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A UNIFIED RANKED SET SAMPLING FOR ESTIMATING THE POPULATION MEAN

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Abstract: A unified ranked set sampling scheme is proposed to estimate the population mean. In such a scheme various existing ranked set sampling schemes are combined in order to minimizing the error of ranking and cost of sampling. It is shown that the sample weighted mean of the proposed scheme is more efficient than simple random sample mean. Also, assuming the underlying distribution is normal, the existence and uniqueness of maximum likelihood estimator of the location parameter are investigated. The pairwise relative precisions of the derived estimators are compared using simulation and numerical computations. It is concluded that a combination of existing sampling schemes may be considered as a good suggestion with rather high efficiency. A cost analysis is also performed. Some conclusions are eventually stated.

Key words: Cost analysis; Maximum likelihood estimator; Normal distribution; Order statistics; Ranking error

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1. Introduction

A sampling method based on rankings is called ranked set sampling (RSS) which was introduced by [9]. In this scheme, first of all, k independent sets each containing k samples are considered from an infinity population. Each set is ranked by visual inspection or by some other relatively inexpensive methods, without actual measurement of the variable of interest. Then, the jth ($j = 1, 2, ..., k$) judged order statistic from the jth set is measured. If such process is repeated m times, the following data set is obtained

$$\mathbf{X}_{RSS}^{(m)} = \{X_{1(1:k)1}, X_{2(2:k)1}, ..., X_{k(k:k)1}, ..., X_{1(1:k)m}, X_{2(2:k)m}, ..., X_{k(k:k)m}\},$$

where for $r = 1, ..., k$ and $i = 1, ..., m$, $X_{r(j:k)i}$ stands for the recorded observation of the rth set in the ith cycle and the expression within parenthesis represents the order of observation. That is, for $j = 1, ..., k$, $X_{r(j:k)i}$ means that in the rth set of the ith cycle, the jth order statistic in a random sample of size k is recorded. We use this notation throughout the paper. For more details about order statistics and their applications, one may refer to the books by [6] and [3].

[17] showed that the mean of RSS is unbiased for the population mean and it is more efficient than that of simple random sample (SRS). This method can reduce cost of sampling and increase the accuracy of results. Using concomitant variable, various estimators for estimating the population mean are presented by [14], [8] and [16]. [7] investigated the group sequential comparison of two binomial proportions under RSS. [12] studied the problem of prediction of order statistics and record values based on ordered RSS.

Here, we recall some existing sampling based on the idea of RSS. These schemes can be used to estimate various parameters of the underlying population such as mean, median and range.

[13] introduced extreme RSS (ERSS) scheme for estimating the mean of population. We explain this scheme when k is even. Assume that there are k independent sets each involves k samples

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from the population of interest. We select visually the sample minimum from the first $k/2$ sets and the sample maximum from the last $k/2$ sets. If k is odd, the scheme can be also illustrated. The corresponding data set of ERSS with m cycles is given by

$$\mathbf{X}_{ERSS}^{(m)} = \{X_{1(1:k)1}, X_{2(1:k)1}, ..., X_{\frac{k}{2}(1:k)1}, X_{\frac{k+2}{2}(k:k)1}, ..., X_{k(k:k)1}, \dots, X_{1(1:k)m}, X_{2(1:k)m}, ..., X_{\frac{k}{2}(1:k)m}, X_{\frac{k+2}{2}(k:k)m}, ..., X_{k(k:k)m}\}.$$

The moving extreme RSS (MERSS) was introduced by [1]. They consider k independent sets for which the jth one includes j sample points ($j = 1, 2, ..., k$). That is, the first set contains 1 observation, the second set has 2 sample points and so the kth set has k observations. If only the largest or smallest judged order statistics from each set is measured, then MERSS Type-I or MERSS Type-II schemes are obtained, respectively. These data sets with m cycles are given by

$$\mathbf{X}_{MERSS-I}^{(m)} = \{X_{1(1:1)1}, X_{2(2:2)1}, ..., X_{k(k:k)1}, ..., X_{1(1:1)m}, X_{2(2:2)m}, ..., X_{k(k:k)m}\},$$

and

$$\mathbf{X}_{MERSS-II}^{(m)} = \{X_{1(1:1)1}, X_{2(1:2)1}, ..., X_{k(1:k)1}, X_{1(1:1)m}, X_{2(1:2)m}, ..., X_{k(1:k)m}\},$$

respectively. The aforementioned data sets can be combined together which is known in general as MERSS. This scheme has been considered by some authors, such as [5].

[10] proposed a new scheme for estimating the mean of population based on median of ranked sets and called it as median RSS (MRSS). Assume that k (which is even) independent sets, each contains k independent and identically distribution observations, are taken from an infinity population. In an MRSS, the $(k/2)$th judged order statistic from the first $k/2$ sets and the $(k/2+1)$th judged order statistic from the last $k/2$ sets are measured. If such scheme is repeated m times, the data set

$$\mathbf{X}_{MRSS}^{(m)} = \left\{ \{X_{j(k/2:k)i}\}_{j=1}^{k/2} \cup \{X_{j(k/2+1:k)i}\}_{j=k/2+1}^{k}, \ \ i = 1, 2, ..., m \right\},$$

is obtained, where $A \cup B$ stands for the union of the sets $A = \{X_{j(k/2:k)i}\}_{j=1}^{k/2}$ and $B = \{X_{j(k/2+1:k)i}\}_{j=k/2+1}^{k}$. [2] used the auxiliary information in MRSS scheme to derive the ratio estimator of the population mean.

