İSTATİSTİK: JOURNAL OF THE TURKISH STATISTICAL ASSOCIATION

İSTATİSTİK

Vol. 9, No. 2, July 2016, pp. 67–77

ISSN 1300-4077 | 16 | 2 | 67 | 77

ON THE NUMBER OF l-OVERLAPPING SUCCESS RUNS OF LENGTH k UNDER q- SEQUENCE OF BINARY TRIALS

Ismail Kinaci, Coşkun Kuş*, Kadir Karakaya and Yunus Akdoğan

Department of Statistics, Faculty of Science,

Selcuk University, 42250, Konya, Turkey

Abstract: Let $X_1, X_2, \ldots, X_n$ be a $\{0,1\}$-valued Bernoulli trials with a geometrically varying success probability. The probability mass function, first and second moments of the number of l-overlapping success runs of length k in $X_1, X_2, \ldots, X_n$ are obtained. The new distribution generalizes, Type I and Type II q-binomial distributions which were recently studied in the literature.

Key words: Discrete distributions; q-Distributions; Runs

History: Submitted: 4 May 2016; Revised: 27 June 2016; Accepted: 8 July 2016

1. Introduction

The distribution of the number of success runs has been widely studied in the literature under various assumptions on binary sequences. See, e.g. [1]-[7].

Let $\{X_i\}_{i\geq 1}$ be a sequence of Bernoulli trials such that the trials of subsequences after the $(i-1)$st zero until the i-th zero are independent with failure probability

$$q_i = 1 - \theta q^{i-1},\ i = 1, 2, \ldots,\ 0 < \theta < 1,\ 0 < q \leq 1. \tag{1.1}$$

Charalambides [8] studied discrete $q-$ distributions under the above mentioned Bernoulli scheme. In particular, he has obtained the pmf of the number Z_n of successes in $X_1, X_2, \ldots, X_n$ as

$$P(Z_n = s) = \begin{bmatrix} n \\ s \end{bmatrix}_q \theta^s \prod_{i=1}^{n-s} \left(1 - \theta q^{i-1}\right),\ s = 0, 1, \ldots, n, \tag{1.2}$$

where

$$\begin{bmatrix} n \\ s \end{bmatrix}_q = \frac{[n]_{s,q}}{[s]_q!}$$

and $[n]_{s,q} = [n]_q [n-1]_q \times \cdots \times [n-s+1]_q$, $[s]_q = \frac{1-q^s}{1-q}$, $[s]_q! = [1]_q [2]_q \times \cdots \times [s]_q$. Note that $q-$binomial distribution converges to the usual binomial distribution as $q \to 1$.

According to the nonoverlapping enumeration scheme, the distribution of the number of success runs of length k in n trials follows a Type I binomial distribution of order k which reduces to the well-known binomial distribution when $k = 1$. Recently, Yalcin and Eryilmaz [9] and Yalcin [10] introduced and studied q-geometric distribution of order k and q-binomial distribution of order k under q-sequence of binary trials.

In this study, we consider a new generalization of a q-binomial distribution of order k. Let $N^q_{n,k,l}$ denote the number of l-overlapping success runs of length k in n Bernoulli trials under q-sequence of binary trials. For example, let $n = 15$ and the trials be

Outcomes: 1 1 1 0 1 1 1 1 0 1 1 1 1 1 0

* Corresponding author. E-mail address: coskunkus@gmail.com

Then, the number of $l=2$ overlapping success run of length $k=3$ is $N^q_{15,3,2}=6$ and the corresponding success runs are

$$1\ 2\ 3,\ \ 5\ 6\ 7,\ \ 6\ 7\ 8,\ \ 10\ 11\ 12,\ \ 11\ 12\ 13,\ \ 12\ 13\ 14.$$

For $l=1$ and $k=3$, the corresponding success runs are

$$1\ 2\ 3,\ \ 5\ 6\ 7,\ \ 10\ 11\ 12,\ \ 12\ 13\ 14$$

and hence $N^q_{15,3,1}=4$.

The distribution of random variable $N^q_{n,k,l}$ reduces

- to the distribution of $N_{n,k,l}$ (Makri and Philippou [4]) when $q\to 1$
- to the distribution of $M_{n,k}$ (Yalcin [10]) when $l=k-1$
- to the distribution of $M_n^{(k)}$ (Ling [3]) when $q\to 1$ and $l=k-1$
- to the distribution of $N_{n,k}$ (Yalcin and Eryilmaz [9]) when $l=0$
- to the distribution of $N_n^{(k)}$ (Hirano [1] and Philippou and Makri [2]) when $q\to 1$ and $l=0$.

The paper is organized as follows. In Section 2, probability mass function (pmf) of $N^q_{n,k,l}$ is obtained. First and second moments of introduced distribution are also derived.

2. Distribution and the first two moments of $N^q_{n,k,l}$

We first prove the following lemma which will be useful in the sequel.

