İSTATİSTİK: JOURNAL OF THE TURKISH STATISTICAL ASSOCIATION

İSTATİSTİK

Vol. 9, No. 3, December 2016, pp. 93–106
ISSN 1300-4077 | 16 | 3 | 106 | 106

SOME IMPROVED CLASSIFICATION-BASED RIDGE PARAMETER OF HOERL AND KENNARD ESTIMATION TECHNIQUES

Adewale F. Lukman* and Kayode Ayinde
Department of Statistics
Ladoke Akintola University of Technology,
P.M.B. 4000, Ogbomoso, Oyo State, Nigeria.

Abstract: In a linear regression model, it is often assumed that the explanatory variables are independent. This assumption is often violated and Ridge Regression estimator introduced by Hoerl and Kennard (1970) has been identified to be more efficient than ordinary least square (OLS) in handling it. However, it requires a ridge parameter, K, of which many have been proposed. In this study, estimators based on Hoerl and Kennard were classified into different forms and various types and some modifications were proposed to improve it. Investigation were done by conducting 1000 Monte-Carlo experiments under five (5) levels of multicollinearity, three (3) levels of error variance and five levels of sample size. For the purpose of comparing the performance of the improved ridge parameter with the existing ones, the number of times the MSE of the improved ridge parameter is less than the existing ones is counted over the levels of multicollinearity (5) and error variance (3). Also, a maximum of fifteen (15) counts is expected. Results show that the improved ridge parameters proposed in this study are better than the existing ones most especially with the quantity $\frac{1}{VIF_{max}}$. Finally, results from the real life data support the simulation results to some extent.

Key words: Linear Regression Model; Multicollinearity; Ridge Parameter Estimation Techniques; Relative Efficiency
History: Submitted: 10 February 2016; Revised: 1 June 2016; Accepted: 3 October 2016

---

1. Introduction

A general linear regression model is defined in matrix form as:

$$Y = X\beta + U \tag{1.1}$$

where X is an $n \times p$ matrix with full rank, Y is a $n \times 1$ vector of dependent variable, β is a $p \times 1$ vector of unknown parameters, and U is the error term such that $E(U) = 0$ and $E(UU') = \sigma^2 I_n$. The Ordinary Least Square (OLS) estimator commonly used to estimate the regression parameter β in (1.1) is defined as:

$$\hat{\beta} = (X'X)^{-1} X'Y \tag{1.2}$$

Gauss Markov theorem states that the OLS estimator in the class of unbiased estimators has minimum variance, that is, they are best linear unbiased estimator (Gujarati, 1995). The theorem holds as long as the assumptions of classical linear regression model are satisfied. However, if one or more of these assumptions do not hold, OLS is no longer the best linear unbiased estimator (BLUE). A pertinent example is when two or more explanatory variables are linearly related. Consequently, the performance of OLS estimator is unsatisfactory when the explanatory variables are related. The regression coefficients is determinate but cannot be estimated with great precision and sometimes have wrong signs (Gujarati, 1995). Several methods have been suggested in literature to solve this problem. The method of ridge regression, introduced by Hoerl and Kennard (1970) is generally

* E-mail address: wale3005@yahoo.com

acceptable as alternative to the OLS estimator to handle the problem of multicollinearity. They suggested the addition of ridge parameter K to the diagonal of $X'X$ matrix in (1.2). Therefore, ridge estimator is defined as:

$$\hat{\beta} = (X'X + KI)^{-1} X'Y \tag{1.3}$$

where K is a diagonal matrix of non-negative constants that is $K \geq 0$. Though this estimator is biased but it gives a smaller mean squared error when compared to the OLS estimator for a positive value of K (Hoerl and Kennard, 1970). The use of the estimator depends largely on the ridge parameter, K. Several methods for estimating this ridge parameter have been proposed as follows: Hoerl and Kennard (1970); McDonald and Galarneau (1975); Lawless and Wang (1976); Hocking *et al*. (1976); Wichern and Churchill (1978); Gibbons (1981); Nordberg (1982); Kibria (2003), Khalaf and Shukur (2005), Alkhamisi *et al*. (2006), Muniz and Kibria (2009), Mansson *et al*. (2010), Dorugade (2014) and recently, Lukman and Ayinde (2015). The purpose of this study is to apply the modification in Alkhamisi and Shukur (2007), Dorugade and Kashid (2010) and proposed another modification to improve the various types and different forms of Hoerl and Kennard (1970) as classified by Lukman and Ayinde (2015). A Simulation study is conducted and the performances of the estimators examined via mean square error (MSE).

2. Review of methods of estimating the Ridge parameter

Hoerl and Kennard (1970) defined the ridge parameter as:

$$K_i = \frac{\sigma^2}{\sigma_i^2} \tag{2.1}$$

They suggested estimating the ridge parameter by taking the maximum (Fixed Maximum) of α_i^2 such that the estimator of K is:

$$\hat{K}_{HK}^{FM} = \frac{\hat{\sigma}^2}{Max(\hat{\alpha_i^2})} \tag{2.2}$$

Hoerl *et al*. (1975) proposed a different estimator of K by taking the Harmonic Mean of the ridge parameter K_{HK_i}. This estimator is given as:

$$\hat{K}_{HK}^{FM} = \frac{P\hat{\sigma}^2}{\sum_{i=1}^{p} \alpha_i^2} \tag{2.3}$$

Lukman and Ayinde (2015) reviewed that the several methods of estimating the ridge parameters earlier mentioned and observed that the existing ridge parameters followed some different forms such as Fixed Maximum, Varying Maximum, Arithmetic Mean, Harmonic Mean, Geometric Mean and Median and various types such as Original, Reciprocal, Square Root and Reciprocal of Square Root. This is further illustrated in Table 1.

	Various Types of K			
Different Forms	O	R	SR	RSR
FM	$\hat{K}_{HK}^{FMO} = \frac{\hat{\sigma}^2}{Max(\hat{\alpha}_i^2)}$ Hoerl *et al.* (1970)	$\hat{K}_{HK}^{FMR} = \frac{1}{\hat{K}_{HK}^{FMO}}$ Lukman and Ayinde (2015)	$\hat{K}_{HK}^{FMSR} = \sqrt{\hat{K}_{HK}^{FMO}}$ Lukman and Ayinde (2015)	$\hat{K}_{HK}^{FMRSR} = \frac{1}{\sqrt{\hat{K}_{HK}^{FMO}}}$ Lukman and Ayinde (2015)
VM	$\hat{K}_{HK}^{VMO} = Max(\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2})$ Lukman and Ayinde (2015)	$\hat{K}_{HK}^{VMR} = Max(\frac{1}{(\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2})})$ Lukman and Ayinde (2015)	$\hat{K}_{HK}^{VMSR} = Max(\sqrt{\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2}})$ Muniz and Kibria (2009)	$\hat{K}_{HK}^{VMRSR} = Max(\frac{1}{\sqrt{\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2}}})$ Muniz and Kibria (2009)
AM	$\hat{K}_{HK}^{AMO} = \frac{1}{p}\sum_{i=1}^{p}\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2}$ Kibria (2003)	$\hat{K}_{HK}^{AMR} = \frac{1}{\hat{K}_{HK}^{AMO}}$ Lukman and Ayinde (2015)	$\hat{K}_{HK}^{AMSR} = \sqrt{\hat{K}_{HK}^{AMO}}$ Lukman and Ayinde (2015)	$\hat{K}_{HK}^{AMRSR} = \frac{1}{\sqrt{\hat{K}_{HK}^{AMO}}}$ Lukman and Ayinde (2015)
HM	$\hat{K}_{HK}^{HMO} = \frac{p\hat{\sigma}^2}{\sum_{i=1}^{p}\hat{\alpha}_i^2}$ Hoerl *et al.* (1975)	$\hat{K}_{HK}^{HMR} = \frac{1}{\hat{K}_{HK}^{HMO}}$ Lukman and Ayinde (2015)	$\hat{K}_{HK}^{HMSR} = \sqrt{\hat{K}_{HK}^{HMO}}$ Lukman and Ayinde (2015)	$\hat{K}_{HK}^{HMRSR} = \frac{1}{\sqrt{\hat{K}_{HK}^{HMO}}}$ Lukman and Ayinde (2015)
GM	$\hat{K}_{HK}^{GMO} = \frac{\hat{\sigma}^2}{(\prod_{i=1}^{p}\hat{\alpha}_i^2)^{\frac{1}{p}}}$ Kibria (2003)	$\hat{K}_{HK}^{GMR} = \frac{1}{\hat{K}_{HK}^{GMO}}$ Lukman and Ayinde (2015)	$\hat{K}_{HK}^{GMSR} = \sqrt{\hat{K}_{HK}^{GMO}}$ Muniz and Kibria (2009)	$\hat{K}_{HK}^{GMRSR} = \frac{1}{\sqrt{\hat{K}_{HK}^{GMO}}}$ Muniz and Kibria (2009)
M	$\hat{K}_{HK}^{MO} = Median(\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2})$ Kibria (2003)	$\hat{K}_{HK}^{MR} = Median(\frac{1}{(\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2})})$ Lukman and Ayinde (2015)	$\hat{K}_{HK}^{MSR} = Median(\sqrt{(\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2})})$ Muniz and Kibria (2009)(2015)	$\hat{K}_{HK}^{MRSR} = Median(\frac{1}{\sqrt{(\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2})}})$ Muniz and Kibria (2009)

TABLE 1. Summary of Different Forms and Various Types for $\hat{K}_{HKi} = \frac{\hat{\sigma}^2}{\hat{\alpha}_i^2}$

Furthermore, Alkhamisi and Shukur (2007) present new methods of estimating the ridge parameter K as:

$$\hat{K}_{AS} = \frac{\hat{\sigma}^2}{Max(\hat{\alpha_i^2})} + \frac{1}{Max(\lambda_i)} \tag{2.4}$$

Dorugade and Kashid (2010) suggested the improvement of ridge parameter by introducing variance inflation factor, which is defined as:

$$\hat{K}_{DK} = Max(0, \frac{\hat{\sigma}^2}{Max(\hat{\alpha_i^2})} - \frac{1}{(VIF_j)max}) \tag{2.5}$$

where $VIF_j = \frac{1}{1-R_j^2}$; $j = 1, 2, 3, ..., p$ is the variance inflation factor of j^{th} regressor
Also, the quantity $\frac{1}{n\lambda_{max}}$ is suggested in this study to improve the ridge estimator.