Note that when the ranking error is occurred in each of the above mentioned schemes, the validity of the results is reduced. On the other hand, the error of ranking in the ERSS, MERSS Type-I and MERSS Type-II, in which only the extreme order statistics (maximum or minimum) in each set are measured, is less than other schemes. However, the ordinary RSS and MRSS schemes are more appropriate to inference about the population mean. Hence, this question arises: Does there exist any sampling scheme containing all aforementioned schemes such that in spite of reducing the ranking error leads to rather high efficiency for estimating the population mean? Toward this end, we consider m_1, m_2, m_3, m_4 and m_5 cycles of RSS, ERSS, MERSS Type-I, MERSS Type-II and MRSS schemes, respectively. We call this procedure unified RSS (URSS) scheme with the following data set

$$\mathbf{X}_{URSS} = \{\mathbf{X}_{RSS}^{(m_1)} \cup \mathbf{X}_{ERSS}^{(m_2)} \cup \mathbf{X}_{MERSS-I}^{(m_3)} \cup \mathbf{X}_{MERSS-II}^{(m_4)} \cup \mathbf{X}_{MRSS}^{(m_5)}\},$$

such that $M = \sum_{i=1}^{5} m_i$. This scheme may be used to minimize the error of ranking and cost of sampling. Any permutation of $\mathbf{m}$=$(m_1, m_2, m_3, m_4, m_5)$ may be suggested to use in practice, if its efficiency is a little less than the most efficient permutation, but its ranking error or cost of sampling is less than it. It is clear that the URSS may reduce to the existing schemes if all of

m_i's $(i=1,...,5)$ equal to zero except one. For example, when $m_1=m_2=m_3=m_4=0$, the URSS reduces to MRSS scheme with $M=m_5$ cycles. [15] obtained an estimator for distribution function via combining the ERSS and MRSS schemes, i.e., when $m_1=m_3=m_4=0$ while $m_2, m_5 \neq 0$.

The rest of the paper is as follows. In Section 2, the weighted mean of $\mathbf{X}_{URSS}$ is considered as an estimator of the population mean. It is shown that the proposed estimator is unbiased and it is more efficient than the mean of an SRS. In Section 3, we focus on the normal distribution. The sample mean and maximum likelihood estimator (MLE) are derived for the location parameter based on URSS and their properties are investigated. It is shown that the MLE exists and it is unique. In Section 4, the relative precisions (RPs) of the proposed estimators in Section 3 are compared via simulation and numerical computations. It is concluded that a combination of existing sampling schemes may be suggested as a good choice in practice with rather high precision. Cost analysis based on the relative efficiency (RE) is presented in Section 5. In Section 6, some conclusions are stated.

2. Estimation of the population mean

Let μ and σ^2 be the mean and variance of the underlying population, respectively. Here, we focus on estimating μ based on URSS scheme. First of all, note that the the population mean may be estimated using the sample mean of the RSS, ERSS, MERSS Type-I, MERSS Type-II and MRSS data sets separately as

$$\bar{X}^{(m)}_{RSS} = \frac{1}{mk}\sum_{i=1}^{m}\sum_{j=1}^{k} X_{j(j:k)i},$$

$$\bar{X}^{(m)}_{ERSS} = \frac{1}{mk}\sum_{i=1}^{m}\left\{\sum_{j=1}^{\frac{k}{2}} X_{j(1:k)i} + \sum_{j=\frac{k}{2}+1}^{k} X_{j(k:k)i}\right\},$$

$$\bar{X}^{(m)}_{MERSS-I} = \frac{1}{mk}\sum_{i=1}^{m}\sum_{j=1}^{k} X_{j(j:j)i},$$

$$\bar{X}^{(m)}_{MERSS-II} = \frac{1}{mk}\sum_{i=1}^{m}\sum_{j=1}^{k} X_{j(1:j)i},$$

and

$$\bar{X}^{(m)}_{MRSS} = \frac{1}{mk}\sum_{i=1}^{m}\left\{\sum_{j=1}^{\frac{k}{2}} X_{j(\frac{k}{2}:k)i} + \sum_{j=\frac{k}{2}+1}^{k} X_{j(\frac{k}{2}+1:k)i}\right\},$$

respectively. Therefore, by averaging all of the above statistics, a reasonable estimator for the population mean based on URSS scheme can be introduced as follows

$$\bar{X}_{URSS} = \frac{m_1}{M}\bar{X}^{(m_1)}_{RSS} + \frac{m_2}{M}\bar{X}^{(m_2)}_{ERSS} + \frac{m_3}{M}\bar{X}^{(m_3)}_{MERSS-I} + \frac{m_4}{M}\bar{X}^{(m_4)}_{MERSS-II} + \frac{m_5}{M}\bar{X}^{(m_5)}_{MRSS}. \tag{2.1}$$

The proposed estimator can be interpreted as both weighted mean of all means of different schemes and arithmetic mean from URSS data set. Therefore, the $\bar{X}_{URSS}$ is a moment estimator for the population mean.

In the following theorem, we prove analytically that under some conditions, the statistic $\bar{X}_{URSS}$ is an unbiased estimator for μ. Then its efficiency is compared with the mean of an SRS data set, denoted by $\bar{X}_{SRS}$.

THEOREM 1. *Suppose that $m_3 = m_4$ and the underlying distribution is symmetric. Then, $\bar{X}_{URSS}$ is unbiased for μ with variance*

$$Var(\bar{X}_{URSS}) = \frac{1}{M^2 k}\left\{m_1\left(\sigma^2 - \frac{1}{k}\sum_{j=1}^{k}(\mu_{j:k}-\mu)^2\right) + m_2\sigma^2_{1:k} + \frac{2m_3}{k}\sum_{j=1}^{k}\sigma^2_{1:j} + m_5\sigma^2_{\frac{k}{2}:k}\right\}, \quad (2.2)$$

where $\mu_{j:k}$ and $\sigma^2_{j:k}$ are the mean and variance of the jth order statistic in a random sample of size k, respectively.

PROOF. First of all, note that $\bar{X}_{RSS}$ is an unbiased estimator of μ [17]. Next, using the symmetric property of the underlying distribution around μ, we have (see, for example, [3, p.26])

$$X_{j:k} \stackrel{\mathrm{d}}{=} 2\mu - X_{k-j+1:k}. \quad (2.3)$$

Therefore,

$$\begin{aligned} E(\bar{X}^{(m_2)}_{ERSS}) &= \frac{1}{m_2 k}E\left\{\sum_{i=1}^{m_2}\left(\sum_{j=1}^{\frac{k}{2}}X_{j(1:k)i} + \sum_{j=\frac{k}{2}+1}^{k}X_{j(k:k)i}\right)\right\} \\ &= \frac{1}{k}E\left\{\sum_{j=1}^{\frac{k}{2}}X_{j(1:k)1} + \sum_{j=\frac{k}{2}+1}^{k}X_{j(k:k)1}\right\} \\ &= \frac{1}{2}E(X_{1:k}+X_{k:k}) \\ &= \mu. \end{aligned} \quad (2.4)$$