Lemma 1. *For $0<q\le 1$, define*

$$A_q(r,s,t)=\sum\cdots\sum_{B} q^{y_2+2y_3+\cdots+(r-1)y_r},$$

where

$$B=\{\mathbf{y}: y_1+y_2+\cdots+y_r=s,\ c_1+c_2+\cdots+c_r=t,\ y_1\ge 0,\ldots,y_r\ge 0\},$$

$$c_j=\begin{cases}\left[\frac{y_j-l}{k-l}\right], & if\ y_j\ge k\\ 0, & otherwise,\end{cases} \tag{2.1}$$

$\mathbf{y}=(y_1,y_2,\ldots,y_r)$, $[x]$ *denotes the integer part of x and $y_i s$ $(i=1,2,\ldots,r)$ are all integers. Then $A_q(r,s,t)$ obeys the following recurrence relation*

$$A_q(r,s,t)=\begin{cases}\sum_{j=0}^{s} q^{(r-1)j}A_q(r-1,s-j,t-d_j), & if\ r>1, s\ge 0, t\ge 0\\ 1, & if\ \left(r=1, s\ge k, t=\frac{s-l}{k-l}\right)\ or\\ & (r=1, 0\le s<k, t=0)\\ 0, & otherwise,\end{cases}$$

where

$$d_j=\begin{cases}\left[\frac{j-l}{k-l}\right], & if\ j\ge k\\ 0, & otherwise,\end{cases}$$

$j=0,1,\ldots,s$.

PROOF. For $r>1$, consider the values that y_r can take, it can be written

$$\begin{aligned}A_q(r,s,t) &= \sum_{B_0}\cdots\sum q^{y_2+2y_3+\cdots+(r-2)y_{r-1}} + q^{r-1}\sum_{B_1}\cdots\sum q^{y_2+2y_3+\cdots+(r-2)y_{r-1}} \\ &\quad + q^{2(r-1)}\sum_{B_2}\cdots\sum q^{y_2+2y_3+\cdots+(r-2)y_{r-1}} + \cdots + q^{s(r-1)}\sum_{B_s}\cdots\sum q^{y_2+2y_3+\cdots+(r-2)y_{r-1}} \\ &= A_q(r-1,s,t-d_0) \\ &\quad + q^{r-1}A_q(r-1,s-1,t-d_1)+\cdots+q^{s(r-1)}A_q(r-1,0,t-d_s) \\ &= \sum_{j=0}^{s} q^{(r-1)j}A_q(r-1,s-j,t-d_j),\end{aligned}$$

where

$$B_j=\{\mathbf{y}: y_1+y_2+\cdots+y_{r-1}=s-j,\ c_1+c_2+\cdots+c_{r-1}=t-d_j,\ y_1\geq 0,\ldots,y_{r-1}\geq 0\}$$

for $j=0,1,\ldots,s$ such that $B=\cup_{j=0}^{s}B_j$. The other parts of recurrence are obvious.■

THEOREM 1. *Let $N_{n,k,l}^{q}$ denote the number of l-overlapping success runs of length k in n Bernoulli trials under q-sequence of binary trials. Then , for $0<q\leq 1$, the probability mass function of $N_{n,k,l}^{q}$ is given by*

$$P\left\{N_{n,k,l}^{q}=x\right\}=\sum_{i=0}^{\nu_n(x)}\theta^{n-i}\prod_{j=1}^{i}\left(1-\theta q^{j-1}\right)A_q(i+1,n-i,x),$$

$x=0,1,\ldots,\left[\frac{n-l}{k-l}\right]$, *where*

$$\nu(x)=\begin{cases} n, & if\ x=0 \\ n-(x(k-l)+l), & otherwise,\end{cases}$$

and $A_q(r,s,t)$ is defined as in Lemma 1.

PROOF. Let S_n denote the total number of zeros (failures) in n trials. Then

$$P\left\{N_{n,k,l}^{q}=x\right\}=\sum_{i}P\left\{N_{n,k,l}^{q}=x,S_n=i\right\}.$$

The joint event $\left\{N_{n,k,l}^{q}=x,S_n=i\right\}$ can be described with the following binary sequence which consists of i zeros.

$$\underbrace{1\ldots1}_{y_1}0\underbrace{1\ldots1}_{y_2}0\ldots0\underbrace{1\ldots1}_{y_i}0\underbrace{1\ldots1}_{y_{i+1}},$$

where $y_j\geq 0,\ j=1,2,\ldots,i+1$,

$$\begin{aligned} y_1+y_2+\cdots+y_{i+1} &= n-i \\ c_1+c_2+\cdots+c_{i+1} &= x\end{aligned}$$

and c_js are defined as Eq. (2.1).

It is clear that i can take the values at least 0 and at most $\nu(x)$. Under the model (1.1),

$$\begin{aligned}
&P\left\{N_{n,k,l}^{q}=x\right\} \\
&=\sum_{i=0}^{\nu(x)} \sum_{\substack{\sum_{j=1}^{i+1} y_j=n-i \\ \sum_{j=1}^{i+1} c_j=x \\ y_1\geq 0,\ldots,y_{i+1}\geq 0}} \cdots \sum \left(\theta q^{0}\right)^{y_1}\left(1-\theta q^{0}\right)(\theta q)^{y_2}(1-\theta q)\times\cdots\times\left(\theta q^{i-1}\right)^{y_i}\left(1-\theta q^{i-1}\right)\left(\theta q^{i}\right)^{y_{i+1}} \\
&=\sum_{i=0}^{\nu(x)} \theta^{n-i}\prod_{j=1}^{i}\left(1-\theta q^{j-1}\right) \sum_{\substack{\sum_{j=1}^{i+1} y_j=n-i \\ \sum_{j=1}^{i+1} c_j=x \\ y_1\geq 0,\ldots,y_{i+1}\geq 0}} \cdots \sum q^{y_2+2y_3+\cdots+(i-1)y_i+iy_{i+1}} \\
&=\sum_{i=0}^{\nu(x)} \theta^{n-i}\prod_{j=1}^{i}\left(1-\theta q^{j-1}\right) A_q(i+1,n-i,x). \blacksquare
\end{aligned}$$

As a special case, the pmf of the $N_{12,3,2}^{q}$ is given for different parameters settings in Table 1.