3. Proposed Ridge parameter

Following Alkhamisi and Shukur (2007), Dorugade and Kashid (2010), the following quantities are used to improve ridge parameter in this study:
$\frac{1}{\lambda_{max}}, \frac{1}{n\lambda_{max}}$ and $\frac{1}{nVIF_{max}}$ and considered in their different forms and various types as classified by Lukman and Ayinde (2015). The improved version of Hoerl and Kennard using the quantity $\frac{1}{\lambda_{max}}$ is summarized in Table 2.

	Various Types of K			
Different Forms	O	R	SR	RSR
FM	$\hat{K}_{HK}^{FMO} = \frac{\hat{\sigma}^2}{Max(\hat{\alpha}_i^2)} + \frac{1}{\lambda_{max}}$ Alkhamisi and Shukur (2007)	$\hat{K}_{HK}^{FMR} = \frac{1}{\hat{K}_{HK}^{FMO}} + \frac{1}{\lambda_{max}}$ Proposed	$\hat{K}_{HK}^{FMSR} = \sqrt{\hat{K}_{HK}^{FMO}} + \frac{1}{\lambda_{max}}$ Proposed	$\hat{K}_{HK}^{FMRSR} = \frac{1}{\sqrt{\hat{K}_{HK}^{FMO}}} + \frac{1}{\lambda_{max}}$ Proposed
VM	$\hat{K}_{HK}^{VMO} = Max(\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2}) + \frac{1}{\lambda_{max}}$ Proposed	$\hat{K}_{HK}^{VMR} = Max(\frac{1}{(\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2})}) + \frac{1}{\lambda_{max}}$ Proposed	$\hat{K}_{HK}^{VMSR} = Max(\sqrt{\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2}}) + \frac{1}{\lambda_{max}}$ Proposed	$\hat{K}_{HK}^{VMRSR} = Max(\frac{1}{\sqrt{\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2}}}) + \frac{1}{\lambda_{max}}$ Proposed
AM	$\hat{K}_{HK}^{AMO} = \frac{1}{p}\sum_{i=1}^{p}\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2} + \frac{1}{\lambda_{max}}$ Proposed	$\hat{K}_{HK}^{AMR} = \frac{1}{\hat{K}_{HK}^{AMO}} + \frac{1}{\lambda_{max}}$ Proposed	$\hat{K}_{HK}^{AMSR} = \sqrt{\hat{K}_{HK}^{AMO}} + \frac{1}{\lambda_{max}}$ Proposed	$\hat{K}_{HK}^{AMRSR} = \frac{1}{\sqrt{\hat{K}_{HK}^{AMO}}} + \frac{1}{\lambda_{max}}$ Proposed
HM	$\hat{K}_{HK}^{HMO} = \frac{p\hat{\sigma}^2}{\sum_{i=1}^{p}\hat{\alpha}_i^2} + \frac{1}{\lambda_{max}}$ Proposed	$\hat{K}_{HK}^{HMR} = \frac{1}{\hat{K}_{HK}^{HMO}} + \frac{1}{\lambda_{max}}$ Proposed	$\hat{K}_{HK}^{HMSR} = \sqrt{\hat{K}_{HK}^{HMO}} + \frac{1}{\lambda_{max}}$ Proposed	$\hat{K}_{HK}^{HMRSR} = \frac{1}{\sqrt{\hat{K}_{HK}^{HMO}}} + \frac{1}{\lambda_{max}}$ Proposed
GM	$\hat{K}_{HK}^{GMO} = \frac{\hat{\sigma}^2}{(\prod_{i=1}^{p}\hat{\alpha}_i^2)^{\frac{1}{p}}} + \frac{1}{\lambda_{max}}$ Proposed	$\hat{K}_{HK}^{GMR} = \frac{1}{\hat{K}_{HK}^{GMO}} + \frac{1}{\lambda_{max}}$ Proposed	$\hat{K}_{HK}^{GMSR} = \sqrt{\hat{K}_{HK}^{GMO}} + \frac{1}{\lambda_{max}}$ Proposed	$\hat{K}_{HK}^{GMRSR} = \frac{1}{\sqrt{\hat{K}_{HK}^{GMO}}} + \frac{1}{\lambda_{max}}$ Proposed
M	$\hat{K}_{HK}^{MO} = Median(\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2}) + \frac{1}{\lambda_{max}}$ Proposed	$\hat{K}_{HK}^{MR} = Median(\frac{1}{(\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2})}) + \frac{1}{\lambda_{max}}$ Proposed	$\hat{K}_{HK}^{MSR} = Median(\sqrt{(\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2})}) + \frac{1}{\lambda_{max}}$ Proposed	$\hat{K}_{HK}^{MRSR} = Median(\frac{1}{\sqrt{(\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2})}}) + \frac{1}{\lambda_{max}}$ Proposed

TABLE 2. Summary of Different Forms and Various Types for $\hat{K}_{HK_{Ai}} = \frac{\hat{\sigma}^2}{\hat{\alpha}_i^2} + \frac{1}{\lambda_{max}}$

4. Simulation study

Simulation procedure used by McDonald and Galarneau (1975), Wichern and Churchill (1978), Gibbons (1981), Kibria (2003), Muniz and Kibria (2009), Lukman and Ayinde (2015) was also used to generate the explanatory variables in this study: This is given as:

$$X_{ij} = (1-\rho^2)^{\frac{1}{2}} Z_{ij} + \rho Z_{ip} i = 1,2,3,...,n; j = 1,2,...,p. \quad (4.1)$$

where Z_{ij} is independent standard normal distribution with mean zero and unit variance, ρ is the correlation between any two explanatory variables and p is the number of explanatory variables. The values of ρ were taken as $0.8, 0.9, 0.95, 0.99$ and 0.999. Thus, the correlation between the variables is the same. In this study, the number of explanatory variable (p) was taken to be three (3) and seven (7).
The considered regression model is of the form:

$$Y_t = \beta_0 + \beta_1 X_{1t} + \beta_2 X_{2t} + ... + \beta_p X_{pt} + U_t \quad (4.2)$$

where $t = 1,2,...,n; p = 3,7$
The error term U_t was generated to be normally distributed with mean zero and variance σ^2, $U_t \sim N(0,\sigma^2)$. In this study, σ were taken to be 0.5, 1 and 5.

The model was studied with fixed regressors, $X_{it}, i = 1,2,...,p; t = 1,2,...,n$ such that there exist different levels of multicollinearity among the regressors.

4.1. Procedure for generating the error terms

The error term U_t was generated to be normally distributed with mean zero and variance $U_t \sim N(0,\sigma^2)$. In this study, σ values were 0.5, 1 and 5.
β_0 was taken to be identically zero. When $p = 3$, the values of β were chosen to be: $\beta_1 = 0.8$, $\beta_2 = 0.1$,

$\beta_3 = 0.6$. When $p = 7$, the values of β were chosen to be: $\beta_1 = 0.4, \beta_2 = 0.1, \beta_3 = 0.6, \beta_4 = 0.2, \beta_5 = 0.25, \beta_6 = 0.3, \beta_7 = 0.53$. The parameter values were chosen such that $\beta'\beta = 1$ which is a common restriction in simulation studies of this type (Muniz and Kibria, 2009). We varied the sample sizes between $10, 20, 30, 40$ and 50. Three different values of $\sigma : 0.5, 1$ and 5 were also used. At a specified value of n, p and σ, the fixed Xs are first generated; followed by the U, and the values of Y are then obtained using the regression model. The performance of this model is evaluated using mean square error (MSE). The MSE for ridge and OLS are calculated using the equation defined as:

$$MSE(\hat{\beta}_{ridge}) = \hat{\sigma}^2 \sum_{i=1}^{p} \frac{\lambda_i}{(\lambda_i + \hat{K})^2} + \hat{K}^2 \sum_{i=1}^{p} \frac{\hat{\alpha}_i^2}{(\lambda_i + \hat{K})^2} \tag{4.3}$$

$$MSE(\hat{\beta}_{OLS}) = \hat{\sigma}^2 \sum_{i=1}^{p} \frac{1}{\lambda_i} \tag{4.4}$$

where $\lambda_1, \lambda_2, ..., \lambda_p$ are the eigenvalues of $X'X, \hat{K}$ is the estimator of the ridge parameter $K, \hat{\alpha}_i$ is the i^{th} element of the vector $\hat{\alpha} = Q'\hat{\beta}$ where Q is an orthogonal matrix. For the purpose of comparing the performance of the improved ridge parameter with the existing ones, the number of times the estimated MSE of each of the improved ridge parameters is less than the existing ones is counted over the levels of multicollinearity (5) and error variances (3) and the overall expected number of count should be 15. An estimator with a minimum of atleast 8 is generally preferred to others. A sample of the estimated mean square error of OLS and relative efficiency of the ridge estimators based on $\hat{K}_{HKA}$ and $\hat{K}_{HK}$ are presented in Appendix 1 and 2. The relative efficiency of the ridge estimators relative to OLS estimator.

$$RelativeEfficiency(RE) = \frac{MSE(\hat{\beta}_{ridge})}{MSE(\hat{\beta}_{OLS})} \tag{4.5}$$

where $0 < RE < 1$.

5. Results and discussion

The number of times the improved ridge parameters estimator is better than the existing ridge parameter with multicollinearity (5 levels) and error variances (3 levels) effect partial out is summarized in Table 3 and 4. From Table 3 and 4, the best five improved techniques by introducing the quantity $\frac{1}{\lambda_{max}}$ are [FMR, VMR], [FMRSR, VMRSR] and FMO. The best five improved techniques by introducing the quantity $\frac{1}{n\lambda_{max}}$ are FMO, [FMR, VMR], [FMRSR and VMRSR]. Consequently, by introducing the quantity $\frac{1}{nVIF_{max}}$, results show that the best five improved techniques are HMO, MO, GMO, FMO and GMSR. Generally, the results show that the improved ridge parameters perform better than the existing ones especially with the ones identified as the best five. Results show that the quantity $\frac{1}{nVIF_{max}}$ generally perform better than other quantities. However, the performance of the quantity proposed in this study is also good especially with Fixed Maximum Original. As the sample sizes increases to 200, the performances of the estimators are not too different.