By considering $m_3 = m_4$ and using (2.3) we can write

$$\begin{aligned} E\left\{\bar{X}^{(m_3)}_{MERSS-I} + \bar{X}^{(m_4)}_{MERSS-II}\right\} &= \frac{1}{m_3 k}\sum_{i=1}^{m_3}E\left\{\sum_{j=1}^{k}X_{j(j:j)i} + \sum_{j=1}^{k}X_{j(1:j)i}\right\} \\ &= \frac{1}{m_3 k}\sum_{i=1}^{m_3}E\left\{\sum_{j=1}^{k}\left[2\mu - X_{j(1:j)i}\right] + \sum_{j=1}^{k}X_{j(1:j)i}\right\} \\ &= 2\mu. \end{aligned} \quad (2.5)$$

Similarly,

$$\begin{aligned} E\left\{\bar{X}^{(m_5)}_{MRSS}\right\} &= \frac{1}{mk}\sum_{i=1}^{m}E\left\{\sum_{j=1}^{\frac{k}{2}}X_{j(\frac{k}{2}:k)i} + \sum_{j=\frac{k}{2}+1}^{k}X_{j(\frac{k}{2}+1:k)i}\right\} \\ &= \frac{1}{2}E\left\{X_{\frac{k}{2}:k} + X_{\frac{k}{2}+1:k}\right\} \\ &= \frac{1}{2}E\left\{X_{\frac{k}{2}:k} + 2\mu - X_{\frac{k}{2}:k}\right\} \\ &= \mu. \end{aligned} \quad (2.6)$$

Therefore, using (2.1), (2.4), (2.5) and (2.6) we have

$$E\left\{\bar{X}_{URSS}\right\} = \frac{m_1}{M}\mu + \frac{m_2}{M}\mu + 2\frac{m_3}{M}\mu + \frac{m_5}{M}\mu = \mu.$$

On the other hand, the variance of the $\bar{X}^{(m_1)}_{RSS}$ is [17]

$$Var(\bar{X}^{(m_1)}_{RSS}) = \frac{1}{m_1 k}\left\{\sigma^2 - \frac{1}{k}\sum_{j=1}^{k}(\mu_{j:k}-\mu)^2\right\}.$$

Also, using (2.3), for ERSS scheme we get

$$Var(\bar{X}^{(m_2)}_{ERSS}) = \frac{1}{(m_2 k)^2} \sum_{i=1}^{m_2} \left\{ \sum_{j=1}^{\frac{k}{2}} Var(X_{j(1:k)i}) + \sum_{j=\frac{k}{2}+1}^{k} Var(X_{j(k:k)i}) \right\}$$
$$= \frac{1}{m_2 k} \sigma^2_{1:k}.$$

Similarly, it can be shown that

$$Var(\bar{X}^{(m_3)}_{MERSS-I} + \bar{X}^{(m_4)}_{MERSS-II}) = \frac{2}{m_3 k^2} \sum_{j=1}^{k} \sigma^2_{1:j},$$

and

$$Var(\bar{X}^{(m_5)}_{MRSS}) = \frac{1}{m_5 k} \sigma^2_{\frac{k}{2}:k}.$$

Now, since the different sampling schemes are independent, the proof is complete. □

Remark 1. If remove the condition $m_3 = m_4$ in the Theorem 1, then the $\bar{X}_{URSS}$ will not be unbiased and it can be easily shown that

$$E\left\{\bar{X}_{URSS}\right\} = \frac{m_1 + m_2 + 2m_3 + m_5}{M} \mu + \frac{m_4 - m_3}{Mk} \sum_{j=1}^{k} \mu_{1:j}.$$

Note that the mean of an SRS data set of size Mk is an unbiased estimator of μ with variance $\sigma^2/(Mk)$. Hence, under the assumptions of Theorem 1, $\bar{X}_{URSS}$ is more efficient than $\bar{X}_{SRS}$, provided that

$$-\frac{m_1}{k} \sum_{j=1}^{k} (\mu_{j:k} - \mu)^2 + m_2 \left(\sigma^2_{1:k} - \sigma^2\right) + 2m_3 \left(\frac{1}{k} \sum_{j=1}^{k} \sigma^2_{1:j} - \sigma^2\right) + m_5 \left(\sigma^2_{\frac{k}{2}} - \sigma^2\right) \le 0. \qquad (2.7)$$

In the next section, we show that the condition (2.7) holds for the normal distribution. Moreover, the MLE of the location parameter is investigated.

3. Normal distribution

In this section, the normal distribution with the location and scale parameters μ and σ, respectively, denoted by $N(\mu, \sigma^2)$, is considered. It is assumed that σ is known and so without loss of generality, we take $\sigma = 1$. The estimator $\bar{X}_{URSS}$ in (2.1) and also the MLE of μ are investigated based on URSS scheme.

From Theorem 4.9.1 of [3, p.91], for the $N(\mu, 1)$ distribution, we have

$$\sum_{j=1}^{k} \sigma_{i,j:k} = 1, \quad i = 1, 2, ..., k,$$

where $\sigma_{i,j:k}$ stands for the covariance between the ith and jth order statistics. Moreover, assuming $E(X^2_{i:k}) + E(X^2_{j:k}) < \infty$, [4] showed that $\sigma_{i,j:k} \ge 0$. Therefore, for $i = 1, 2, ..., k$, we get $\sigma^2_{i:k} = \sigma_{i,i:k} \le 1$. That is, the variances of the order statistics in a random sample from $N(\mu, \sigma^2)$ distribution is less than σ^2. Hence, the inequality in (2.7) holds and so we get

$$Var(\bar{X}_{URSS}) \le Var(\bar{X}_{SRS}). \qquad (3.1)$$

On the other hand, according to Theorem 1, $\bar{X}_{URSS}$ is unbiased for μ, when $m_3 = m_4$. Therefore, in this case, $\bar{X}_{URSS}$ is more efficient than $\bar{X}_{SRS}$. Other cases are studied in Section 4.