Let $X_1,X_2,\ldots,X_n$ be the Bernoulli random variables with geometrically varying success probability. For $i=1,k-l+1,2(k-l)+1,\ldots,\left[\frac{n-k}{k-l}\right](k-l)+1$, let E_i be the event that "$X_1X_2\cdots X_{i+k-1}=1$" and for $i=2,\ldots,n-k+1$ and $j=0,1,\ldots,\left[\frac{i-2}{k-l}\right]$, let $E_{i,j}$ be the event that "$X_{i-j(k-l)-1}=0, X_{i-j(k-l)}\cdots X_{i+k-1}=1$". Next A_1,A_2,A_3 be a partition of the index set $I=\{1,2,\ldots,n-k+1\}$, where

$$\begin{aligned}
A_1 &= \{1\}, \\
A_2 &= \left\{k-l+1, 2(k-l)+1,\ldots,\left[\tfrac{n-k}{k-l}\right](k-l)+1\right\}, \\
A_3 &= I-(A_1\cup A_2).
\end{aligned}$$

In the following Theorem, we obtain an expression for the expected value of the random variable $N_{n,k,l}^{q}$.

Theorem 2. *Let $N_{n,k,l}^{q}$ be a random variable as in Theorem 1. Then for $n\leq k-1$, $E\left(N_{n,k,l}^{q}\right)=0$ and for $n\geq k>l\geq 0$,*

$$E\left(N_{n,k,l}^{q}\right)=\theta^{k}+\sum_{i\in A_2}\theta^{i+k-1}+\sum_{i=2}^{n-k+1}\sum_{j=0}^{\left[\frac{i-2}{k-l}\right]}\sum_{s=0}^{\nu_{i,j}}P\left(Z_{\nu_{i,j}}=s\right)\left(1-\theta q^{\nu_{i,j}-s}\right)\left(\theta q^{\nu_{i,j}-s+1}\right)^{k+j(k-l)},$$

where $\nu_{i,j}=i-j(k-l)-2$ and $P(Z_y=s)$ is defined as Eq. (1.2)

Proof. Let us define random variables Y_i, $1\leq i\leq n-k+1$, as follows:

$$Y_1=\begin{cases}1, & if\ E_1\ occurs\\ 0, & otherwise,\end{cases}$$

for $i\in A_2$,

$$Y_i=\begin{cases}1, & if\ E_i\cup\bigcup_{j=0}^{\left[\frac{i-2}{k-l}\right]}E_{i,j}\ occurs\\ 0, & otherwise,\end{cases}$$

and for $i \in A_3$,

$$Y_i = \begin{cases} 1, & if \bigcup_{j=0}^{\left[\frac{i-2}{k-l}\right]} E_{i,j} \; occurs \\ 0, & otherwise. \end{cases}$$

In this case $N_{n,k,l}^q$ can be written by using Ys as $N_{n,k,l}^q = \sum_{i=1}^{n-k+1} Y_i$. Therefore, expected value of $N_{n,k,l}^q$ is obtained by

$$E\left(N_{n,k,l}^q\right) = P\left(Y_1 = 1\right) + \sum_{i \in A_2} P\left(Y_i = 1\right) + \sum_{i \in A_3} P\left(Y_i = 1\right), \tag{2.2}$$

where

$$P\left(Y_1 = 1\right) = \theta^k, \tag{2.3}$$

for $i \in A_2$

$$\begin{aligned} P\left(Y_i = 1\right) &= P\left(E_{i,0} \cup E_{i,1} \cup \cdots \cup E_{i,\left[\frac{i-2}{k-l}\right]} \cup E_i\right) \\ &= P\left(E_{i,0}\right) + P\left(E_{i,1}\right) + \cdots + P\left(E_{i,\left[\frac{i-2}{k-l}\right]}\right) + P\left(E_i\right) \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} P\left(E_{i,j}\right) &= P\left(X_{i-j(k-l)-1} = 0, X_{i-j(k-l)} \cdots X_{i+k-1} = 1\right) \\ &= \sum_{s=0}^{\nu_{i,j}} P\left(Z_{\nu_{i,j}} = s\right)\left(1 - \theta q^{\nu_{i,j}-s}\right)\left(\theta q^{\nu_{i,j}-s+1}\right)^{k+j(k-l)}, \; j = 0, 1, \ldots, \left[\tfrac{i-2}{k-l}\right], \end{aligned} \tag{2.5}$$

$$P\left(E_i\right) = P\left(X_1 \cdots X_{i+k-1} = 1\right) = \theta^{i+k-1}. \tag{2.6}$$