P=3																		
methods	10			20			30			40			50			200		
	$\hat{K}_{HKA}$	$\hat{K}_{HKB}$	$\hat{K}_{HKC}$	$\hat{K}_{HKA}$	$\hat{K}_{HKB}$	$\hat{K}_{HKC}$	$\hat{K}_{HKA}$	$\hat{K}_{HKB}$	$\hat{K}_{HKC}$	$\hat{K}_{HKA}$	$\hat{K}_{HKB}$	$\hat{K}_{HKC}$	$\hat{K}_{HKA}$	$\hat{K}_{HKB}$	$\hat{K}_{HKC}$	$\hat{K}_{HKA}$	$\hat{K}_{HKB}$	$\hat{K}_{HKC}$
FMO	9	12	0	10	13	0	10	15	0	12	14	0	15	14	0	15	3	0
FMR	15	15	0	12	11	0	12	12	0	9	9	0	9	8	0	6	3	0
FMSR	3	3	12	6	6	7	6	4	8	9	5	6	9	2	6	11	0	3
FMRSR	15	15	5	10	10	2	12	12	2	9	9	3	9	7	2	6	3	1
VMO	0	0	15	0	0	13	0	0	15	0	0	15	0	0	15	0	0	0
VMR	15	15	0	12	11	0	12	12	0	9	9	0	9	8	0	6	3	0
VMSR	0	0	15	0	0	13	1	0	14	1	0	9	2	0	9	1	0	1
VMRSR	15	15	5	10	10	2	12	12	2	9	9	3	9	7	2	6	3	1
AMO	0	0	15	0	0	13	0	0	15	0	0	13	0	0	13	0	0	4
AMR	15	15	0	9	9	0	11	10	0	8	6	0	6	6	0	3	3	3
AMSR	0	0	15	2	0	10	3	1	12	6	0	9	5	0	6	5	0	3
AMRSR	15	15	0	9	9	3	11	9	2	8	6	0	6	6	0	3	2	1
HMO	2	2	13	5	2	8	3	2	12	6	1	9	9	0	6	12	0	3
HMR	15	15	0	10	10	0	12	11	0	8	8	0	8	6	0	6	3	0
HMSR	0	0	12	3	3	8	4	0	9	7	2	6	9	0	6	7	0	3
HMRSR	15	15	3	10	10	4	11	11	3	8	7	3	6	6	2	5	3	0
GMO	0	0	15	0	0	13	0	0	14	3	0	11	3	0	9	4	0	3
GMR	15	15	0	10	10	0	11	11	1	8	6	0	8	6	0	5	3	0
GMSR	0	0	15	3	2	10	3	0	11	6	0	8	7	1	6	1	0	3
GMRSR	15	15	5	10	10	4	11	10	5	8	6	3	6	6	3	3	3	3
MO	0	0	15	2	1	11	3	1	12	6	0	9	8	1	6	8	0	3
MR	15	15	2	10	10	1	11	9	2	7	6	0	6	6	2	3	3	0
MSR	0	0	13	3	0	8	4	1	10	7	0	6	9	1	6	7	0	3
MRSR	14	14	5	10	10	5	11	8	5	6	6	5	6	6	5	2	3	1

TABLE 3. Number of times the improved ridge parameters estimators are better than the existing ridge parameter $\hat{K}_{HK_i} = \frac{\hat{\sigma}^2}{\hat{\alpha}_i^2}$ estimator with Multicollinearity (5 Levels) and Error Variances (3 levels) effect partial out when the number of regressor is three

P=7																		
methods	10			20			30			40			50			200		
	$\hat{K}_{HKA}$	$\hat{K}_{HKB}$	$\hat{K}_{HKC}$	$\hat{K}_{HKA}$	$\hat{K}_{HKB}$	$\hat{K}_{HKC}$	$\hat{K}_{HKA}$	$\hat{K}_{HKB}$	$\hat{K}_{HKC}$	$\hat{K}_{HKA}$	$\hat{K}_{HKB}$	$\hat{K}_{HKC}$	$\hat{K}_{HKA}$	$\hat{K}_{HKB}$	$\hat{K}_{HKC}$	$\hat{K}_{HKA}$	$\hat{K}_{HKB}$	$\hat{K}_{HKC}$
FMO	0	5	0	11	14	0	13	15	0	15	14	0	15	12	0	15	3	3
FMR	15	15	0	15	15	4	15	12	1	9	8	3	11	9	0	6	5	5
FMSR	0	0	15	2	2	11	6	4	9	9	5	6	9	4	6	5	0	3
FMRSR	15	15	9	15	15	9	12	12	5	8	8	5	9	9	3	6	4	4
VMO	0	0	15	0	0	15	0	0	12	0	0	11	0	0	6	0	0	1
VMR	15	15	0	15	15	4	15	12	1	9	8	3	11	9	0	6	5	5
VMSR	0	0	15	0	0	15	0	0	15	0	0	14	0	0	10	0	0	1
VMRSR	15	15	9	15	15	9	12	12	5	8	8	5	9	9	3	6	4	4
AMO	0	0	15	0	0	15	0	0	15	0	0	15	0	0	8	0	0	0
AMR	15	15	1	10	10	1	6	6	0	6	6	0	6	6	0	3	3	3
AMSR	0	0	15	0	0	15	0	0	15	2	0	12	0	0	11	0	0	2
AMRSR	15	15	7	10	10	4	8	6	1	6	6	1	6	6	1	3	2	2
HMO	0	0	15	0	0	15	0	0	15	6	1	9	3	0	10	5	0	4
HMR	15	15	0	14	14	6	12	11	3	6	6	2	8	7	0	6	3	3
HMSR	0	0	15	0	0	14	3	1	12	7	1	6	6	0	8	2	0	4
HMRSR	15	15	11	14	14	7	12	12	5	6	6	5	9	7	3	6	3	3
GMO	0	0	15	0	0	15	0	0	15	0	0	15	0	0	13	0	0	4
GMR	15	15	7	11	11	6	8	6	4	6	6	2	6	6	3	3	3	3
GMSR	0	0	15	0	0	15	0	0	15	4	1	9	0	0	9	0	0	2
GMRSR	15	15	12	13	12	8	9	9	6	6	6	5	6	6	6	3	3	3
MO	0	0	15	0	0	15	0	0	15	0	0	14	0	0	12	0	0	4
MR	2	3	9	7	7	6	9	9	4	12	11	3	12	12	3	12	0	1
MSR	0	0	15	0	0	15	2	0	13	7	4	8	6	1	8	7	2	3
MRSR	0	0	13	7	7	8	9	9	6	10	10	5	9	9	6	11	0	4

TABLE 4. Number of times the improved ridge parameters estimators are better than the existing ridge parameter $\hat{K}_{HK_i} = \frac{\hat{\sigma}^2}{\hat{\alpha}_i^2}$ estimator with Multicollinearity (5 Levels) and Error Variances (3 levels) effect partial out whne the number of regressor is seven

6. Real life application

To illustrate the theoretical results a real life data set was used. This study uses a secondary data on Gross Domestic Product (GDP), Government capital formation, import, export and foreign direct investment for the year 1981 to 2013 extracted from the publication of World Bank (2015). The long run empirical model for this study is obtained by regressing the natural logarithm of gross domestic product on the natural logarithm of other indicator variables. The condition number is $255.764 > 30$ which shows that there is multicollinearity. Results from simulation study are in line with the results in the application of real life data. The results of the simulation shows that HMO is generally improved by $\frac{1}{nVIF_{max}}$ while FMO is improved by $\frac{1}{\lambda_{max}}$ and $\frac{1}{n\lambda_{max}}$. Results from Table 5 shows that the most improved estimators are HMO, AMRSR, MRSR, [FMO and MO]. It can be seen that HMO has the best performance in terms of the mean square error. The quantity $\frac{1}{nVIF_{max}}$ performs best while the performance of the quantity the quantity $\frac{1}{\lambda_{max}}$ and $\frac{1}{n\lambda_{max}}$ are not too different.

Different Forms	Various Types	Methods	MSE H_{HK}	MSE H_{HKA}	MSE H_{HKB}	MSE H_{HKC}
OLS		OLS	0.049658	0.049658	0.049658	0.049658
fixed Maximum	Original	FMO	0.045205	0.045205	0.045205	0.044086
	Reciprocal	FMR	0.310630	0.310620	0.310630	0.136880
	Square root	FMSR	0.048133	0.048133	0.048133	0.042877
	Reciprocal of Square root	FMRSR	0.171440	0.171430	0.171440	0.089886
Varying Maximum	Original	VMO	0.490560	0.490560	0.490560	0.195200
	Reciprocal	VMR	0.310630	0.310620	0.310630	0.136880
	Square root	VMSR	0.297030	0.297030	0.297030	0.123690
	Reciprocal of Square root	VMRSR	0.171440	0.171430	0.171440	0.089886
Arithmetic Mean	Original	AMO	0.393700	0.393700	0.393700	0.137010
	Reciprocal	AMR	0.046473	0.046473	0.046473	0.044094
	Square root	AMSR	0.214090	0.214090	0.214090	0.089948
	Reciprocal of Square root	AMRSR	0.045742	0.045742	0.045742	0.042873
Harmonic Mean	Original	HMO	0.046850	0.046850	0.046850	0.042612
	Reciprocal	HMR	0.189080	0.189070	0.189080	0.098102
	Square root	HMSR	0.055833	0.055833	0.055833	0.045149
	Reciprocal of Square root	HMRSR	0.123140	0.123140	0.123140	0.072692
Geometric Mean	Original	GMO	0.133200	0.133200	0.133200	0.051878
	Reciprocal	GMR	0.053461	0.053460	0.053461	0.057441
	Square root	GMSR	0.102150	0.102150	0.102150	0.053126
	Reciprocal of Square root	GMRSR	0.063234	0.063234	0.063234	0.055907
Median	Original	MO	0.045205	0.045205	0.045205	0.044086
	Reciprocal	MR	0.048394	0.048394	0.048394	0.047757
	Square root	MSR	0.048133	0.048133	0.048133	0.042880
	Reciprocal of Square root	MRSR	0.045086	0.045086	0.045086	0.043322

TABLE 5

7. Conclusion

In this study, some improved ridge parameters are classified into different forms and various types. The performances of these estimators are evaluated through Monte-Carlo Simulation where levels of multicollinearity, sample sizes and error variances have been varied. The number of times the MSE of the improved ridge parameter is less than the existing ones is counted over the levels of multicollinearity (5) and error variance (3). Also, a maximum of fifteen (15) counts is expected. Having counted over all the levels of multicollinearity, sample sizes and error variances, the best five with the highest counts is selected. Results show that the improved ridge parameters proposed in this study are better than the existing ones most especially with the quantity $\frac{1}{nVIF_{max}}$. Results based on real life application shows that the most improved ridge estimator is HMO which is consistent with the study of Lukman and Ayinde (2015).

References

[1] Alkhamisi, M., Khalaf, G. and Shukur, G. (2006). Some modifications for choosing ridge parameters, *Communications in Statistics- Theory and Methods*, 35(11), 2005-2020.