In what follows, the MLE of μ is investigated. Let us denote the cumulative distribution function (cdf) and probability density function (pdf) of underlying population by $F(x;\boldsymbol{\theta})$ and $f(x;\boldsymbol{\theta})$, respectively, where $\boldsymbol{\theta}$ is a vector of parameters. Then, the likelihood functions of $\boldsymbol{\theta}$ based on the data sets $\mathbf{X}^{(m)}_{RSS}, \mathbf{X}^{(m)}_{ERSS}, \mathbf{X}^{(m)}_{MERSS-I}, \mathbf{X}^{(m)}_{MERSS-II}$ and $\mathbf{X}^{(m)}_{MRSS}$ are given by

$$L^{(m)}_{RSS}(\boldsymbol{\theta}) = \left\{k!\prod_{j=1}^{k}\binom{k}{j}\right\}^{m}\prod_{i=1}^{m}\prod_{j=1}^{k} f(x_{j(j:k)i};\boldsymbol{\theta})\big(F(x_{j(j:k)i};\boldsymbol{\theta})\big)^{j-1}\big(\bar{F}(x_{j(j:k)i};\boldsymbol{\theta})\big)^{k-j},$$

$$L^{(m)}_{ERSS}(\boldsymbol{\theta}) = \prod_{i=1}^{m}\left\{\prod_{j=1}^{\frac{k}{2}} kf(x_{j(1:k)i};\boldsymbol{\theta})\big(\bar{F}(x_{j(1:k)i};\boldsymbol{\theta})\big)^{k-1}\prod_{j=\frac{k}{2}+1}^{k} kf(x_{j(k:k)i};\boldsymbol{\theta})\big(F(x_{j(k:k)i};\boldsymbol{\theta})\big)^{k-1}\right\},$$

$$L^{(m)}_{MERSS-I}(\boldsymbol{\theta}) = \prod_{i=1}^{m}\prod_{j=1}^{k} jf(x_{j(j:j)i};\boldsymbol{\theta})\big(F(x_{j(j:j)i};\boldsymbol{\theta})\big)^{j-1},$$

$$L^{(m)}_{MERSS-II}(\boldsymbol{\theta}) = \prod_{i=1}^{m}\prod_{j=1}^{k} jf(x_{j(1:j)i};\boldsymbol{\theta})\big(\bar{F}(x_{j(1:j)i};\boldsymbol{\theta})\big)^{j-1},$$

and

$$\begin{aligned}L^{(m)}_{MRSS}(\boldsymbol{\theta}) =& \left\{\frac{k!}{(\frac{k}{2})!(\frac{k}{2}-1)!}\right\}^{mk}\prod_{i=1}^{m}\left\{\prod_{j=1}^{\frac{k}{2}} f(x_{j(\frac{k}{2}:k)i};\boldsymbol{\theta})(F(x_{j(\frac{k}{2}:k)i};\boldsymbol{\theta}))^{\frac{k}{2}-1}(\bar{F}(x_{j(\frac{k}{2}:k)i};\boldsymbol{\theta}))^{\frac{k}{2}}\right.\\ &\left.\times\prod_{j=\frac{k}{2}+1}^{k} f(x_{j(\frac{k}{2}+1:k)i};\boldsymbol{\theta})(F(x_{j(\frac{k}{2}+1:k)i};\boldsymbol{\theta}))^{\frac{k}{2}}(\bar{F}(x_{j(\frac{k}{2}+1:k)i};\boldsymbol{\theta}))^{\frac{k}{2}-1}\right\},\end{aligned}$$

respectively, where $\bar{F}(\cdot) = 1 - F(\cdot)$ stands for the survival function. Therefore, likelihood function of $\boldsymbol{\theta}$ based on $\mathbf{X}_{URSS}$ is as follows

$$L_{URSS}(\boldsymbol{\theta}) = \{L^{(m_1)}_{RSS}(\boldsymbol{\theta})\}^{\delta_1}\{L^{(m_2)}_{ERSS}(\boldsymbol{\theta})\}^{\delta_2}\{L^{(m_3)}_{MERSS-I}(\boldsymbol{\theta})\}^{\delta_3}\{L^{(m_4)}_{MERSS-II}(\boldsymbol{\theta})\}^{\delta_4}\{L^{(m_5)}_{MRSS}(\boldsymbol{\theta})\}^{\delta_5}, \tag{3.2}$$

where, for $i = 1, ..., 5$,

$$\delta_i = \begin{cases} 1, & m_i > 0, \\ 0, & m_i = 0. \end{cases}$$

[5] obtained the MLE of the scale parameter of normal distribution based on MERSS scheme. Here, we show that the MLE of the location parameter of normal distribution based on a URSS data set exists and it is unique. Let us denote the pdf and cdf of the standard normal distribution by $\phi(\cdot)$ and $\Phi(\cdot)$, respectively. Using (3.2) and doing some algebraic calculations, the likelihood equation of μ in $N(\mu,1)$ distribution, on the basis of URSS, is obtained as

$$\frac{\partial l(\mu)}{\partial\mu} = -Mk\mu + \sum_{i=1}^{m_1}\sum_{j=1}^{k}\left\{x_{j(j:k)i} - (j-1)\frac{\phi(x_{j(j:k)i}-\mu)}{\Phi(x_{j(j:k)i}-\mu)} + (k-j)\frac{\phi(x_{j(j:k)i}-\mu)}{\Phi(\mu - x_{j(j:k)i})}\right\}$$

$$+\sum_{i=1}^{m_2}\left\{\sum_{j=1}^{\frac{k}{2}}\left(x_{j(1:k)i}+(k-1)\frac{\phi(x_{j(1:k)i}-\mu)}{\Phi(\mu-x_{j(1:k)i})}\right)+\sum_{j=\frac{k}{2}+1}^{k}\left(x_{j(k:k)i}-(k-1)\frac{\phi(x_{j(k:k)i}-\mu)}{\Phi(x_{j(k:k)i}-\mu)}\right)\right\}$$
$$+\sum_{i=1}^{m_3}\sum_{j=1}^{k}\left\{x_{j(j:j)i}-(j-1)\frac{\phi(x_{j(j:j)i}-\mu)}{\Phi(x_{j(j:j)i}-\mu)}\right\}+\sum_{i=1}^{m_4}\sum_{j=1}^{k}\left\{x_{j(1:j)i}+(j-1)\frac{\phi(x_{j(1:j)i}-\mu)}{\Phi(\mu-x_{j(1:j)i})}\right\}$$
$$+\sum_{i=1}^{m_5}\left\{\sum_{j=1}^{\frac{k}{2}}\left(x_{j(\frac{k}{2}:k)i}-(\frac{k}{2}-1)\frac{\phi(x_{j(\frac{k}{2}:k)i}-\mu)}{\Phi(x_{j(\frac{k}{2}:k)i}-\mu)}+\frac{k}{2}\frac{\phi(x_{j(\frac{k}{2}:k)i}-\mu)}{\Phi(\mu-x_{j(\frac{k}{2}:k)i})}\right)\right.$$
$$\left.+\sum_{j=\frac{k}{2}+1}^{k}\left(x_{j(\frac{k}{2}+1:k)i}-\frac{k}{2}\frac{\phi(x_{j(\frac{k}{2}+1:k)i}-\mu)}{\Phi(x_{j(\frac{k}{2}+1:k)i}-\mu)}+(\frac{k}{2}-1)\frac{\phi(x_{j(\frac{k}{2}+1:k)i}-\mu)}{\Phi(\mu-x_{j(\frac{k}{2}+1:k)i})}\right)\right\}=0, \tag{3.3}$$

