02}^2}{z_{01}^2}>f_1(\alpha_2^2). \tag{5.6}$$

It is obvious that if (5.3) and (5.4) have opposite signs, the right hand side of (5.6) is negative, thus (5.6) always holds. Consequently, at that region the 2PR estimator is uniformly superior to the OLS estimator. The condition for the positiveness of (5.3) can be written as

$$\alpha_2^2>\frac{\sigma^2\left((\lambda_2+k)^2-q^2\lambda_2^2\right)}{\lambda_2((1-q)\lambda_2+k)^2} \tag{5.7}$$

and the condition for the positiveness of (5.4) can be written as

$$(\lambda_1+k)^2>q^2\lambda_1^2. \tag{5.8}$$

Of course, the opposite conditions are needed for the negativeness of (5.3) and (5.4). The vertical asymptote of the hyperbola is at the point

$$\alpha_2^2=\frac{\sigma^2\left((\lambda_2+k)^2-q^2\lambda_2^2\right)}{\lambda_2((1-q)\lambda_2+k)^2}. \tag{5.9}$$

Therefore, we get the results of Friedman and Montgomery [1] which are corrected by Özbey and Kaçıranlar [8] if $q=1$.

Appendix 2

If the 2PR estimator is superior to the PCR estimator in terms of the PMSE criterion, we have $J_{2PR}<J_r$. That is,

$$\sigma^2+\sigma^2q^2\left[\frac{z_{01}^2\lambda_1}{(\lambda_1+k)^2}+\frac{z_{02}^2\lambda_2}{(\lambda_2+k)^2}\right]+\frac{((1-q)\lambda_2+k)^2z_{02}^2\alpha_2^2}{(\lambda_2+k)^2}<$$
$$\sigma^2+\frac{\sigma^2z_{01}^2}{\lambda_1}+z_{02}^2\alpha_2^2. \tag{5.10}$$

Rearranging this inequality we will obtain

$$z_{02}^2\left(\frac{\sigma^2q^2\lambda_2}{(\lambda_2+k)^2}+\frac{((1-q)\lambda_2+k)^2\alpha_2^2}{(\lambda_2+k)^2}-\alpha_2^2\right)<$$
$$z_{01}^2\sigma^2\left(\frac{1}{\lambda_1}-\frac{q^2\lambda_1}{(\lambda_1+k)^2}\right). \tag{5.11}$$

If both

$$\frac{\sigma^2q^2\lambda_2}{(\lambda_2+k)^2}+\frac{((1-q)\lambda_2+k)^2\alpha_2^2}{(\lambda_2+k)^2}-\alpha_2^2 \tag{5.12}$$

and

$$\sigma^2\left(\frac{1}{\lambda_1}-\frac{q^2\lambda_1}{(\lambda_1+k)^2}\right) \tag{5.13}$$

have the same signs, the condition for the superiority of the 2PR estimator over the PCR estimator is

$$\frac{z_{02}^2}{z_{01}^2}<f_2(\alpha_2^2). \tag{5.14}$$

If (5.12) and (5.13) have opposite signs, the condition for the superiority of the 2PR estimator over the PCR estimator is

$$\frac{z_{02}^2}{z_{01}^2}>f_2(\alpha_2^2). \tag{5.15}$$

It is obvious that if (5.12) and (5.13) have opposite signs, the right hand side of (5.15) is negative, thus (5.15) always holds. Consequently, at that region the 2PR estimator is uniformly superior to the PCR estimator. The condition for the positiveness of (5.12) can be written as

$$\alpha_2^2<\frac{\sigma^2q^2\lambda_2}{(\lambda_2+k)^2-((1-q)\lambda_2+k)^2} \tag{5.16}$$

and the condition for the positiveness of (5.13) can be written as

$$(\lambda_1+k)^2>q^2\lambda_1^2. \tag{5.17}$$

Of course, the opposite conditions are needed for the negativeness of (5.12) and (5.13). The vertical asymptote of the hyperbola $f_2(\alpha_2^2)$ is at the point

$$\alpha_2^2=\frac{\sigma^2q^2\lambda_2}{(\lambda_2+k)^2-((1-q)\lambda_2+k)^2}. \tag{5.18}$$

Therefore, we get the results of Friedman and Montgomery [1] if $q=1$.