[2] Alkhamisi, M. and Shukur, G. (2007). A Monte Carlo study of recent ridge parameters, *Commun. Statist. Simulation and Computation*, 36(3), 535-547.

[3] Dorugade, A. V. (2014). On comparison of some ridge parameters in Ridge Regression, *Sri Lankan Journal of Applied Statistics*, 15(1), 31-46.

[4] Dorugade, A. V. and Kashid, D. N. (2010). Alternative method for choosing ridge parameter for regression, *International Journal of Applied Mathematical Sciences*, 4(9), 447-456.

[5] Gibbons, D. G. (1981). A simulation study of some ridge estimators,*Journal of the American Statistical Association*, 76, 131-139.

[6] Gujarati, D.N.(1995). *Basic Econometrics*. McGraw-Hill, New York.

[7] Hocking, R., Speed, F. M. and Lynn, M. J. (1976). A class of biased estimators in linear regression, *Technometrics*, 18(4), 425-437.

[8] Hoerl, A.E. and Kennard, R.W. (1970). Ridge regression: biased estimation for non-orthogonal problems, *Technometrics*, 12, 55-67.

[9] Khalaf, G. and Shukur, G. (2005). Choosing ridge parameters for regression problems, *Communications in Statistics- Theory and Methods*, 34, 1177-1182.

[10] Kibria, B. M. G. (2003). Performance of some new ridge regression estimators, *Communications in Statistics-Simulation and Computation*, 32, 419-435.

[11] Lawless, J. F. and Wang, P. (1976). A simulation study of ridge and other regression estimators, *Communications in Statistics A*, 5, 307-323.

[12] Lukman, A. F. and Ayinde, K. (2015). Review and classification of the Ridge Parameter Estimation Techniques, *Hacettepe Journal of Mathematics and Statistics*, Accepted for Publication.

[13] Mansson, K., Shukur, G. and Kibria, B. M. G. (2010). A simulation study of some ridge regression estimators under different distributional assumptions,*Communications in Statistics-Simulations and Computations*, 39(8), 1639-1670.

[14] McDonald, G. C. and Galarneau, D. I. (1975). A Monte Carlo evaluation of some ridge-type estimators, *Journal of the American Statistical Association*, 70, 407-416.

[15] Muniz, G. and Kibria, B. M. G. (2009). On some ridge regression estimators: An empirical comparison,*Communications in Statistics-Simulation and Computation*, 38, 621-630.

[16] Muniz, G., Kibria, B.M.G., Mansson, K., Shukur, G. (2012). On Developing Ridge Regression Parameters: A Graphical Investigation, *SORT*. 36(2), 115-138.

[17] Nordberg, L. (1982). A procedure for determination of a good ridge parameter in linear regression, *Communications in Statistics A11*, 285-309.

[18] Saleh, A. K. Md. E. and Kibria, B. M. G. (1993). Performances of some new preliminary test ridge regression estimators and their properties, *Communications in Statistics-Theory and Methods*, 22, 2747-2764.

[19] Wichern, D. and Churchill, G. (1978). A Comparison of Ridge Estimators, *Technometrics*, 20, 301–311.

Appendix 1: MSE of OLS and relative efficiency of the Ridge parameter based on $\hat{K}_{HKA}$

n=10 p=3															
Method	$\sigma = 0.5$					$\sigma = 1$					$\sigma = 5$				
	0.8	0.9	0.95	0.99	0.999	0.8	0.9	0.95	0.99	0.999	0.8	0.9	0.95	0.99	0.999
MSE OLS	0.920	1.908	4.030	22.661	248.616	0.985	2.043	4.315	24.262	266.182	3.034	6.292	13.291	74.738	819.954
FMO	0.517	0.435	0.398	0.394	0.673	0.509	0.430	0.394	0.390	0.656	0.474	0.430	0.403	0.377	0.495
FMR	0.761	0.826	0.921	0.997	0.997	0.764	0.817	0.902	0.968	0.965	0.736	0.698	0.718	0.706	0.666
FMSR	0.533	0.449	0.440	0.623	0.914	0.528	0.445	0.436	0.611	0.885	0.507	0.445	0.428	0.501	0.622
FMRSR	0.657	0.682	0.796	0.966	0.994	0.660	0.677	0.781	0.939	0.962	0.651	0.601	0.635	0.686	0.665
VMO	0.867	0.734	0.732	0.881	0.980	0.847	0.723	0.724	0.862	0.950	0.818	0.730	0.695	0.679	0.663
VMR	0.761	0.826	0.921	0.997	0.997	0.764	0.817	0.902	0.968	0.965	0.736	0.698	0.718	0.706	0.666
VMSR	0.588	0.599	0.672	0.881	0.982	0.571	0.591	0.664	0.860	0.951	0.615	0.621	0.637	0.668	0.662
VMRSR	0.657	0.682	0.796	0.966	0.994	0.660	0.677	0.781	0.939	0.962	0.651	0.601	0.635	0.686	0.665
AMO	0.704	0.616	0.621	0.782	0.957	0.683	0.607	0.615	0.769	0.928	0.725	0.665	0.644	0.648	0.657
AMR	0.896	0.843	0.830	0.918	0.988	0.895	0.837	0.818	0.896	0.957	0.933	0.858	0.775	0.702	0.676
AMSR	0.537	0.543	0.610	0.834	0.973	0.523	0.536	0.603	0.815	0.943	0.569	0.578	0.601	0.649	0.659
AMRSR	0.790	0.706	0.708	0.885	0.985	0.789	0.702	0.702	0.862	0.953	0.832	0.715	0.636	0.625	0.650
HMO	0.443	0.408	0.402	0.438	0.698	0.439	0.406	0.399	0.433	0.680	0.456	0.433	0.420	0.413	0.514
HMR	0.798	0.800	0.880	0.989	0.996	0.801	0.796	0.863	0.960	0.964	0.803	0.705	0.692	0.702	0.666
HMSR	0.490	0.448	0.476	0.672	0.919	0.486	0.444	0.472	0.659	0.964	0.478	0.449	0.459	0.537	0.627
HMRSR	0.687	0.664	0.756	0.955	0.994	0.689	0.661	0.743	0.928	0.962	0.695	0.601	0.608	0.678	0.665
GMO	0.488	0.478	0.502	0.620	0.822	0.480	0.475	0.499	0.611	0.801	0.574	0.560	0.558	0.570	0.607
GMR	0.856	0.812	0.839	0.965	0.995	0.855	0.807	0.828	0.937	0.963	0.889	0.787	0.704	0.670	0.665
GMSR	0.470	0.476	0.540	0.761	0.943	0.467	0.473	0.535	0.745	0.914	0.497	0.506	0.538	0.608	0.645
GMRSR	0.739	0.672	0.720	0.928	0.992	0.738	0.667	0.710	0.901	0.960	0.768	0.644	0.589	0.646	0.662
MO	0.456	0.461	0.518	0.647	0.798	0.451	0.460	0.519	0.640	0.776	0.549	0.552	0.569	0.575	0.586
MR	0.819	0.762	0.797	0.948	0.994	0.815	0.755	0.779	0.919	0.962	0.845	0.740	0.661	0.660	0.664
MSR	0.470	0.473	0.548	0.766	0.938	0.469	0.471	0.546	0.750	0.909	0.501	0.507	0.545	0.608	0.641
MRSR	0.715	0.651	0.704	0.922	0.992	0.713	0.644	0.690	0.895	0.960	0.737	0.619	0.571	0.644	0.663
n=20															
MSE OLS	0.468	0.928	1.889	10.031	106.098	0.482	0.956	1.947	10.338	109.346	0.952	1.888	3.845	20.420	215.986
FMO	0.372	0.519	0.441	0.383	0.495	0.634	0.520	0.444	0.390	0.513	0.613	0.561	0.536	0.522	0.735
FMR	0.762	0.722	0.771	0.916	0.935	0.768	0.738	0.799	0.962	0.984	0.801	0.783	1.002	1.507	1.580
FMSR	0.661	0.535	0.453	0.489	0.783	0.661	0.536	0.458	0.505	0.822	0.652	0.576	0.550	0.713	1.282
FMRSR	0.724	0.638	0.649	0.856	0.930	0.727	0.647	0.669	0.898	0.978	0.753	0.688	0.818	1.393	1.571
VMO	1.050	0.799	0.722	0.786	0.909	1.057	0.815	0.737	0.823	0.957	1.013	0.989	1.093	1.389	1.558
VMR	0.762	0.722	0.771	0.916	0.935	0.768	0.738	0.799	0.962	0.984	0.801	0.783	1.002	1.507	1.580
VMSR	0.724	0.638	2.647	0.856	0.930	0.727	0.647	0.669	0.898	0.978	0.753	0.688	0.818	1.393	1.571
VMRSR	0.613	0.557	0.586	0.759	0.907	0.608	0.570	0.602	0.794	0.955	0.654	0.701	0.853	1.292	1.546
AMO	0.819	0.653	0.608	0.682	0.871	0.824	0.667	0.618	0.709	0.917	0.799	0.778	0.868	1.211	1.520
AMR	0.926	0.859	0.794	0.811	0.907	0.927	0.863	0.803	0.841	0.957	0.924	0.834	0.764	1.029	1.480
AMSR	0.581	0.517	0.534	0.703	0.892	0.574	0.527	0.547	0.733	0.938	0.618	0.628	0.737	1.179	1.524
AMRSR	0.861	0.760	0.678	0.749	0.907	0.863	0.765	0.686	0.778	0.954	0.854	0.744	0.694	1.061	1.505
HMO	0.531	0.452	0.414	0.407	0.537	0.530	0.455	0.419	0.415	0.557	0.558	0.538	0.538	0.565	0.802
HMR	0.838	0.761	0.749	0.892	0.934	0.841	0.770	0.771	0.936	0.983	0.860	0.775	0.872	1.462	1.579
HMSR	0.602	0.496	0.451	0.544	0.800	0.602	0.499	0.457	0.563	0.840	0.609	0.560	0.573	0.805	1.309
HMRSR	0.773	0.666	0.633	0.829	0.928	0.775	0.672	0.649	0.869	0.977	0.792	0.695	0.753	1.343	1.569
GMO	0.541	0.482	0.480	0.551	0.715	0.537	0.490	0.490	0.566	0.746	0.604	0.609	0.655	0.826	1.151
GMR	0.895	0.816	0.760	0.854	0.930	0.897	0.822	0.775	0.891	0.979	0.899	0.799	0.789	1.335	1.573
GMSR	0.543	0.477	0.476	0.632	0.847	0.544	0.481	0.487	0.656	0.890	0.583	0.571	0.633	0.975	1.408
GMRSR	0.823	0.711	0.642	0.787	0.923	0.824	0.717	0.654	0.824	0.971	0.825	0.714	0.710	1.241	1.560
MO	0.525	0.467	0.471	0.590	0.695	0.526	0.471	0.479	0.607	0.722	0.575	0.583	0.628	0.732	0.934
MR	0.862	0.765	0.699	0.817	0.927	0.864	0.772	0.715	0.853	0.975	0.877	0.785	0.824	1.379	1.571
MSR	0.563	0.485	0.479	0.647	0.840	0.563	0.489	0.489	0.670	0.881	0.591	0.567	0.616	0.901	1.344
MRSR	0.798	0.682	0.611	0.773	0.922	0.800	0.686	0.623	0.809	0.970	0.808	0.703	0.725	1.285	1.563