where $l(\mu)$ stands for the log-likelihood function of μ. It is clear that the left hand side of equation (3.3) is a continuous function of μ and it converges to $+\infty$ and $-\infty$ when μ tends to $-\infty$ and $+\infty$, respectively. It means that there is at least one root and so the MLE of μ, denoted by $\hat{\mu}$, exists. Now, we will show that the MLE of μ is unique. To prove the uniqueness of $\hat{\mu}$, it is enough to show that $\frac{\partial^2 l(\mu)}{\partial\mu^2}$ is negative. By performing some calculations, it is not hard to show that

$$\frac{\partial^2 l(\mu)}{\partial\mu^2}=-Mk-\eta_1+\eta_2$$

where

$$\eta_1=\sum_{i=1}^{m_1}\sum_{j=1}^{k}(j-1)\frac{\phi(y_{j(j:k)i})}{\Phi(y_{j(j:k)i})}\left(y_{j(j:k)i}+\frac{\phi(y_{j(j:k)i})}{\Phi(y_{j(j:k)i})}\right)+(k-1)\sum_{i=1}^{m_2}\sum_{j=\frac{k}{2}+1}^{k}\frac{\phi(y_{j(k:k)i})}{\Phi(y_{j(k:k)i})}$$
$$\times\left(y_{j(k:k)i}+\frac{\phi(y_{j(k:k)i})}{\Phi(y_{j(k:k)i})}\right)+\sum_{i=1}^{m_3}\sum_{j=1}^{k}(j-1)\frac{\phi(y_{j(j:j)i})}{\Phi(y_{j(j:j)i})}\left(y_{j(j:j)i}+\frac{\phi(y_{j(j:j)i})}{\Phi(y_{j(j:j)i})}\right)$$
$$+(\frac{k}{2}-1)\sum_{i=1}^{m_5}\left\{\sum_{j=1}^{\frac{k}{2}}\frac{\phi(y_{j(\frac{k}{2}:k)i})}{\Phi(y_{j(\frac{k}{2}:k)i})}\left(y_{j(\frac{k}{2}:k)i}+\frac{\phi(y_{j(\frac{k}{2}:k)i})}{\Phi(y_{j(\frac{k}{2}:k)i})}\right)+\frac{k}{2}\sum_{j=\frac{k}{2}+1}^{k}\frac{\phi(y_{j(\frac{k}{2}+1:k)i})}{\Phi(y_{j(\frac{k}{2}+1:k)i})}\right.$$
$$\left.\times\left(y_{j(\frac{k}{2}+1:k)i}+\frac{\phi(y_{j(\frac{k}{2}+1:k)i})}{\Phi(y_{j(\frac{k}{2}+1:k)i})}\right)\right\},$$

and

$$\eta_2=\sum_{i=1}^{m_1}\sum_{j=1}^{k}(k-j)\frac{\phi(y_{j(j:k)i})}{\Phi(-y_{j(j:k)i})}\left(y_{j(j:k)i}-\frac{\phi(y_{j(j:k)i})}{\Phi(-y_{j(j:k)i})}\right)+(k-1)\sum_{i=1}^{m_2}\sum_{j=1}^{\frac{k}{2}}\frac{\phi(y_{j(1:k)i})}{\Phi(-y_{j(1:k)i})}$$
$$\times\left(y_{j(1:k)i}-\frac{\phi(y_{j(1:k)i})}{\Phi(-y_{j(1:k)i})}\right)+\sum_{i=1}^{m_4}\sum_{j=1}^{k}(j-1)\frac{\phi(y_{j(1:j)i})}{\Phi(-y_{j(1:j)i})}\left(y_{j(1:j)i}-\frac{\phi(y_{j(1:j)i})}{\Phi(-y_{j(1:j)i})}\right)$$
$$+\frac{k}{2}\sum_{i=1}^{m_5}\left\{\sum_{j=1}^{\frac{k}{2}}\frac{\phi(y_{j(\frac{k}{2}:k)i})}{\Phi(-y_{j(\frac{k}{2}:k)i})}\left(y_{j(\frac{k}{2}:k)i}-\frac{\phi(y_{j(\frac{k}{2}:k)i})}{\Phi(-y_{j(\frac{k}{2}:k)i})}\right)+(\frac{k}{2}-1)\sum_{j=\frac{k}{2}+1}^{k}\frac{\phi(y_{j(\frac{k}{2}+1:k)i})}{\Phi(-y_{j(\frac{k}{2}+1:k)i})}\right.$$
$$\left.\times\left(y_{j(\frac{k}{2}+1:k)i}-\frac{\phi(y_{j(\frac{k}{2}+1:k)i})}{\Phi(-y_{j(\frac{k}{2}+1:k)i})}\right)\right\},$$

such that $y_{r(j:k)i}=x_{r(j:k)i}-\mu$, for each r,j,k and i. Since $\left(x+\frac{\phi(x)}{\Phi(x)}\right)>0$ and $\left(x-\frac{\phi(x)}{\Phi(-x)}\right)<0$ for all x (see, [5]), so it is clear that $\eta_1>0$ and $\eta_2<0$ and therefore $\frac{\partial^2 l(\mu)}{\partial\mu^2}$ is always negative. Hence, the MLE of μ is unique. But, since there is no any closed form for $\hat{\mu}$, it can be obtained numerically.

