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Lyapunov inequalities for dynamic equations via new Opial type inequalities

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Abstract

In this paper, we prove some new dynamic inequalities of Opial type on time scales. By employing these new inequalities we establish some new Lyapunov type inequalities for a second order dynamic equation with a damping term. These new Lyapunov inequalities give lower bounds on the distance between zeros of a solution and/or its derivative.

Keywords: Disfocality, Opial's inequality, Lyapunov inequality, dynamic equations, time scales.

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1. Opial's type inequalities

In this section, we prove some new dynamic Opial-type inequalities on a time scale $\mathbb{T}$. The main results in this section present a slight improvement on some Opial inequalities proved in the literature. Before doing this, we give a brief introduction on the development of Opial's type inequalities and some concepts related to time scales calculus. In 1960, Z. Opial [18] published an inequality involving integrals of a function and its derivative of the form

$$(1.1)\qquad \int_0^h \left|f(t)f^{'}(t)\right| dt \leq \frac{h}{4}\int_0^h \left|f^{'}(t)\right|^2 dt,$$

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where $f \in C^1[0,h]$, $f(0) = f(h) = 0$, and $f > 0$ on $(0,h)$, and the constant $h/4$ is the best possible. Olech [17] extended the inequality (1.1) and proved that if f is absolutely continuous on $[0,h]$ and $f(0) = 0$, then

$$\int_0^h \left|f(t)f'(t)\right| dt \leq \frac{h}{2} \int_0^h \left|f'(t)\right|^2 dt. \tag{1.2}$$

These two inequalities were generalized and extended on time scales; for more details, we refer the reader to the book [2]. Time scales were introduced in [13] to unify the study of differential and difference equations. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus, that is, when $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}$ and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$ where $q > 1$. The forward jump operator and the backward jump operator are defined by:

$$\sigma(t) := \inf\{s \in \mathbb{T} : \ s > t\}, \quad \rho(t) := \sup\{s \in \mathbb{T} : \ s < t\}.$$

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be right–dense continuous (rd–continuous) provided f is continuous at right–dense points and at left–dense points in $\mathbb{T}$, left hand limits exist and are finite. The set of all such rd–continuous functions is denoted by $C_{rd}(\mathbb{T})$. For more details of time scale analysis, we refer the reader to the books by Bohner and Peterson [5, 6]. We will use the following product and quotient rules for the derivative of the product fg and the quotient f/g (where $gg^\sigma \neq 0$, here $g^\sigma = g \circ \sigma$) of two differentiable function f and g

$$(fg)^\Delta = f^\Delta g + f^\sigma g^\Delta = fg^\Delta + f^\Delta g^\sigma, \text{ and } \left(\frac{f}{g}\right)^\Delta = \frac{f^\Delta g - fg^\Delta}{gg^\sigma}. \tag{1.3}$$

The chain rule formula is given by

$$(x^\gamma(t))^\Delta = \gamma \int_0^1 [hx^\sigma + (1-h)x]^{\gamma-1} \, dh x^\Delta(t), \tag{1.4}$$

which is a simple consequence of Keller's chain rule [5, Theorem 1.90]. In this paper, we will refer to the (delta) integral which we can define as follows. If $G^\Delta(t) = g(t)$, then the Cauchy (delta) integral of g is defined by $\int_a^t g(s)\Delta s := G(t) - G(a)$. It can be shown (see [5]) that if $g \in C_{rd}(\mathbb{T})$, then the Cauchy integral $G(t) := \int_{t_0}^t g(s)\Delta s$ exists, $t_0 \in \mathbb{T}$, and satisfies $G^\Delta(t) = g(t)$, $t \in \mathbb{T}$. An infinite integral is defined as $\int_a^\infty g(t)\Delta t = \lim_{b\to\infty} \int_a^b g(t)\Delta t$. On discrete time scales $\int_a^b g(t)\triangle t = \sum_{t\in[a,b)} \mu(t)g(t)$. The integration by parts formula on time scales is given by

$$\int_a^b u(t)v^\Delta(t)\Delta t = [u(t)v(t)]_a^b - \int_a^b u^\Delta(t)v^\sigma(t)\Delta t. \tag{1.5}$$

Hölder's inequality on time scale [5, Theorem 6.13] is given by

$$\int_a^b |u(t)v(t)| \, \Delta t \leq \left[\int_a^b |u(t)|^q \, \Delta t\right]^{\frac{1}{q}} \left[\int_a^b |v(t)|^p \, \Delta t\right]^{\frac{1}{p}}, \tag{1.6}$$

where a, $b \in \mathbb{T}$, u, $v \in C_{rd}([a,b]_\mathbb{T}, \mathbb{R})$, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Throughout this paper, we will assume that the functions in the statements of the theorems are positive and rd-continuous functions and the integrals considered are assumed to exist. Without loss of generality, we assume that $\sup \mathbb{T} = \infty$, and define the time scale interval $[a,b]_\mathbb{T}$ by $[a,b]_\mathbb{T} := [a,b] \cap \mathbb{T}$.