n=30 p=3															
Method	$\sigma = 0.5$					$\sigma = 1$					$\sigma = 5$				
	0.8	0.9	0.95	0.99	0.999	0.8	0.9	0.95	0.99	0.999	0.8	0.9	0.95	0.99	0.999
MSE OLS	0.386	0.804	1.700	9.536	104.240	0.396	0.826	1.746	9.794	107.059	0.732	1.526	3.226	18.094	197.792
FMO	0.638	0.522	0.000	0.397	0.476	0.633	0.516	0.444	0.393	0.469	0.472	0.376	0.325	0.292	0.319
FMR	0.757	0.731	0.839	1.031	1.049	0.759	0.726	0.823	1.007	1.025	0.833	0.728	0.651	0.592	0.590
FMSR	0.663	0.539	0.472	0.541	0.855	0.659	0.535	0.467	0.533	0.836	0.558	0.425	0.351	0.354	0.492
FMRSR	0.722	0.646	0.707	0.967	1.044	0.723	0.642	0.695	0.945	1.020	0.765	0.625	0.534	0.550	0.587
VMO	1.007	0.829	0.799	0.904	1.021	0.994	0.817	0.786	0.887	0.999	0.682	0.549	0.512	0.539	0.581
VMR	0.757	0.731	0.839	1.031	1.049	0.759	0.726	0.823	1.007	1.025	0.833	0.728	0.651	0.592	0.590
VMSR	0.722	0.646	0.707	0.967	1.044	0.723	0.642	0.695	0.945	1.020	0.765	0.625	0.534	0.550	0.587
VMRSR	0.607	0.584	0.660	0.871	1.018	0.595	0.576	0.648	0.853	0.995	0.439	0.405	0.424	0.511	0.577
AMO	0.798	0.677	0.678	0.804	0.983	0.786	0.669	0.668	0.790	0.962	0.552	0.464	0.449	0.493	0.565
AMR	0.913	0.844	0.797	0.880	1.032	0.914	0.844	0.793	0.865	1.000	0.945	0.880	0.785	0.584	0.571
AMSR	0.584	0.538	0.601	0.817	1.001	0.574	0.532	0.592	0.801	0.979	0.440	0.389	0.398	0.486	0.569
AMRSR	0.850	0.753	0.691	0.824	1.020	0.851	0.752	0.686	0.807	0.992	0.878	0.769	0.638	0.501	0.566
HMO	0.540	0.468	0.441	0.444	0.544	0.535	0.464	0.437	0.439	0.535	0.397	0.343	0.322	0.317	0.356
HMR	0.831	0.754	0.789	1.004	1.048	0.833	0.752	0.778	0.981	1.024	0.892	0.793	0.690	0.589	0.590
HMSR	0.608	0.510	0.485	0.620	0.883	0.604	0.505	0.480	0.610	0.863	0.498	0.391	0.349	0.392	0.508
HMRSR	0.769	0.665	0.673	0.937	1.042	0.770	0.662	0.663	0.915	1.018	0.810	0.671	0.552	0.537	0.586
GMO	0.537	0.504	0.539	0.659	0.818	0.530	0.499	0.534	0.649	0.803	0.401	0.371	0.377	0.425	0.493
GMR	0.884	0.804	0.773	0.946	1.044	0.886	0.804	0.766	0.926	1.018	0.926	0.845	0.736	0.578	0.586
GMSR	0.555	0.496	0.530	0.740	0.952	0.550	0.491	0.524	0.726	0.931	0.445	0.373	0.366	0.449	0.545
GMRSR	0.814	0.706	0.664	0.880	1.035	0.815	0.705	0.656	0.861	1.011	0.848	0.721	0.586	0.512	0.581
MO	0.526	0.486	0.530	0.713	0.815	0.521	0.482	0.525	0.701	0.798	0.397	0.363	0.379	0.449	0.484
MR	0.850	0.749	0.707	0.899	1.039	0.851	0.748	0.699	0.879	1.014	0.903	0.803	0.679	0.542	0.582
MSR	0.578	0.505	0.532	0.763	0.947	0.574	0.501	0.525	0.748	0.926	0.471	0.386	0.374	0.460	0.540
MRSR	0.788	0.674	0.635	0.860	1.033	0.789	0.673	0.628	0.841	1.009	0.826	0.688	0.550	0.498	0.580
n=50															
MSE OLS	0.165	0.333	0.686	3.711	39.738	0.168	0.338	0.696	3.768	40.352	0.249	0.502	1.035	5.604	60.002
FMO	0.826	0.711	0.586	0.432	0.406	0.824	0.708	0.583	0.429	0.404	0.736	0.600	0.476	0.350	0.331
FMR	0.894	0.807	0.744	0.953	1.051	0.895	0.809	0.742	0.940	1.036	0.925	0.854	0.759	0.691	0.717
FMSR	0.847	0.734	0.603	0.453	0.666	0.846	0.733	0.600	0.450	0.659	0.805	0.672	0.528	0.363	0.481
FMRSR	0.881	0.782	0.677	0.819	1.037	0.881	0.782	0.676	0.808	1.022	0.900	0.808	0.680	0.589	0.707
VMO	1.810	1.123	0.863	0.789	0.988	1.794	1.109	0.854	0.781	0.975	1.371	0.860	0.662	0.585	0.686
VMR	0.894	0.807	0.744	0.953	1.051	0.895	0.809	0.742	0.940	1.036	0.925	0.854	0.759	0.691	0.717
VMSR	0.881	0.782	0.677	0.819	1.037	0.881	0.782	0.676	0.808	1.022	0.900	0.808	0.680	0.589	0.707
VMRSR	0.807	0.659	0.597	0.713	0.979	0.814	0.649	0.590	0.705	0.966	0.684	0.537	0.475	0.522	0.676
AMO	1.333	0.870	0.703	0.670	0.907	1.322	0.858	0.695	0.664	0.896	1.021	0.681	0.555	0.510	0.642
AMR	0.979	0.946	0.883	0.783	39.105	0.979	0.947	0.884	0.777	0.968	0.985	0.960	0.907	0.707	0.673
AMSR	0.768	0.637	0.561	0.646	0.941	0.776	0.629	0.555	0.639	0.929	0.681	0.536	0.458	0.480	0.654
AMRSR	0.953	0.897	0.804	0.686	0.966	0.953	0.897	0.804	0.679	0.953	0.960	0.910	0.821	0.584	0.657
HMO	0.727	0.598	0.499	0.425	0.437	0.724	0.595	0.496	0.422	0.434	0.623	0.492	0.405	0.345	0.350
HMR	0.941	0.877	0.789	0.887	1.047	0.941	0.878	0.789	0.876	1.032	0.959	0.910	0.823	0.679	0.715
HMSR	0.802	0.676	0.553	0.490	0.738	0.800	0.674	0.551	0.486	0.729	0.751	0.607	0.474	0.383	0.528
HMRSR	0.910	0.826	0.713	0.762	1.030	0.911	0.827	0.712	0.752	1.015	0.925	0.849	0.728	0.568	0.702
GMO	0.697	0.582	0.520	0.545	0.692	0.700	0.575	0.516	0.541	0.685	0.599	0.480	0.427	0.430	0.515
GMR	0.966	0.922	0.845	0.811	1.029	0.967	0.923	0.846	0.801	1.015	0.976	0.943	0.876	0.677	0.702
GMSR	0.737	0.613	0.522	0.568	0.859	0.735	0.611	0.519	0.563	0.848	0.681	0.541	0.439	0.432	0.604
GMRSR	0.935	0.866	0.760	0.705	1.009	0.935	0.866	0.760	0.697	0.995	0.945	0.884	0.778	0.558	0.686
MO	0.686	0.565	0.492	0.571	0.695	0.684	0.562	0.490	0.567	0.688	0.588	0.471	0.407	0.447	0.514
MR	0.949	0.892	0.797	0.746	1.017	0.950	0.893	0.798	0.738	1.002	0.965	0.920	0.834	0.612	0.690
MSR	0.764	0.632	0.529	0.581	0.849	0.763	0.630	0.526	0.576	0.839	0.710	0.565	0.453	0.441	0.598
MRSR	0.921	0.845	0.732	0.681	1.004	0.921	0.845	0.731	0.673	0.990	0.934	0.864	0.747	0.526	0.682