4. Simulation and numerical computations

In the previous section, we proposed two different estimators for the location parameter of the normal distribution based on the URSS scheme. Further, it was deduced that in the case of $m_3 = m_4$, $\bar{X}_{URSS}$ is more efficient than $\bar{X}_{SRS}$. In this case, we now compute the relative precision (RP) of the MLE based on URSS scheme, $\hat{\mu}$, with respect to $\bar{X}_{URSS}$. Such a comparison helps us to determine the best choice of **m** in practice. The RP of the estimator T_1 with respect to T_2 is generally defined as

$$\mathrm{RP(T_1, T_2)} = \frac{\mathrm{MSE(T_2)}}{\mathrm{MSE(T_1)}},$$

where MSE(T) stands for the mean squared error of T. To compute $\mathrm{RP}(\hat{\mu}, \bar{\mathrm{X}}_{\mathrm{URSS}})$, we obtain $\mathrm{MSE}(\bar{\mathrm{X}}_{\mathrm{URSS}})$ from (2.2) using numerical computations. The MLE of μ though has not any closed form, but we can obtain it by simulating the equation (3.3). To this purpose, we are used **uniroot** command to find the root of the equation (3.3) using simulation in R software. The simulation algorithm is repeated 10^5 for all permutations of $\mathbf{m} = (m_1, m_2, m_3, m_4, m_5)$ that $m_3 = m_4$ and then the $\mathrm{MSE}(\hat{\mu})$ is computed. The results are presented in Table 1 for $M = 5$ and $k = 4, 6$. In this case, there are 34 permutations for **m** which are sorted in descending order of RPs.

From Table 1, in the problem of estimation the mean of normal distribution, for all permutations of **m** with $m_3 = m_4$, it is deduced that

- The RP of the estimator $\hat{\mu}$ is more than $\bar{X}_{URSS}$.
- By increasing k, the RPs increase, which is trivial.
- Combining some of various existing sampling schemes leads to the most RP for MLE; they are (0,0,2,2,1) for $k = 4$ and (0,0,1,1,3) for $k = 6$. Nevertheless, there are some other permutations of **m** for which the $\mathrm{RP}(\hat{\mu}, \bar{\mathrm{X}}_{\mathrm{URSS}})$ is at least 98 percent of the highest RP. So, with a little connivance, there are more than one choice for the permutations of **m** in practice. Among these choices, one may select the one with the minimum ranking error and cost of sampling.

Table 1. Values of $\mathrm{RP}(\hat{\mu}, \bar{\mathrm{X}}_{\mathrm{URSS}})$ for all permutations of **m** with $m_3 = m_4$, when $k = 4, 6$ and $M = 5$.

$k=4$		$k=4$		$k=6$		$k=6$	
m	RP	**m**	RP	**m**	RP	**m**	RP
(0,0,2,2,1)	1.13778	(2,3,0,0,0)	1.04158	(0,0,1,1,3)	1.24255	(3,2,0,0,0)	1.08563
(0,0,1,1,3)	1.12709	(3,1,0,0,1)	1.04121	(0,1,1,1,2)	1.22414	(0,2,0,0,3)	1.08198
(1,0,1,1,2)	1.12296	(5,0,0,0,0)	1.03989	(1,0,1,1,2)	1.22371	(4,1,0,0,0)	1.08159
(2,0,1,1,1)	1.12237	(2,2,0,0,1)	1.03876	(0,0,2,2,1)	1.22062	(1,3,0,0,1)	1.08051
(0,1,1,1,2)	1.11842	(0,2,0,0,3)	1.03667	(2,0,1,1,1)	1.21281	(2,3,0,0,0)	1.07995
(1,0,2,2,0)	1.11624	(1,2,0,0,2)	1.03479	(1,1,1,1,1)	1.19851	(3,1,0,0,1)	1.07909
(3,0,1,1,0)	1.11264	(3,2,0,0,0)	1.03391	(3,0,1,1,0)	1.19668	(2,1,0,0,2)	1.07406
(0,2,1,1,1)	1.11217	(0,4,0,0,1)	1.03305	(1,0,2,2,0)	1.18392	(1,4,0,0,0)	1.06957
(1,1,1,1,1)	1.11003	(2,1,0,0,2)	1.02989	(2,1,1,1,0)	1.17284	(1,1,0,0,3)	1.06422
(0,1,2,2,0)	1.10630	(3,0,0,0,2)	1.02985	(0,2,1,1,1)	1.16993	(4,0,0,0,1)	1.06103
(2,1,1,1,0)	1.09863	(1,1,0,0,3)	1.02978	(1,2,1,1,0)	1.14771	(5,0,0,0,0)	1.05807
(1,2,1,1,0)	1.08123	(1,4,0,0,0)	1.02943	(0,1,2,2,0)	1.12564	(3,0,0,0,2)	1.05257
(0,3,1,1,0)	1.07445	(0,5,0,0,0)	1.02567	(0,3,1,1,0)	1.10036	(0,1,0,0,4)	1.05154
(1,3,0,0,1)	1.04491	(2,0,0,0,3)	1.02274	(1,2,0,0,2)	1.09473	(0,5,0,0,0)	1.04334
(4,1,0,0,0)	1.04449	(0,1,0,0,4)	1.01981	(0,3,0,0,2)	1.09010	(2,0,0,0,3)	1.03457
(4,0,0,0,1)	1.04196	(1,0,0,0,4)	1.01473	(2,2,0,0,1)	1.08979	(1,0,0,0,4)	1.02077
(0,3,0,0,2)	1.04162	(0,0,0,0,5)	1.00367	(0,4,0,0,1)	1.08592	(0,0,0,0,5)	1.00510

When the condition $m_3 = m_4$ is omitted, the values of $\mathrm{RP}(\hat{\mu}, \bar{\mathrm{X}}_{\mathrm{URSS}})$ and $\mathrm{RP}(\bar{\mathrm{X}}_{\mathrm{URSS}}, \bar{\mathrm{X}}_{\mathrm{SRS}})$ should be computed. For $M = 5$, the number of all permutations of **m** is equal to 126. Table 2 shows the RPs for $k = 4$ and 6. The RPs in this table are computed as illustrated in Table 1. To summarize the results of these tables, only the first ten highest and the last ten lowest RPs are reported.

Table 2. The RPs for some permutations of **m**, when $k=4,6$ and $M=5$.