Hence, substituting Eqs. (2.5)-(2.6) in Eq. (2.4), for $i \in A_2$

$$P\left(Y_i = 1\right) = \sum_{j=0}^{\left[\frac{i-2}{k-l}\right]} \sum_{s=0}^{\nu_{i,j}} P\left(Z_{\nu_{i,j}} = s\right)\left(1 - \theta q^{\nu_{i,j}-s}\right)\left(\theta q^{\nu_{i,j}-s+1}\right)^{k+j(k-l)} + \theta^{i+k-1}. \tag{2.7}$$

Furthermore for $i \in A_3$, $P\left(Y_i = 1\right)$ can be similarly obtained as

$$P\left(Y_i = 1\right) = \sum_{j=0}^{\left[\frac{i-2}{k-l}\right]} \sum_{s=0}^{\nu_{i,j}} P\left(Z_{\nu_{i,j}} = s\right)\left(1 - \theta q^{\nu_{i,j}-s}\right)\left(\theta q^{\nu_{i,j}-s+1}\right)^{k+j(k-l)}. \tag{2.8}$$

The proof is completed by using Eqs. (2.3), (2.7) and (2.8) in Eq. (2.2).

Theorem 3. *Let $N_{n,k,l}^q$ be a random variable as in Theorem 1. Then for $n \le k-1$, $E\left(N_{n,k,l}^q\right)^2 = 0$ and $n \ge k > l \ge 0$,*

$$E\left(N_{n,k,l}^q\right)^2 = E\left(N_{n,k,l}^q\right) + 2\left(I_1 + I_2\right),$$

where

$$I_1 = \sum_{r=2}^{n-k+1} \sum_{j=0}^{\left[\frac{r-k-2}{k-l}\right]} \sum_{s=0}^{\eta_{1,r,j}} P\left(Z_{\eta_{1,r,j}} = s\right)\theta^k\left(1 - \theta q^{\eta_{1,r,j}-s}\right)\left(\theta q^{\eta_{1,r,j}-s+1}\right)^{k+j(k-l)} + \sum_{r \in A_2} \theta^{r+k-1},$$

$$I_2=\sum_{i=2}^{n-k}\sum_{r=i+1}^{n-k+1}\sum_{t_1=0}^{\left[\frac{i-2}{k-l}\right]}\Bigg\{\Bigg(\sum_{t_2=0}^{\left[\frac{r-i-k-1}{k-l}\right]}\Bigg[\sum_{s_1=0}^{\nu_{i,t_1}}\sum_{s_2=0}^{\eta_{i,r,t_2}}P\left(Z_{\nu_{i,t_1}}=s_1\right)\left(1-\theta q^{\nu_{i,t_1}-s_1}\right)\left(\theta q^{\nu_{i,t_1}-s_1+1}\right)^{k+t_1(k-l)}$$
$$\times P\left(W_{\eta_{i,r,t_2}}=s_2\right)\left(1-\theta q^{\nu_{i,t_1}+\eta_{i,r,t_2}-(s_1+s_2)+1}\right)\left(\theta q^{\nu_{i,t_1}+\eta_{i,r,t_2}-(s_1+s_2)+2}\right)^{k+t_2(k-l)}\Bigg]\Bigg)$$
$$+\delta\sum_{s_1=0}^{\nu_{r,t_1}}P\left(Z_{\nu_{r,t_1}}=s_1\right)\left(1-\theta q^{\nu_{r,t_1}-s_1}\right)\left(\theta q^{\nu_{r,t_1}-s_1+1}\right)^{r+k-i+t_1(k-l)}\Bigg\}$$
$$+\sum_{i\in A_2}\sum_{r=i+1}^{n-k+1}\sum_{t_2=0}^{\left[\frac{r-i-k-1}{k-l}\right]}\sum_{s=0}^{\eta_{i,r,t_2}}P\left(Z_{\eta_{i,r,t_2}}=s\right)\theta^{i+k-1}\left(1-\theta q^{\eta_{i,r,t_2}-s}\right)\left(\theta q^{\eta_{i,r,t_2}-s+1}\right)^{k+t_2(k-l)}$$
$$+\sum_{i\in A_2}\sum_{\substack{r\in A_2\\ r>i}}\theta^{r+k-1},$$

$$\delta=\begin{cases}1, & if\ \frac{r-i+t_1(k-l)}{k-l}\ is\ integer\\ 0, & otherwise,\end{cases}$$

$\nu_{i,j}=i-j(k-l)-2$, $\eta_{i,r,j}=r-i-k-j(k-l)-1$. *Also* $P(Z_a=s_1)$ *and* $P(W_b=s_2)$ *are pmf of the* $q-$*binomial distribution (see Eq. (1.2)) with* a *and* b *trials and initial success probability* θ *and* θq^{a-s_1+1}, *respectively. Note that* $E\left(N^q_{n,k,l}\right)$ *is already obtained in Theorem 2.*

Proof. It is clear that

$$\left(N^q_{n,k,l}\right)^2=\left(\sum_{i=1}^{n-k+1}Y_i\right)^2=\sum_{i=1}^{n-k+1}Y_i^2+2\sum_{i=1}^{n-k}\sum_{r=i+1}^{n-k+1}Y_iY_r$$

and one can also write

$$E\left(N^q_{n,k,l}\right)^2=E\left(N^q_{n,k,l}\right)+2\sum_{i=1}^{n-k}\sum_{r=i+1}^{n-k+1}E\left(Y_iY_r\right). \tag{2.9}$$