n=200 p=3															
Method	$\sigma = 0.5$					$\sigma = 1$					$\sigma = 5$				
	0.8	0.9	0.95	0.99	0.999	0.8	0.9	0.95	0.99	0.999	0.8	0.9	0.95	0.99	0.999
MSE OLS	0.036	0.073	0.150	0.814	8.746	0.036	0.073	0.151	0.817	8.780	0.041	0.082	0.169	0.919	9.872
FMO	0.035	0.067	0.126	0.459	3.713	0.035	0.067	0.126	0.460	3.721	0.039	0.074	0.139	0.492	3.996
FMR	0.035	0.069	0.134	0.591	9.108	0.035	0.069	0.135	0.593	9.120	0.040	0.078	0.153	0.660	9.405
FMSR	0.035	0.067	0.128	0.470	4.259	0.035	0.068	0.129	0.471	4.266	0.039	0.076	0.143	0.512	4.521
FMRSR	0.035	0.068	0.132	0.531	8.440	0.035	0.069	0.133	0.533	8.451	0.040	0.078	0.150	0.593	8.699
VMO	0.167	0.204	0.239	0.663	7.648	0.166	0.205	0.240	0.666	7.666	0.172	0.216	0.256	0.696	8.087
VMR	0.035	0.069	0.134	0.591	9.108	0.035	0.069	0.135	0.593	9.120	0.040	0.078	0.153	0.660	9.405
VMSR	0.043	0.069	0.113	0.494	7.293	0.044	0.069	0.115	0.493	7.308	0.046	0.074	0.125	0.516	7.658
VMRSR	0.035	0.068	0.132	0.531	8.440	0.035	0.069	0.133	0.533	8.451	0.040	0.078	0.150	0.593	8.699
AMO	0.111	0.141	0.176	0.559	6.468	0.111	0.141	0.179	0.561	6.485	0.114	0.148	0.193	0.584	6.908
AMR	0.036	0.072	0.147	0.709	7.722	0.036	0.073	0.148	0.712	7.718	0.041	0.082	0.166	0.806	7.950
AMSR	0.039	0.065	0.110	0.461	6.665	0.040	0.065	0.112	0.459	6.680	0.043	0.070	0.122	0.482	7.011
AMRSR	0.036	0.071	0.143	0.641	7.067	0.036	0.072	0.144	0.644	7.054	0.041	0.081	0.162	0.727	7.278
HMO	0.033	0.061	0.110	0.392	3.736	0.033	0.062	0.110	0.393	3.743	0.037	0.068	0.119	0.419	4.012
HMR	0.036	0.071	0.141	0.620	8.795	0.036	0.071	0.142	0.622	8.806	0.040	0.080	0.161	0.704	9.081
HMSR	0.034	0.065	0.121	0.433	4.917	0.034	0.065	0.121	0.434	4.925	0.039	0.073	0.135	0.469	5.183
HMRSR	0.036	0.070	0.137	0.557	8.057	0.036	0.070	0.138	0.559	8.067	0.040	0.079	0.155	0.628	8.301
GMO	0.034	0.058	0.099	0.424	5.125	0.035	0.059	0.100	0.422	5.141	0.038	0.063	0.108	0.449	5.500
GMR	0.036	0.072	0.145	0.671	8.182	0.036	0.072	0.146	0.675	8.171	0.041	0.081	0.165	0.765	8.462
GMSR	0.033	0.061	0.110	0.416	5.902	0.034	0.062	0.110	0.415	5.915	0.037	0.068	0.121	0.444	6.203
GMRSR	0.036	0.071	0.141	0.598	7.485	0.036	0.071	0.141	0.601	7.486	0.040	0.080	0.160	0.679	7.707
MO	0.032	0.057	0.098	0.390	5.614	0.032	0.057	0.098	0.391	5.626	0.036	0.062	0.106	0.420	5.916
MR	0.036	0.071	0.143	0.626	7.682	0.036	0.072	0.144	0.628	7.686	0.041	0.081	0.162	0.714	7.892
MSR	0.034	0.063	0.114	0.417	6.066	0.034	0.063	0.114	0.418	6.076	0.038	0.070	0.126	0.451	6.331
MRSR	0.036	0.070	0.139	0.572	7.293	0.036	0.070	0.140	0.575	7.300	0.040	0.079	0.157	0.646	7.512
n=500															
MSE OLS	0.015	0.030	0.063	0.340	3.654	0.015	0.031	0.063	0.341	3.660	0.016	0.032	0.066	0.357	3.836
FMO	0.015	0.029	0.058	0.238	1.565	0.015	0.029	0.058	0.238	1.566	0.016	0.031	0.060	0.246	1.619
FMR	0.015	0.030	0.060	0.267	3.373	0.015	0.030	0.060	0.267	3.371	0.016	0.031	0.063	0.282	3.412
FMSR	0.015	0.029	0.058	0.245	1.633	0.015	0.029	0.058	0.245	1.634	0.016	0.031	0.061	0.255	1.686
FMRSR	0.015	0.030	0.059	0.259	2.890	0.015	0.030	0.059	0.260	2.888	0.016	0.031	0.062	0.273	2.925
VMO	0.252	0.204	0.189	0.317	2.771	0.253	0.203	0.189	0.317	2.770	0.254	0.203	0.185	0.320	2.850
VMR	0.015	0.030	0.060	0.267	3.373	0.015	0.030	0.060	0.267	3.371	0.016	0.031	0.063	0.282	3.412
VMSR	0.028	0.038	0.060	0.207	2.504	0.030	0.037	0.059	0.206	2.503	0.032	0.038	0.062	0.210	2.575
VMRSR	0.015	0.030	0.059	0.259	2.890	0.015	0.030	0.059	0.260	2.888	0.016	0.031	0.062	0.273	2.925
AMO	0.161	0.131	0.133	0.257	2.321	0.162	0.130	0.132	0.256	2.321	0.164	0.130	0.128	0.259	2.405
AMR	0.015	0.030	0.062	0.320	2.864	0.015	0.030	0.062	0.321	2.863	0.016	0.032	0.066	0.337	2.949
AMSR	0.022	0.033	0.057	0.202	2.252	0.024	0.033	0.056	0.202	2.252	0.025	0.034	0.059	0.206	2.328
AMRSR	0.015	0.030	0.062	0.303	2.496	0.015	0.030	0.062	0.303	2.496	0.016	0.032	0.065	0.318	2.571
HMO	0.015	0.028	0.053	0.197	1.536	0.015	0.028	0.054	0.197	1.537	0.015	0.029	0.056	0.203	1.590
HMR	0.015	0.030	0.061	0.294	3.124	0.015	0.030	0.061	0.295	3.122	0.016	0.032	0.064	0.310	3.168
HMSR	0.015	0.029	0.057	0.224	1.765	0.015	0.029	0.057	0.224	1.765	0.016	0.030	0.059	0.233	1.816
HMRSR	0.015	0.030	0.060	0.276	2.680	0.015	0.030	0.060	0.276	2.679	0.016	0.031	0.063	0.290	2.720
GMO	0.017	0.028	0.050	0.185	1.903	0.017	0.029	0.050	0.185	1.904	0.018	0.030	0.052	0.189	1.990
GMR	0.015	0.030	0.062	0.312	2.915	0.015	0.030	0.062	0.312	2.915	0.016	0.032	0.065	0.328	2.972
GMSR	0.015	0.028	0.053	0.200	1.999	0.015	0.028	0.053	0.201	1.999	0.015	0.029	0.056	0.208	2.065
GMRSR	0.015	0.030	0.061	0.291	2.524	0.015	0.030	0.061	0.292	2.522	0.016	0.032	0.064	0.306	2.567
MO	0.014	0.027	0.050	0.182	1.990	0.014	0.027	0.050	0.182	1.992	0.015	0.028	0.052	0.188	2.063
MR	0.015	0.030	0.062	0.300	2.668	0.015	0.030	0.062	0.300	2.667	0.016	0.032	0.065	0.316	2.709
MSR	0.015	0.029	0.055	0.209	2.040	0.015	0.029	0.055	0.209	2.041	0.016	0.030	0.058	0.217	2.098
MRSR	0.015	0.030	0.061	0.283	2.429	0.015	0.030	0.061	0.283	2.427	0.016	0.031	0.064	0.297	2.468

Appendix 2: MSE of OLS and relative efficiency of the Ridge parameter based on $\hat{K}_{HK}$