$k=4$				$k=6$			
$RP(\hat{\mu}, \bar{X}_{URSS})$		$RP(\bar{X}_{URSS}, \bar{X}_{SRS})$		$RP(\hat{\mu}, \bar{X}_{URSS})$		$RP(\bar{X}_{URSS}, \bar{X}_{SRS})$	
m	RP	**m**	RP	**m**	RP	**m**	RP
(0,0,0,3,2)	1.16215	(0,0,0,0,5)	2.77426	(0,0,2,1,2)	1.26525	(0,0,0,0,5)	4.06153
(0,0,2,1,2)	1.15203	(1,0,0,0,4)	2.67679	(0,0,0,3,2)	1.26458	(1,0,0,0,4)	3.84983
(0,0,3,0,2)	1.14670	(2,0,0,0,3)	2.58593	(0,0,1,2,2)	1.26112	(2,0,0,0,3)	3.65911
(0,0,1,2,2)	1.14204	(0,1,0,0,4)	2.58593	(0,0,3,0,2)	1.25591	(0,1,0,0,4)	3.56944
(0,0,1,3,1)	1.14111	(3,0,0,0,2)	2.50104	(1,0,1,2,1)	1.24331	(3,0,0,0,2)	3.48640
(0,0,0,2,3)	1.13986	(1,1,0,0,3)	2.50104	(0,0,1,1,3)	1.24255	(1,1,0,0,3)	3.40490
(0,0,0,4,1)	1.13941	(4,0,0,0,1)	2.42154	(0,0,2,0,3)	1.24185	(4,0,0,0,1)	3.32925
(1,0,1,2,1)	1.13892	(2,1,0,0,2)	2.42154	(1,0,0,3,1)	1.23798	(2,1,0,0,2)	3.25486
(1,0,0,3,1)	1.13871	(0,2,0,0,3)	2.42154	(0,0,0,2,3)	1.23758	(5,0,0,0,0)	3.18566
(0,0,2,2,1)	1.13778	(0,0,0,1,4)	2.35289	(1,0,2,1,1)	1.23396	(0,2,0,0,3)	3.18371
(0,4,0,0,1)	1.03305	(0,1,1,3,0)	1.55058	(1,4,0,0,0)	1.06957	(0,1,1,3,0)	1.77715
(2,1,0,0,2)	1.02989	(0,1,2,2,0)	1.55058	(1,1,0,0,3)	1.06422	(0,1,2,2,0)	1.77715
(3,0,0,0,2)	1.02985	(0,1,3,1,0)	1.55058	(4,0,0,0,1)	1.06103	(0,1,3,1,0)	1.77715
(1,1,0,0,3)	1.02978	(0,1,4,0,0)	1.55058	(5,0,0,0,0)	1.05807	(0,1,4,0,0)	1.77715
(1,4,0,0,0)	1.02943	(0,0,0,5,0)	1.46366	(3,0,0,0,2)	1.05257	(0,0,0,5,0)	1.66836
(0,5,0,0,0)	1.02567	(0,0,1,4,0)	1.46366	(0,1,0,0,4)	1.05154	(0,0,1,4,0)	1.66836
(2,0,0,0,3)	1.02274	(0,0,2,3,0)	1.46366	(0,5,0,0,0)	1.04334	(0,0,2,3,0)	1.66836
(0,1,0,0,4)	1.01981	(0,0,3,2,0)	1.46366	(2,0,0,0,3)	1.03457	(0,0,3,2,0)	1.66836
(1,0,0,0,4)	1.01473	(0,0,4,1,0)	1.46366	(1,0,0,0,4)	1.02077	(0,0,4,1,0)	1.66836
(0,0,0,0,5)	1.00367	(0,0,5,0,0)	1.46366	(0,0,0,0,5)	1.00510	(0,0,5,0,0)	1.66836

From Table 2, the additional results are deduced as follows

- The RP of $\bar{X}_{URSS}$ is more than $\bar{X}_{SRS}$, for all permutations of **m** not only when $m_3=m_4$.
- For $k=4$, using the permutation (0,0,0,3,2) leads to the most precise URSS scheme. However, if one of (0,0,2,1,2), (0,0,3,0,2), (0,0,1,2,2), (0,0,1,3,1), (0,0,0,2,3), (0,0,0,4,1) or (1,0,1,2,1) has less ranking error or cost of sampling than the most efficient permutation, it may be used in practice. Note that the precisions of the later 7 permutations are at least 97 percent of (0,0,0,3,2).
- For $k=6$, the most precise permutation is (0,0,2,1,2), whereas one of (0,0,0,3,2), (0,0,1,2,2), (0,0,3,0,2), (1,0,1,2,1), (0,0,1,1,3) or (0,0,2,0,3) may also be recommended with the precision at least 98 percent of (0,0,2,1,2), if their ranking error or cost of sampling is less than (0,0,2,1,2).

If only minimizing the ranking error is of interest, then it is better to use the permutations with smaller values of m_1 and m_5. This issue should be considered for choosing the best URSS scheme.

In Tables 1 and 2, the results are presented for $k=4,6$ and $M=5$. For other values of k and M, the results may be derived to determine the best choice of **m**.

5. Cost analysis

It is obvious that sampling, ranking and specially measuring the units in each scheme are costly. This issue is studied by some authors in the field of RSS scheme. [11] investigated the optimal set size based on the RSS procedure with cost considerations. They considered the relative efficiency (RE) of RSS with respect to SRS method as a function of the associated RP and required costs. An extension to these results had been done by [18] to show that taking two or more observations from each ranked set is beneficial. In this context, we investigate the cost of URSS in comparison with SRS scheme for estimating the mean of normal distribution when the sample sizes are the same, i.e., the sample size of the SRS scheme is also equal to $N{=}Mk$. First of all, let us introduce the following notations:

[11] showed that the needed number of pairwise comparisons for visual ranking in a set of size k is approximately $f(k)\approx\frac{(k+2)(k-1)}{2}$. On the other hand, the number of pairwise comparisons for RSS and MRSS schemes are the same, which are equal to $f(k)$, while it is easy to see that the needed number of pairwise comparisons for visual ranking in ERSS, MERSS-I and MERSS-II is $g(k)\approx k-1$. It is important to highlight that the cost of ranking in each set using RSS and MRSS is more than cost of ranking in ERSS. Also, the cost of ranking based on the MERSS-I and MERSS-II

C_{ERSS}	total cost of ERSS scheme,
$C_{MERSS-I}$	total cost of MERSS-I scheme,
$C_{MERSS-II}$	total cost of MERSS-II scheme,
C_{MRSS}	total cost of MRSS scheme,
C_{URSS}	total cost of URSS scheme,
c_i	cost for sampling one unit,
c_q	cost of quantification of the interested variable for one unit,
c_{r1}	cost of one pairwise comparison in RSS and MRSS schemes,
c_{r2}	cost of one pairwise comparison for ERSS scheme,
c_{r3}	cost of one pairwise comparison using MERSS-I and MERSS-II schemes.

is less than the cost of ranking using ERSS. In other words, $c_{r3} \le c_{r2} \le c_{r1}$. Hence, the cost of sampling for various scheme may be given by

$C_{SRS} = N\,(c_i + c_q)$,
$C_{RSS} = m_1 k\,(kc_i + f(k)c_{r1} + c_q)$,
$C_{ERSS} = m_2 k\,(kc_i + g(k)c_{r2} + c_q)$,
$C_{MERSS-I} = m_3 k\left[\frac{k+1}{2}c_i + g(k)c_{r3} + c_q\right]$,
$C_{MERSS-II} = m_4 k\left[\frac{k+1}{2}c_i + g(k)c_{r3} + c_q\right]$,
$C_{MRSS} = m_5 k\,(kc_i + f(k)c_{r1} + c_q)$.