Appendix 3

If the 2PR estimator is superior to the ORR estimator in terms of the PMSE criterion, we have $J_{2PR}<J_k$. That is,

$$\sigma^2+\sigma^2q^2\left[\frac{z_{01}^2\lambda_1}{(\lambda_1+k)^2}+\frac{z_{02}^2\lambda_2}{(\lambda_2+k)^2}\right]+\frac{((1-q)\lambda_2+k)^2z_{02}^2\alpha_2^2}{(\lambda_2+k)^2}<$$
$$\sigma^2+\sigma^2\left[\frac{z_{01}^2\lambda_1}{(\lambda_1+k)^2}+\frac{z_{02}^2\lambda_2}{(\lambda_2+k)^2}\right]+\frac{k^2z_{02}^2\alpha_2^2}{(\lambda_2+k)^2}. \tag{5.19}$$

Rearranging this inequality we will obtain

$$z_{02}^2\left(\frac{\sigma^2q^2\lambda_2}{(\lambda_2+k)^2}+\frac{((1-q)\lambda_2+k)^2\alpha_2^2}{(\lambda_2+k)^2}-\frac{\sigma^2\lambda_2}{(\lambda_2+k)^2}-\frac{k^2\alpha_2^2}{(\lambda_2+k)^2}\right)<$$
$$z_{01}^2\sigma^2\left(\frac{\lambda_1}{(\lambda_1+k)^2}-\frac{q^2\lambda_1}{(\lambda_1+k)^2}\right). \tag{5.20}$$

If both

$$\frac{\sigma^2q^2\lambda_2}{(\lambda_2+k)^2}+\frac{((1-q)\lambda_2+k)^2\alpha_2^2}{(\lambda_2+k)^2}-\frac{\sigma^2\lambda_2}{(\lambda_2+k)^2}-\frac{k^2\alpha_2^2}{(\lambda_2+k)^2} \tag{5.21}$$

and

$$\sigma^2\left(\frac{\lambda_1}{(\lambda_1+k)^2}-\frac{q^2\lambda_1}{(\lambda_1+k)^2}\right) \tag{5.22}$$

have the same signs, the condition for the superiority of the 2PR estimator over the ORR estimator is

$$\frac{z_{02}^2}{z_{01}^2} < f_3(\alpha_2^2). \tag{5.23}$$

If (5.21) and (5.22) have opposite signs, the condition for the superiority of the 2PR estimator over the ORR estimator is

$$\frac{z_{02}^2}{z_{01}^2} > f_3(\alpha_2^2). \tag{5.24}$$

It is obvious that if (5.21) and (5.22) have opposite signs, the right hand side of (5.24) is negative, thus (5.24) always holds. Consequently, at that region the 2PR estimator is uniformly superior to the ORR estimator. The condition for the positiveness of (5.21) can be written as

$$\alpha_2^2 > \frac{\sigma^2\lambda_2(1-q^2)}{((1-q)\lambda_2+k)^2-k^2} \tag{5.25}$$

and the condition for the positiveness of (5.22) can be written as

$$q^2 < 1. \tag{5.26}$$

Of course, the opposite conditions are needed for the negativeness of (5.21) and (5.22). The vertical asymptote of the hyperbola is at the point

$$\alpha_2^2 = \frac{\sigma^2\lambda_2(1-q^2)}{((1-q)\lambda_2+k)^2-k^2}. \tag{5.27}$$