n=10 p=3															
Method	$\sigma=0.5$					$\sigma=1$					$\sigma=5$				
	0.8	0.9	0.95	0.99	0.999	0.8	0.9	0.95	0.99	0.999	0.8	0.9	0.95	0.99	0.999
MSE OLS	0.920	1.908	1.889	22.661	248.616	0.985	2.043	4.315	24.262	266.182	3.034	6.292	13.291	74.738	819.954
FMO	0.520	0.438	0.401	0.380	0.374	0.512	0.433	0.397	0.376	0.370	0.475	0.431	0.405	0.373	0.354
FMR	0.771	0.840	0.935	1.003	0.998	0.773	0.831	0.915	0.974	0.966	0.739	0.705	0.724	0.708	0.667
FMSR	0.534	0.447	0.428	0.538	0.764	0.529	0.443	0.424	0.529	0.744	0.508	0.444	0.422	0.463	0.557
FMRSR	0.662	0.692	0.811	0.975	0.996	0.664	0.687	0.795	0.947	0.964	0.652	0.605	0.642	0.691	0.666
VMO	0.866	0.732	0.729	0.878	0.980	0.846	0.722	0.721	0.859	0.949	0.818	0.729	0.695	0.678	0.663
VMR	0.771	0.840	0.935	1.003	0.998	0.773	0.831	0.915	0.974	0.966	0.739	0.705	0.724	0.708	0.667
VMSR	0.587	0.598	0.671	0.880	0.982	0.571	0.590	0.663	0.859	0.951	0.615	0.621	0.637	0.667	0.662
VMRSR	0.662	0.692	0.8117	0.975	0.996	0.664	0.687	0.795	0.947	0.964	0.652	0.605	0.642	0.691	0.666
AMO	0.703	0.614	0.617	0.773	0.952	0.682	0.605	0.611	0.760	0.925	0.725	0.664	0.643	0.646	0.656
AMR	0.897	0.847	0.834	0.921	0.988	0.896	0.840	0.822	0.898	0.957	0.933	0.859	0.776	0.703	0.676
AMSR	0.537	0.542	0.607	0.830	0.972	0.522	0.535	0.600	0.811	0.942	0.569	0.578	0.601	0.649	0.659
AMRSR	0.791	0.708	0.711	0.887	0.985	0.789	0.704	0.705	0.864	0.954	0.832	0.716	0.637	0.626	0.650
HMO	0.443	0.407	0.397	0.397	0.398	0.439	0.404	0.394	0.394	0.395	0.455	0.432	0.416	0.392	0.375
HMR	0.803	0.809	0.889	0.993	0.997	0.805	0.804	0.872	0.964	0.965	0.804	0.707	0.696	0.704	0.667
HMSR	0.490	0.445	0.468	0.627	0.832	0.486	0.442	0.464	0.616	0.809	0.478	0.448	0.456	0.519	0.592
HMRSR	0.689	0.669	0.764	0.961	0.995	0.691	0.665	0.750	0.933	0.963	0.695	0.602	0.612	0.681	0.666
GMO	0.487	0.476	0.497	0.595	0.736	0.479	0.472	0.493	0.588	0.721	0.573	0.559	0.556	0.562	0.584
GMR	0.858	0.817	0.845	0.968	0.996	0.857	0.812	0.834	0.940	0.964	0.889	0.788	0.705	0.671	0.665
GMSR	0.470	0.474	0.536	0.748	0.927	0.467	0.471	0.531	0.733	0.899	0.496	0.506	0.536	0.604	0.641
GMRSR	0.740	0.675	0.724	0.932	0.993	0.738	0.670	0.715	0.905	0.961	0.768	0.644	0.590	0.647	0.663
MO	0.455	0.459	0.513	0.628	0.667	0.451	0.458	0.514	0.621	0.654	0.549	0.551	0.568	0.569	0.545
MR	0.823	0.767	0.804	0.951	0.995	0.818	0.760	0.785	0.922	0.963	0.846	0.741	0.663	0.661	0.665
MSR	0.470	0.471	0.544	0.750	0.900	0.468	0.470	0.542	0.736	0.875	0.501	0.507	0.543	0.604	0.630
MRSR	0.716	0.654	0.709	0.925	0.992	0.714	0.647	0.694	0.898	0.961	0.737	0.620	0.572	0.645	0.663
n=20															
MSE OLS	0.468	0.928	4.030	10.031	106.098	0.482	0.956	1.947	10.338	109.346	0.952	1.888	3.845	20.420	215.986
FMO	0.639	0.522	0.443	0.386	0.446	0.636	0.522	0.447	0.393	0.385	0.614	0.563	0.538	0.525	0.525
FMR	0.762	0.727	0.781	0.923	0.855	0.769	0.743	0.810	0.969	0.986	0.801	0.789	1.019	1.519	1.582
FMSR	0.663	0.535	0.452	0.458	0.634	0.662	0.537	0.456	0.470	0.684	0.653	0.577	0.548	0.666	1.050
FMRSR	0.723	0.640	0.656	0.866	0.853	0.727	0.649	0.677	0.909	0.981	0.752	0.690	0.828	1.412	1.576
VMO	1.049	0.799	0.720	0.784	0.839	1.057	0.814	0.736	0.821	0.956	1.012	0.988	1.091	1.388	1.558
VMR	0.762	0.727	0.781	0.923	0.855	0.769	0.743	0.810	0.969	0.986	0.801	0.789	1.019	1.519	1.582
VMSR	0.613	0.557	0.585	0.758	0.837	0.608	0.569	0.601	0.793	0.955	0.654	0.701	0.852	1.292	1.546
VMRSR	0.723	0.640	0.656	0.866	0.853	0.727	0.649	0.677	0.909	0.981	0.752	0.690	0.828	1.412	1.576
AMO	0.819	0.652	0.606	0.677	0.797	0.824	0.666	0.617	0.704	0.913	0.798	0.776	0.866	1.206	1.518
AMR	0.926	0.859	0.796	0.814	0.820	0.927	0.864	0.805	0.844	0.958	0.923	0.834	0.765	1.033	1.481
AMSR	0.581	0.517	0.533	0.700	0.820	0.574	0.526	0.546	0.730	0.937	0.618	0.627	0.736	1.176	1.523
AMRSR	0.861	0.760	0.679	0.751	0.828	0.863	0.765	0.687	0.780	0.954	0.854	0.744	0.695	1.064	1.505
HMO	0.532	0.452	0.413	0.395	0.460	0.531	0.455	0.418	0.402	0.404	0.558	0.538	0.535	0.545	0.553
HMR	0.838	0.762	0.755	0.897	0.854	0.841	0.772	0.777	0.942	0.985	0.859	0.777	0.881	1.473	1.581
HMSR	0.602	0.496	0.449	0.527	0.714	0.602	0.499	0.455	0.544	0.765	0.609	0.560	0.571	0.778	1.176
HMRSR	0.772	0.666	0.636	0.835	0.851	0.775	0.673	0.653	0.876	0.979	0.792	0.695	0.758	1.355	1.574
GMO	0.541	0.482	0.478	0.541	0.670	0.537	0.489	0.489	0.556	0.697	0.603	0.607	0.652	0.810	1.070
GMR	0.895	0.817	0.763	0.858	0.852	0.897	0.823	0.778	0.895	0.979	0.898	0.800	0.792	1.343	1.574
GMSR	0.543	0.477	0.475	0.627	0.780	0.544	0.481	0.485	0.650	0.877	0.583	0.570	0.631	0.965	1.386
GMRSR	0.823	0.712	0.643	0.790	0.847	0.824	0.717	0.655	0.828	0.973	0.825	0.714	0.712	1.247	1.562
MO	0.525	0.466	0.469	0.583	0.647	0.527	0.470	0.477	0.600	0.659	0.574	0.582	0.625	0.717	0.757
MR	0.862	0.766	0.702	0.821	0.852	0.864	0.772	0.718	0.857	0.976	0.877	0.786	0.830	1.388	1.573
MSR	0.563	0.485	0.478	0.641	0.770	0.563	0.489	0.487	0.664	0.853	0.591	0.567	0.614	0.883	1.253
MRSR	0.798	0.682	0.612	0.776	0.847	0.800	0.686	0.625	0.812	0.971	0.808	0.703	0.728	1.295	1.567

n=30 p=3															
Method	$\sigma=0.5$					$\sigma=1$					$\sigma=5$				
	0.8	0.9	0.95	0.99	0.999	0.8	0.9	0.95	0.99	0.999	0.8	0.9	0.95	0.99	0.999
MSE OLS	0.386	0.804	1.700	9.536	104.240	0.396	0.826	1.746	9.794	107.059	0.732	1.526	3.226	4.687	197.792
FMO	0.639	0.523	0.449	0.397	0.389	0.634	0.517	0.445	0.393	0.385	0.472	0.376	0.325	0.292	0.286
FMR	0.757	0.734	0.844	1.034	1.050	0.759	0.728	0.829	1.010	1.026	0.833	0.729	0.653	0.594	0.591
FMSR	0.663	0.540	0.470	0.519	0.752	0.660	0.535	0.466	0.512	0.738	0.559	0.426	0.351	0.346	0.451
FMRSR	0.722	0.647	0.711	0.973	1.046	0.723	0.643	0.699	0.951	1.021	0.764	0.625	0.536	0.552	0.588
VMO	1.006	0.828	0.798	0.903	1.021	0.994	0.816	0.786	0.886	0.998	0.682	0.549	0.512	0.538	0.580
VMR	0.757	0.734	0.844	1.034	1.050	0.759	0.728	0.829	1.010	1.026	0.833	0.729	0.653	0.594	0.591
VMSR	0.722	0.647	0.711	0.973	1.046	0.723	0.643	0.699	0.951	1.021	0.764	0.625	0.536	0.552	0.588
VMRSR	0.607	0.584	0.659	0.870	1.018	0.595	0.576	0.648	0.853	0.995	0.439	0.405	0.424	0.511	0.577
AMO	0.797	0.676	0.677	0.802	0.982	0.786	0.668	0.667	0.788	0.961	0.552	0.464	0.449	0.493	0.565
AMR	0.913	0.844	0.798	0.882	1.032	0.914	0.844	0.793	0.866	1.001	0.945	0.880	0.785	0.585	0.571
AMSR	0.584	0.538	0.600	0.816	1.001	0.574	0.532	0.591	0.800	0.978	0.440	0.389	0.397	0.485	0.569
AMRSR	0.850	0.753	0.692	0.825	1.020	0.851	0.752	0.687	0.808	0.992	0.878	0.768	0.638	0.501	0.566
HMO	0.541	0.468	0.440	0.435	0.438	0.535	0.464	0.436	0.430	0.434	0.397	0.343	0.322	0.313	0.312
HMR	0.831	0.754	0.792	1.007	1.049	0.833	0.753	0.781	0.984	1.025	0.892	0.793	0.690	0.591	0.590
HMSR	0.608	0.509	0.484	0.610	0.836	0.604	0.505	0.479	0.600	0.818	0.498	0.391	0.349	0.388	0.491
HMRSR	0.769	0.665	0.675	0.940	1.043	0.770	0.663	0.665	0.918	1.019	0.810	0.671	0.552	0.538	0.587
GMO	0.537	0.503	0.538	0.654	0.797	0.530	0.498	0.533	0.644	0.783	0.401	0.371	0.377	0.423	0.486
GMR	0.884	0.805	0.775	0.948	1.044	0.885	0.804	0.768	0.929	1.019	0.926	0.845	0.736	0.579	0.586
GMSR	0.555	0.496	0.529	0.737	0.946	0.550	0.491	0.523	0.723	0.926	0.445	0.373	0.366	0.448	0.543
GMRSR	0.814	0.707	0.664	0.882	1.036	0.815	0.705	0.657	0.863	1.012	0.848	0.721	0.586	0.513	0.581
MO	0.527	0.486	0.529	0.710	0.789	0.521	0.482	0.524	0.698	0.773	0.397	0.363	0.378	0.447	0.475
MR	0.849	0.749	0.708	0.901	1.039	0.851	0.748	0.700	0.881	1.015	0.903	0.803	0.679	0.543	0.583
MSR	0.578	0.505	0.531	0.760	0.937	0.575	0.501	0.524	0.745	0.917	0.471	0.386	0.373	0.459	0.537
MRSR	0.788	0.674	0.636	0.862	1.034	0.789	0.673	0.628	0.842	1.010	0.826	0.688	0.550	0.499	0.580
n=50															
MSE OLS	0.165	4.030	0.272	3.711	39.738	0.168	0.338	0.696	3.768	40.352	0.249	0.502	1.035	5.604	60.002
FMO	0.827	0.712	0.587	0.433	0.410	0.824	0.709	0.584	0.430	0.407	0.736	0.600	0.476	0.351	0.334
FMR	0.894	0.807	0.745	0.957	1.051	0.895	0.808	0.743	0.944	1.037	0.925	0.853	0.759	0.693	0.718
FMSR	0.847	0.735	0.603	0.450	0.618	0.846	0.733	0.601	0.447	0.611	0.805	0.673	0.528	0.362	0.455
FMRSR	0.881	0.781	0.677	0.823	1.039	0.881	0.782	0.676	0.812	1.024	0.899	0.807	0.680	0.591	0.708
VMO	1.810	1.123	0.862	0.788	0.988	1.794	1.109	0.853	0.780	0.975	1.371	0.860	0.662	0.585	0.686
VMR	0.894	0.807	0.745	0.957	1.051	0.895	0.808	0.743	0.944	1.037	0.925	0.853	0.759	0.693	0.718
VMSR	0.881	0.781	0.677	0.823	1.039	0.881	0.782	0.676	0.812	1.024	0.899	0.807	0.680	0.591	0.708
VMRSR	0.807	0.659	0.596	0.712	0.979	0.814	0.649	0.590	0.705	0.965	0.684	0.537	0.475	0.522	0.676
AMO	1.332	0.870	0.703	0.669	0.906	1.322	0.858	0.695	0.663	0.895	1.021	0.681	0.555	0.510	0.641
AMR	0.979	0.946	0.883	0.784	0.985	0.979	0.947	0.884	0.778	0.969	0.985	0.960	0.907	0.707	0.673
AMSR	0.768	0.637	0.561	0.645	0.941	0.776	0.629	0.555	0.638	0.928	0.681	0.537	0.458	0.480	0.653
AMRSR	0.953	0.897	0.804	0.686	0.967	0.953	0.897	0.804	0.680	0.953	0.960	0.910	0.821	0.584	0.658
HMO	0.727	0.599	0.499	0.424	0.421	0.724	0.595	0.496	0.421	0.419	0.623	0.492	0.405	0.344	0.340
HMR	0.940	0.877	0.789	0.890	1.047	0.941	0.878	0.789	0.878	1.033	0.959	0.910	0.823	0.681	0.715
HMSR	0.802	0.676	0.553	0.489	0.717	0.800	0.674	0.551	0.485	0.708	0.751	0.608	0.474	0.382	0.517
HMRSR	0.910	0.826	0.713	0.764	1.031	0.911	0.827	0.712	0.754	1.016	0.925	0.849	0.728	0.569	0.703
GMO	0.697	0.582	0.519	0.543	0.683	0.701	0.575	0.516	0.539	0.677	0.599	0.480	0.427	0.429	0.511
GMR	0.966	0.922	0.845	0.812	1.029	0.967	0.923	0.846	0.803	1.016	0.976	0.943	0.876	0.678	0.702
GMSR	0.737	0.613	0.522	0.567	0.856	0.735	0.611	0.519	0.562	0.845	0.681	0.541	0.439	0.431	0.602
GMRSR	0.935	0.866	0.760	0.706	1.009	0.935	0.866	0.760	0.698	0.995	0.945	0.884	0.778	0.559	0.686
MO	0.686	0.565	0.492	0.570	0.688	0.684	0.562	0.490	0.566	0.681	0.588	0.471	0.407	0.446	0.510
MR	0.949	0.892	0.797	0.748	1.018	0.950	0.893	0.797	0.739	1.003	0.965	0.920	0.834	0.613	0.690
MSR	0.764	0.632	0.529	0.580	0.843	0.763	0.630	0.526	0.575	0.833	0.710	0.565	0.453	0.441	0.595
MRSR	0.921	0.845	0.731	0.682	1.005	0.921	0.845	0.731	0.674	0.991	0.934	0.864	0.746	0.527	0.682