Last term in RHS of Eq. (2.9) is obtained by

$$\sum_{i=1}^{n-k}\sum_{r=i+1}^{n-k+1}E\left(Y_iY_r\right)=\sum_{i=1}^{n-k}\sum_{r=i+1}^{n-k+1}P\left(Y_iY_r=1\right)$$
$$=\sum_{r=2}^{n-k+1}P\left(Y_1Y_r=1\right)+\sum_{i=2}^{n-k}\sum_{r=i+1}^{n-k+1}P\left(Y_iY_r=1\right),$$

where

$$P\left(Y_1Y_r=1\right)$$
$$=\begin{cases}P\left\{E_1\cap\left(\bigcup_{j=0}^{\left[\frac{r-2}{k-l}\right]}E_{r,j}\cup E_r\right)\right\}, & r\in A_2\\ P\left\{E_1\cap\left(\bigcup_{j=0}^{\left[\frac{r-2}{k-l}\right]}E_{r,j}\right)\right\} & , r\in A_3\end{cases}$$
$$=\begin{cases}\sum_{j=0}^{\left[\frac{r-k-2}{k-l}\right]}\sum_{s=0}^{\eta_{1,r,j}}P\left(Z_{\eta_{1,r,j}}=s\right)\theta^k\left(1-\theta q^{\eta_{1,r,j}-s}\right)\left(\theta q^{\eta_{1,r,j}-s+1}\right)^{k+j(k-l)}+\theta^{r+k-1}, & r\in A_2\\ \sum_{j=0}^{\left[\frac{r-k-2}{k-l}\right]}\sum_{s=0}^{\eta_{1,r,j}}P\left(Z_{\eta_{1,r,j}}=s\right)\theta^k\left(1-\theta q^{\eta_{1,r,j}-s}\right)\left(\theta q^{\eta_{1,r,j}-s+1}\right)^{k+j(k-l)} & , r\in A_3,\end{cases}$$

$$P\left(Y_iY_r=1\right)=\begin{cases} P\left\{\left(\bigcup_{t_1=0}^{\left[\frac{i-2}{k-l}\right]}E_{i,t_1}\cup E_i\right)\cap\left(\bigcup_{t_2=0}^{\left[\frac{r-2}{k-l}\right]}E_{r,t_2}\cup E_r\right)\right\}, & i,r\in A_2 \\ P\left\{\left(\bigcup_{t_1=0}^{\left[\frac{i-2}{k-l}\right]}E_{i,t_1}\cup E_i\right)\cap\left(\bigcup_{t_2=0}^{\left[\frac{r-2}{k-l}\right]}E_{r,t_2}\right)\right\}, & \begin{array}{c} i\in A_2 \\ r\in A_3\end{array} \\ P\left\{\left(\bigcup_{t_1=0}^{\left[\frac{i-2}{k-l}\right]}E_{i,t_1}\right)\cap\left(\bigcup_{t_2=0}^{\left[\frac{r-2}{k-l}\right]}E_{r,t_2}\cup E_r\right)\right\}, & \begin{array}{c} i\in A_3 \\ r\in A_2\end{array} \\ P\left\{\left(\bigcup_{t_1=0}^{\left[\frac{i-2}{k-l}\right]}E_{i,t_1}\right)\cap\left(\bigcup_{t_2=0}^{\left[\frac{r-2}{k-l}\right]}E_{r,t_2}\right)\right\}, & i,r\in A_3 \end{cases}$$

$$=\begin{cases} \sum_{t_1=0}^{\left[\frac{i-2}{k-l}\right]}\left\{\left(\sum_{t_2=0}^{\left[\frac{r-i-k-1}{k-l}\right]}P\left(E_{i,t_1}\cap E_{r,t_2}\right)\right)+\delta P\left(E_{r,\frac{r-i+t_1(k-l)}{k-l}}\right)\right\} \\ \quad +\sum_{t_2=0}^{\left[\frac{r-i-k-1}{k-l}\right]}P\left(E_i\cap E_{r,t_2}\right)+P\left(E_i\cap E_r\right), & \begin{array}{c} i\in A_2 \\ r\in A_2\end{array} \\ \sum_{t_1=0}^{\left[\frac{i-2}{k-l}\right]}\left\{\left(\sum_{t_2=0}^{\left[\frac{r-i-k-1}{k-l}\right]}P\left(E_{i,t_1}\cap E_{r,t_2}\right)\right)+\delta P\left(E_{r,\frac{r-i+t_1(k-l)}{k-l}}\right)\right\} \\ \quad +\sum_{t_2=0}^{\left[\frac{r-i-k-1}{k-l}\right]}P\left(E_i\cap E_{r,t_2}\right), & \begin{array}{c} i\in A_2 \\ r\in A_3\end{array} \\ \sum_{t_1=0}^{\left[\frac{i-2}{k-l}\right]}\left\{\left(\sum_{t_2=0}^{\left[\frac{r-i-k-1}{k-l}\right]}P\left(E_{i,t_1}\cap E_{r,t_2}\right)\right)+\delta P\left(E_{r,\frac{r-i+t_1(k-l)}{k-l}}\right)\right\}, & \begin{array}{c} i\in A_3 \\ r\in A_2\cup A_3\end{array}, \end{cases}$$