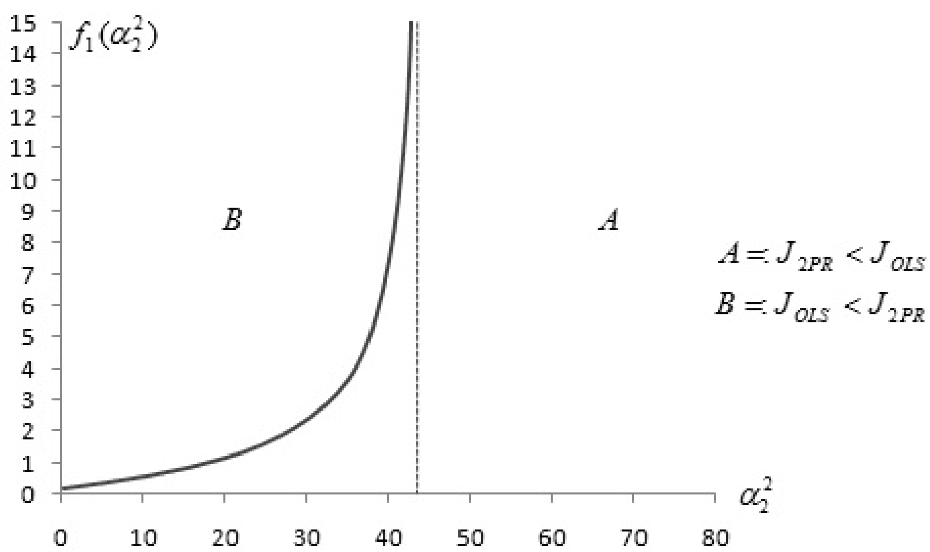

FIGURE 1. The PMSE comparison of the 2PR and the OLS estimators

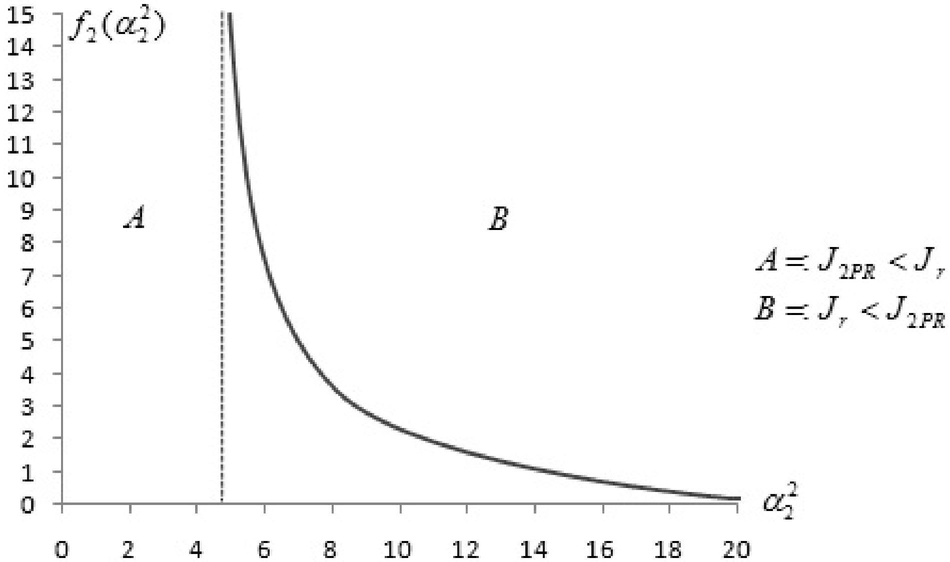

FIGURE 2. The PMSE comparison of the 2PR and the PCR estimators

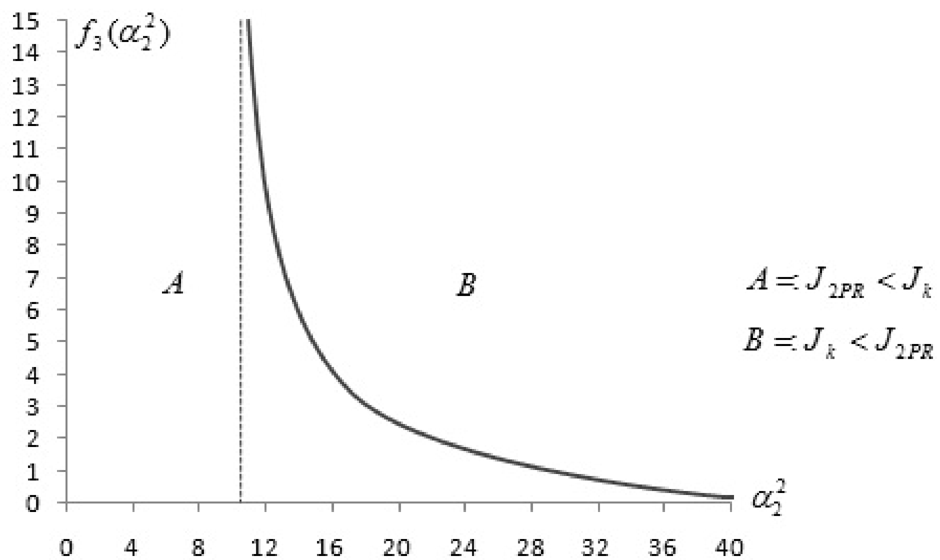

FIGURE 3. The PMSE comparison of the 2PR and the ORR estimators