n=200 p=3															
Method	$\sigma = 0.5$					$\sigma = 1$					$\sigma = 5$				
	0.8	0.9	0.95	0.99	0.999	0.8	0.9	0.95	0.99	0.999	0.8	0.9	0.95	0.99	0.999
MSE OLS	0.036	0.073	0.150	0.814	8.746	0.036	0.073	0.151	0.817	8.780	0.041	0.082	0.169	0.919	9.872
FMO	0.035	0.067	0.126	0.459	3.713	0.035	0.067	0.126	0.460	3.721	0.039	0.074	0.139	0.492	3.996
FMR	0.035	0.069	0.134	0.591	9.108	0.035	0.069	0.135	0.593	9.120	0.040	0.078	0.153	0.660	9.405
FMSR	0.035	0.067	0.128	0.470	4.259	0.035	0.068	0.129	0.471	4.266	0.039	0.076	0.143	0.512	4.521
FMRSR	0.035	0.068	0.132	0.531	8.440	0.035	0.069	0.133	0.533	8.451	0.040	0.078	0.150	0.593	8.699
VMO	0.167	0.204	0.239	0.663	7.648	0.166	0.205	0.240	0.666	7.666	0.172	0.216	0.256	0.696	8.087
VMR	0.035	0.069	0.134	0.591	9.108	0.035	0.069	0.135	0.593	9.120	0.040	0.078	0.153	0.660	9.405
VMSR	0.043	0.069	0.113	0.494	7.293	0.044	0.069	0.115	0.493	7.308	0.046	0.074	0.125	0.516	7.658
VMRSR	0.035	0.068	0.132	0.531	8.440	0.035	0.069	0.133	0.533	8.451	0.040	0.078	0.150	0.593	8.699
AMO	0.111	0.141	0.176	0.559	6.468	0.111	0.141	0.179	0.561	6.485	0.114	0.148	0.193	0.584	6.908
AMR	0.036	0.072	0.147	0.709	7.722	0.036	0.073	0.148	0.712	7.718	0.041	0.082	0.166	0.806	7.950
AMSR	0.039	0.065	0.110	0.461	6.665	0.040	0.065	0.112	0.459	6.680	0.043	0.070	0.122	0.482	7.011
AMRSR	0.036	0.071	0.143	0.641	7.067	0.036	0.072	0.144	0.644	7.054	0.041	0.081	0.162	0.727	7.278
HMO	0.033	0.061	0.110	0.392	3.736	0.033	0.062	0.110	0.393	3.743	0.037	0.068	0.119	0.419	4.012
HMR	0.036	0.071	0.141	0.620	8.795	0.036	0.071	0.142	0.622	8.806	0.040	0.080	0.161	0.704	9.081
HMSR	0.034	0.065	0.121	0.433	4.917	0.034	0.065	0.121	0.434	4.925	0.039	0.073	0.135	0.469	5.183
HMRSR	0.036	0.070	0.137	0.557	8.057	0.036	0.070	0.138	0.559	8.067	0.040	0.079	0.155	0.628	8.301
GMO	0.034	0.058	0.099	0.424	5.125	0.035	0.059	0.100	0.422	5.141	0.038	0.063	0.108	0.449	5.500
GMR	0.036	0.072	0.145	0.671	8.182	0.036	0.072	0.146	0.675	8.171	0.041	0.081	0.165	0.765	8.462
GMSR	0.033	0.061	0.110	0.416	5.902	0.034	0.062	0.110	0.415	5.915	0.037	0.068	0.121	0.444	6.203
GMRSR	0.036	0.071	0.141	0.598	7.485	0.036	0.071	0.141	0.601	7.486	0.040	0.080	0.160	0.679	7.707
MO	0.032	0.057	0.098	0.390	5.614	0.032	0.057	0.098	0.391	5.626	0.036	0.062	0.106	0.420	5.916
MR	0.036	0.071	0.143	0.626	7.682	0.036	0.072	0.144	0.628	7.686	0.041	0.081	0.162	0.714	7.892
MSR	0.034	0.063	0.114	0.417	6.066	0.034	0.063	0.114	0.418	6.076	0.038	0.070	0.126	0.451	6.331
MRSR	0.036	0.070	0.139	0.572	7.293	0.036	0.070	0.140	0.575	7.300	0.040	0.079	0.157	0.646	7.512
n=500															
MSE OLS	0.015	0.030	0.063	0.340	3.654	0.015	0.031	0.063	0.341	3.660	0.016	0.032	0.066	0.357	3.836
FMO	0.015	0.029	0.058	0.238	1.565	0.015	0.029	0.058	0.238	1.566	0.016	0.031	0.060	0.246	1.619
FMR	0.015	0.030	0.060	0.267	3.373	0.015	0.030	0.060	0.267	3.371	0.016	0.031	0.063	0.282	3.412
FMSR	0.015	0.029	0.058	0.245	1.633	0.015	0.029	0.058	0.245	1.634	0.016	0.031	0.061	0.255	1.686
FMRSR	0.015	0.030	0.059	0.259	2.890	0.015	0.030	0.059	0.260	2.888	0.016	0.031	0.062	0.273	2.925
VMO	0.252	0.204	0.189	0.317	2.771	0.253	0.203	0.189	0.317	2.770	0.254	0.203	0.185	0.320	2.850
VMR	0.015	0.030	0.060	0.267	3.373	0.015	0.030	0.060	0.267	3.371	0.016	0.031	0.063	0.282	3.412
VMSR	0.028	0.038	0.060	0.207	2.504	0.030	0.037	0.059	0.206	2.503	0.032	0.038	0.062	0.210	2.575
VMRSR	0.015	0.030	0.059	0.259	2.890	0.015	0.030	0.059	0.260	2.888	0.016	0.031	0.062	0.273	2.925
AMO	0.161	0.131	0.133	0.257	2.321	0.162	0.130	0.132	0.256	2.321	0.164	0.130	0.128	0.259	2.405
AMR	0.015	0.030	0.062	0.320	2.864	0.015	0.030	0.062	0.321	2.863	0.016	0.032	0.066	0.337	2.949
AMSR	0.022	0.033	0.057	0.202	2.252	0.024	0.033	0.056	0.202	2.252	0.025	0.034	0.059	0.206	2.328
AMRSR	0.015	0.030	0.062	0.303	2.496	0.015	0.030	0.062	0.303	2.496	0.016	0.032	0.065	0.318	2.571
HMO	0.015	0.028	0.053	0.197	1.536	0.015	0.028	0.054	0.197	1.537	0.015	0.029	0.056	0.203	1.590
HMR	0.015	0.030	0.061	0.294	3.124	0.015	0.030	0.061	0.295	3.122	0.016	0.032	0.064	0.310	3.168
HMSR	0.015	0.029	0.057	0.224	1.765	0.015	0.029	0.057	0.224	1.765	0.016	0.030	0.059	0.233	1.816
HMRSR	0.015	0.030	0.060	0.276	2.680	0.015	0.030	0.060	0.276	2.679	0.016	0.031	0.063	0.290	2.720
GMO	0.017	0.028	0.050	0.185	1.903	0.017	0.029	0.050	0.185	1.904	0.018	0.030	0.052	0.189	1.990
GMR	0.015	0.030	0.062	0.312	2.915	0.015	0.030	0.062	0.312	2.915	0.016	0.032	0.065	0.328	2.972
GMSR	0.015	0.028	0.053	0.200	1.999	0.015	0.028	0.053	0.201	1.999	0.015	0.029	0.056	0.208	2.065
GMRSR	0.015	0.030	0.061	0.291	2.524	0.015	0.030	0.061	0.292	2.522	0.016	0.032	0.064	0.306	2.567
MO	0.014	0.027	0.050	0.182	1.990	0.014	0.027	0.050	0.182	1.992	0.015	0.028	0.052	0.188	2.063
MR	0.015	0.030	0.062	0.300	2.668	0.015	0.030	0.062	0.300	2.667	0.016	0.032	0.065	0.316	2.709
MSR	0.015	0.029	0.055	0.209	2.040	0.015	0.029	0.055	0.209	2.041	0.016	0.030	0.058	0.217	2.098
MRSR	0.015	0.030	0.061	0.283	2.429	0.015	0.030	0.061	0.283	2.427	0.016	0.031	0.064	0.297	2.468