$$P\left(E_i\cap E_r\right)=\theta^{r+k-1},$$

$$P\left(E_i\cap E_{r,t_2}\right)=\sum_{s=0}^{\eta_{i,r,t_2}}P\left(Z_{\eta_{i,r,t_2}}=s\right)\theta^{i+k-1}\left(1-\theta q^{\eta_{i,r,t_2}-s}\right)\left(\theta q^{\eta_{i,r,t_2}-s+1}\right)^{k+t_2(k-l)},$$

$$\begin{aligned} P\left(E_{i,t_1}\cap E_{r,t_2}\right)=&\sum_{s_1=0}^{\nu_{i,t_1}}\sum_{s_2=0}^{\eta_{i,r,t_2}}P\left(Z_{\nu_{i,t_1}}=s_1\right)\left(1-\theta q^{\nu_{i,t_1}-s_1}\right)\left(\theta q^{\nu_{i,t_1}-s_1+1}\right)^{k+t_1(k-l)} \\ &\times P\left(W_{\eta_{i,r,t_2}}=s_2\right)\left(1-\theta q^{\nu_{i,t_1}+\eta_{i,r,t_2}-(s_1+s_2)+1}\right)\left(\theta q^{\nu_{i,t_1}+\eta_{i,r,t_2}-(s_1+s_2)+2}\right)^{k+t_2(k-l)}, \end{aligned}$$

$$P\left(E_{r,\frac{r-i+t_1(k-l)}{k-l}}\right)=\sum_{s_1=0}^{\nu_{r,t_1}}P\left(Z_{\nu_{r,t_1}}=s_1\right)\left(1-\theta q^{\nu_{r,t_1}-s_1}\right)\left(\theta q^{\nu_{r,t_1}-s_1+1}\right)^{r+k-i+t_1(k-l)},$$

$P\left(Z_a=s_1\right)$ and $P\left(W_b=s_2\right)$ are pmf of the $q-$binomial distribution with initial success probability θ and θq^{a-s_1+1}.■

$E\left(N_{n,k,l}^q\right)$ and $Var\left(N_{n,k,l}^q\right)$ are given for different values of n,k,l,q and θ in Tables 2-3 and some related figures are also given in Figs. 2-3 for $n=10$. From these tables and figures, it can be seen that:

- When l increases, both $E\left(N_{n,k,l}^{q}\right)$ and $Var\left(N_{n,k,l}^{q}\right)$ increase
- When k increases, both $E\left(N_{n,k,l}^{q}\right)$ and $Var\left(N_{n,k,l}^{q}\right)$ decrease
- When θ or q increases, $E\left(N_{n,k,l}^{q}\right)$ increases and $Var\left(N_{n,k,l}^{q}\right)$ is non-monotone.

References

[1] Hirano, K. (1986). Some Properties of the Distributions of Order k. *Fibonacci Numbers and Their Applications* (eds. G.E. Bergum, A.N. Philippou, and A.F. Horadam), 43-53.

[2] Philippou, A.N. and Makri, F.S. (1986). Success Runs and Longest Runs. *Statistics & Probability Letters*, 4, 211-215.

[3] Ling, K.D. (1988). On Binomial Distribution of Order k. *Statistics & Probability Letters*, 6, 247-250.

[4] Makri, F.S. and Philippou, A.N. (2005). On Binomial and Circular Binomial Distributions of order k for l -overlapping success runs of length k. *Statistical Papers*, 46(3), 411-432.

[5] Eryilmaz, S. and Demir, S. (2007). Success runs in a sequence of exchangeable binary trials. *Journal of Statistical Planning and Inference*, 137, 2954–2963.

[6] Demir, S. and Eryilmaz, S. (2010). Run statistics in a sequence of arbitrarily dependent binary trials. *Statistical Papers*, 51, 959–973.

[7] Eryilmaz, S. and Yalcin, F. (2011). Distribution of run statistics in partially exchangeable processes. *Metrika*, 73, 293–304.

[8] Charalambides, C.A. (2010). The q -Bernstein basis as a q -binomial distributions. *Journal of Statistical Planning and Inference*, 140, 2184–2190.

[9] Yalcin, F. and Eryilmaz S. (2014). q -Geometric and q -Binomial Distributions of Order k. *Journal of Computational and Applied Mathematics*, 271, 31–38.

[10] Yalcin, F. (2013). On a Generalization of Ling's Binomial Distribution. *ISTATISTIK, Journal of the Turkish Statistical Association*, 6(3), 110–115.