Since, for any $k \ge 1$, $\frac{k+1}{2} \le k$ and $g(k) \le f(k)$, it is obvious that the costs of sampling of ERSS, MERSS-I and MERSS-II schemes are less than the cost of RSS and MRSS schemes. In addition, it is clear that the total cost of URSS scheme is obtained via

$$\begin{aligned} C_{URSS} &= C_{RSS} + C_{ERSS} + C_{MERSS-I} + C_{MERSS-II} + C_{MRSS} \\ &= k\,(m_1 + m_5)\,[kc_i + f(k)c_{r1} + c_q] + m_2 k(kc_i + g(k)c_{r2} + c_q) \\ &\quad + (m_3 + m_4)\left[\frac{k(k+1)}{2}c_i + k(g(k)c_{r3} + c_q)\right]. \end{aligned}$$

Therefore, considering the cost of sampling for each method, the RE of $\bar{X}_{URSS}$ with respect to $\bar{X}_{SRS}$ may be defined as

$$\mathrm{RE}(\bar{X}_{URSS}, \bar{X}_{SRS}) = \frac{C_{SRS}}{C_{URSS}} \mathrm{RP}(\bar{X}_{URSS}, \bar{X}_{SRS}). \qquad (5.1)$$

Based on the philosophy of the RSS method, it should be noted that cost of sampling one unit (c_i) is low, while the cost of quantification of the interested variable for one unit is high. Here, we consider the costs of sampling, quantifications and pairwise comparisons to be $c_{r1} = 4, c_{r2} = 1.25, c_{r3} = 1, c_q = 31$ and $c_i = 1$. Using (5.1), for a given permutation of $\mathbf{m}$, the RE of $\bar{X}_{URSS}$ with respect to $\bar{X}_{SRS}$ is presented in Table 3 for $k = 4, 6$ and $M = 5$. In this table, only ten lowest and ten highest REs are exhibited.

Table 3. Values of $\mathrm{RE}(\bar{X}_{URSS}, \bar{X}_{SRS})$ for some permutations of $\mathbf{m}$, when $k = 4, 6$ and $M = 5$.

$k=4$				$k=6$			
m	RE	**m**	RE	**m**	RE	**m**	RE
(0,2,3,0,0)	1.67944	(5,0,0,0,0)	1.28321	(0,4,0,1,0)	1.77888	(1,2,1,1,0)	1.24487
(0,0,0,0,5)	1.67503	(1,3,1,0,0)	1.27929	(0,0,0,0,5)	1.68790	(0,1,1,3,0)	1.24104
(0,1,2,0,2)	1.65261	(1,4,0,0,0)	1.27929	(0,1,3,1,0)	1.67546	(0,1,2,0,2)	1.24104
(0,0,0,3,2)	1.65004	(2,0,0,0,3)	1.27929	(1,0,4,0,0)	1.66351	(1,2,2,0,0)	1.21014
(0,0,4,0,1)	1.63909	(2,0,0,1,2)	1.27929	(1,1,0,0,3)	1.66351	(1,3,0,0,1)	1.21014
(0,0,1,2,2)	1.63825	(2,3,0,0,0)	1.27260	(0,0,0,3,2)	1.62594	(1,3,0,1,0)	1.21014
(0,0,0,1,4)	1.61617	(3,0,0,0,2)	1.27260	(0,0,5,0,0)	1.62014	(1,3,1,0,0)	1.21014
(0,2,0,1,2)	1.60728	(3,0,0,1,1)	1.27260	(0,2,1,2,0)	1.61811	(1,0,0,3,1)	1.20563
(0,0,0,5,0)	1.59587	(3,0,0,2,0)	1.27260	(0,0,1,3,1)	1.60439	(1,0,0,4,0)	1.20563
(0,1,0,3,1)	1.59153	(3,0,1,0,1)	1.27260	(0,0,0,1,4)	1.59993	(1,0,1,0,3)	1.20563

From Table 3, it is deduced that the estimator $\bar{X}_{URSS}$ has better performance than $\bar{X}_{SRS}$, for all permutations of $\mathbf{m}$, when $k = 4, 6$. Moreover, ombining some of existing sampling schemes may lead to the highest RE. For example, when $k = 4$ or $k = 6$, the permutations (0,2,3,0,0) or (0,4,0,1,0) have the highest RE, respectively. Such a URSS scheme depends on the values of k, M and also various sampling costs (c_i, c_q, c_{r1}, c_{r2} and c_{r3}), which must be examined in different real situations.

6. Conclusion

In this paper, a URSS scheme including ordinary RSS, ERSS, MERSS Type-I, MERSS Type-II and MRSS was considered. Of course, more other sampling schemes may be included in a unified sampling scheme. In the proposed scheme, it was tried to use the more common ranked set sampling schemes and it was shown that combining some various schemes may lead to more efficiency.

Since, the sample mean is a well-defined estimator of the population mean, the weighted mean of all mentioned sampling schemes, $\bar{X}_{URSS}$, was introduced as a reasonable estimator for the mean of population. It was shown analytically that $\bar{X}_{URSS}$ is unbiased, when the parent distribution is symmetric and the number of cycles of MERSS Type-I and MERSS Type-II schemes are equal. When the underlying distribution is normal, it was deduced that $\bar{X}_{URSS}$ is more efficient than $\bar{X}_{SRS}$, for all permutations of $\mathbf{m}$. Moreover, the MLE of the location parameter of the normal distribution, $\hat{\mu}$, based on $\mathbf{X}_{URSS}$ was investigated and its existence and uniqueness were confirmed. Using simulation and numerical computations, it was deduced that $\hat{\mu}$ is more efficient than $\bar{X}_{URSS}$. Also, through a cost analysis, the most efficient choice of $\mathbf{m}$ was suggested to perform the URSS scheme in practice for the case of normal distribution. This problem may also be studied for more other distributions.

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