TABLE 1. Probability mass function of $N^q_{12,3,2}$

x	$q=0.5,\theta=0.5$	$q=0.5,\theta=0.8$	$q=0.8,\theta=0.5$
0	0.8594	0.4526	0.7668
1	0.0733	0.1169	0.1243
2	0.0343	0.0888	0.0570
3	0.0166	0.0694	0.0270
4	0.0082	0.0550	0.0130
5	0.0041	0.0438	0.0062
6	0.0020	0.0350	0.0030
7	0.0010	0.0312	0.0016
8	0.0005	0.0215	0.0004
9	0.0002	0.0172	0.0003
10	0.0002	0.0687	0.0002

TABLE 2. $E\left(N^q_{n,k,l}\right)$ for different values of θ,q,n,k and l

n	l	k	(θ,q)				
			$(0.2,0.5)$	$(0.5,0.2)$	$(0.5,0.5)$	$(0.5,1)$	$(0.8,0.5)$
8	0	2	0.05509	0.34237	0.41692	1.22265	1.64504
	1	3	0.00948	0.16504	0.18209	0.52734	1.10233
	2	4	0.00177	0.08212	0.08608	0.22265	0.85677
	3	5	0.00034	0.03907	0.03996	0.08984	0.54280
	4	5	0.00041	0.05860	0.05968	0.12499	0.97420
10	0	2	0.05510	0.34348	0.42003	1.55566	1.78102
	1	3	0.00948	0.16702	0.18471	0.69433	1.24823
	2	4	0.00177	0.08310	0.08734	0.30566	0.96948
	3	5	0.00034	0.04102	0.04204	0.13183	0.67952
	4	5	0.00041	0.06153	0.06279	0.18749	1.21949
12	0	2	0.05510	0.34375	0.42083	1.88890	1.86823
	1	3	0.00948	0.16752	0.18537	0.86107	1.34169
	2	4	0.00177	0.08335	0.08765	0.38891	1.04167
	3	5	0.00034	0.04151	0.04256	0.17358	0.76707
	4	5	0.00041	0.06226	0.06357	0.24999	1.37657

TABLE 3. $Var\left(N^q_{n,k,l}\right)$ for different values of θ,q,n,k and l

n	l	k	(θ,q)				
			$(0.2,0.5)$	$(0.5,0.2)$	$(0.5,0.5)$	$(0.5,1)$	$(0.8,0.5)$
8	0	2	0.07342	0.44446	0.60066	2.04772	2.26448
	1	3	0.01035	0.23177	0.25033	0.61644	1.42630
	2	4	0.00191	0.12226	0.12588	0.25120	1.32089
	3	5	0.00037	0.05317	0.05407	0.10521	0.66825
	4	5	0.00061	0.14111	0.14252	0.24268	2.39907
10	0	2	0.08553	0.45503	0.68776	3.31330	3.09906
	1	3	0.01090	0.24494	0.27380	1.29621	2.10510
	2	4	0.00192	0.12894	0.13359	0.41536	1.88385
	3	5	0.00037	0.06278	0.06381	0.16523	1.17570
	4	5	0.00061	0.16908	0.17082	0.40429	4.25853
12	0	2	0.09760	0.45946	0.76340	4.42100	3.84258
	1	3	0.01145	0.24924	0.28704	2.02928	2.71397
	2	4	0.00195	0.13111	0.13700	0.78160	2.37884
	3	5	0.00037	0.06616	0.06731	0.25575	1.65544
	4	5	0.00061	0.17900	0.18093	0.59765	6.00807

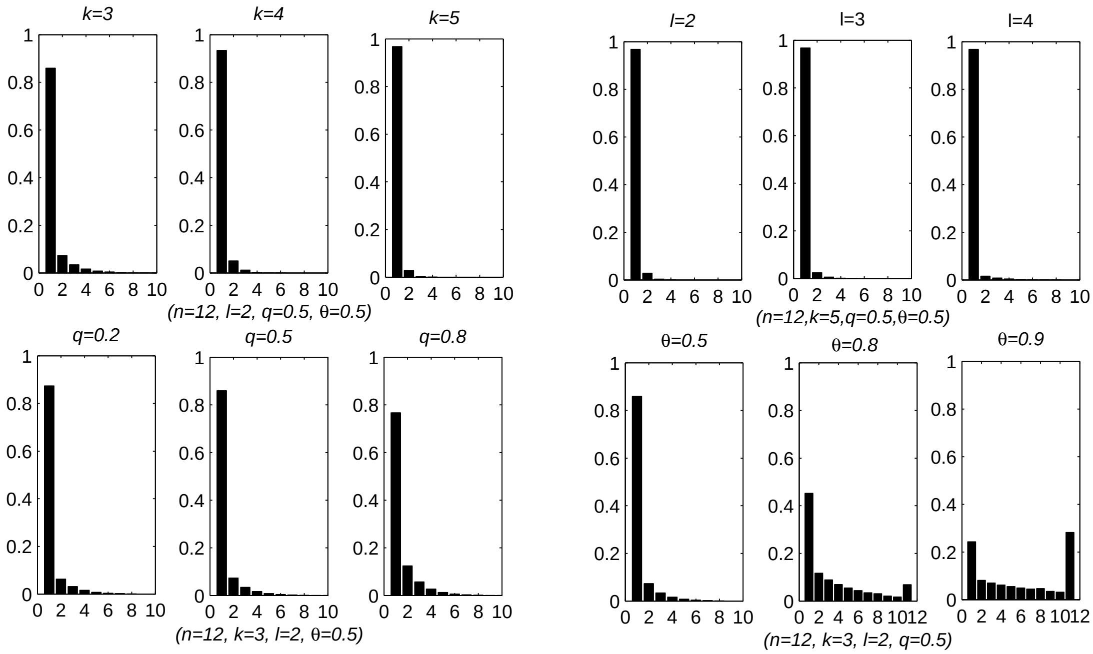

FIGURE 1. Probability mass function of $N^q_{12,k,l}$ for different settings

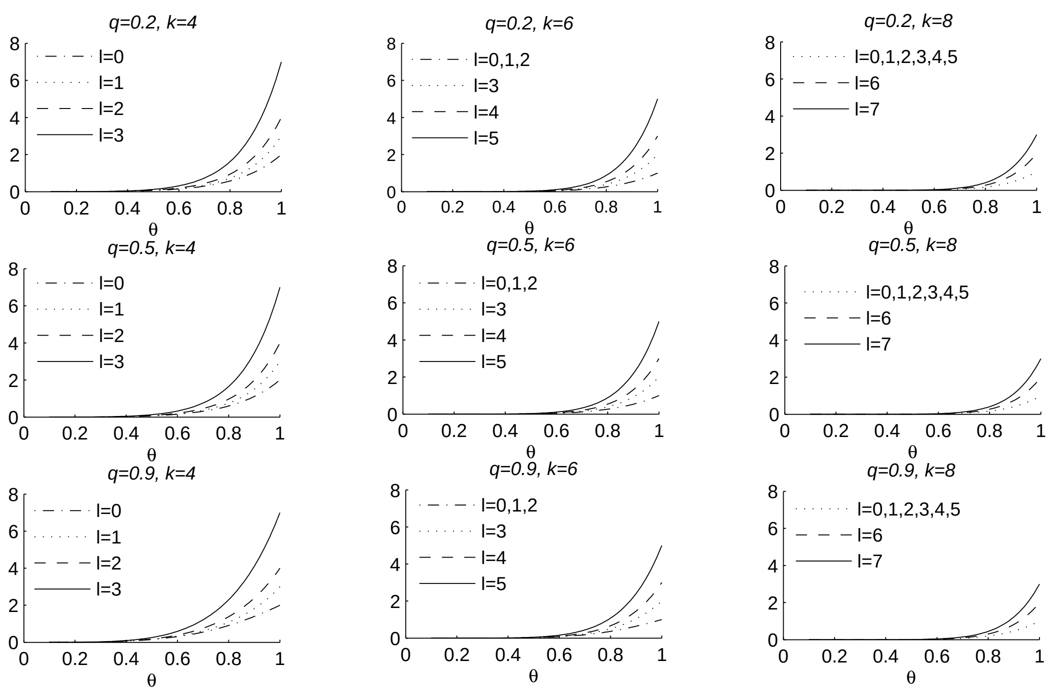

FIGURE 2. $E\left(N^q_{10,k,l}\right)$ for different values of θ, q, k and l

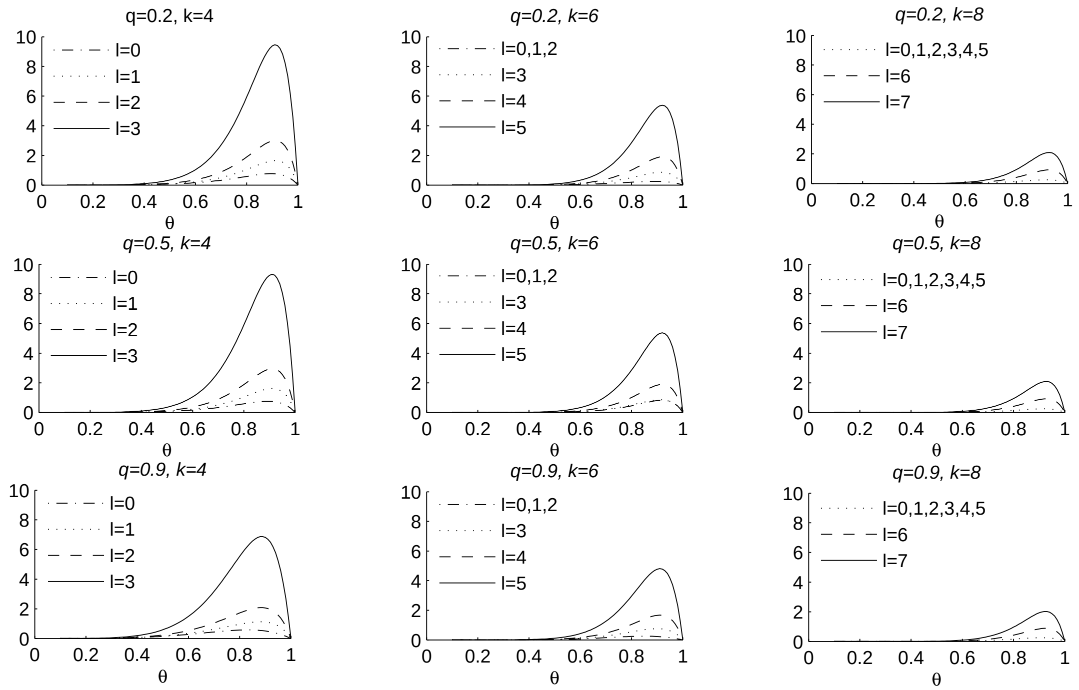

FIGURE 3. $Var\left(N^q_{10,k,l}\right)$ for different values of θ, q, k and l