In the following, we present some results for dynamic Opial type inequalities that serve and motivate the contents of this paper. In [4] the authors extended (1.2) to an arbitrary time scale $\mathbb{T}$ and proved that if $y : [0, b] \cap \mathbb{T} \longrightarrow \mathbb{R}$ is delta differentiable with $y(0) = 0$, then

$$\int_0^h |y(t) + y^\sigma(t)| \left|y^\Delta(t)\right| \Delta t \le h \int_0^h \left|y^\Delta(t)\right|^2 \Delta t. \tag{1.7}$$

In [24] the authors showed that if $p,\ q \in C_{rd}\left([a, b]_{\mathbb{T}}, \mathbb{R}^+\right)$ and $y \in C^1_{rd}\left([a, b]_{\mathbb{T}}, \mathbb{R}\right)$ with $y(a) = 0$, then for $k > 1$, $\lambda > 0$ and $0 < \gamma < k$

$$\int_a^b q(t) |y(t)|^\lambda \left|y^\Delta(t)\right|^\gamma \Delta t \le K(\lambda, \gamma, k) \left[\int_a^b p(t) \left|y^\Delta(t)\right|^k \Delta t\right]^{(\lambda+\gamma)/k}, \tag{1.8}$$

where

$$K(\lambda, \gamma, k) := \left(\frac{\gamma}{\lambda + \gamma}\right)^{\gamma/k} \left[\int_a^b \left(\frac{q^k(t)}{p^\gamma(t)}\right)^{\frac{1}{k-\gamma}} \left(\int_a^t p^{\frac{-1}{k-1}}(s)\Delta s\right)^{\frac{\lambda(k-1)}{(k-\gamma)}} \Delta t\right]^{\frac{k-\gamma}{k}}.$$

In [1] the authors established that if $p,\ q \in C_{rd}\left([a, b]_{\mathbb{T}}, \mathbb{R}^+\right)$ and $y \in C^1_{rd}\left([a, b]_{\mathbb{T}}, \mathbb{R}\right)$ with $y(a) = 0$, then for $\lambda \ge 1$, $\gamma \ge 0$ and $k > \gamma + 1$

$$\int_a^b q(t) \left|(y^\lambda)^\Delta(t)(y^\Delta(t))^\gamma\right| \Delta t \le K \left\{\int_a^b p(t) \left|y^\Delta(t)\right|^k \Delta t\right\}^{\frac{\lambda+\gamma}{k}}, \tag{1.9}$$

where

$$K := c \left\{\int_a^b (q(t))^{\frac{k}{k-\gamma-1}} \frac{\left(R^{\frac{k\lambda-\lambda-\gamma}{k-\gamma-1}}\right)^\Delta(t)}{(p(t))^{\frac{k\gamma}{(k-1)(k-\gamma-1)}}} \Delta t\right\}^{\frac{k-\gamma-1}{k}}$$

with

$$c = \lambda \left(\frac{k-\gamma-1}{k\lambda - \lambda - \gamma}\right)^{\frac{k-\gamma-1}{k}} \left(\frac{\gamma+1}{\lambda+\gamma}\right)^{\frac{\gamma+1}{k}} \quad \text{and} \quad R(t) = \int_a^t \frac{\Delta s}{(p(s))^{\frac{1}{k-1}}}.$$

As a special case of (1.9) if $\gamma = 0$, then

$$\int_a^b q(t) \left|(y^\lambda)^\Delta(t)\right| \Delta t \le K_1 \left\{\int_a^b p(t) \left|y^\Delta(t)\right|^k \Delta t\right\}^{\frac{\lambda}{k}}, \tag{1.10}$$

where

$$K_1 =: \left\{\int_a^b (q(t))^{\frac{k}{k-1}} \left(R^\lambda\right)^\Delta (t)\Delta t\right\}^{\frac{k-1}{k}}.$$

In the following, we prove some new inequalities of Opial's type on time scales.

1.1. Theorem. *Assume that $\mathbb{T}$ be a time scale with $a,\ \tau \in \mathbb{T}$, $k > 1$, $\lambda \ge 1$, $0 < \gamma < k$ and $k \ge \lambda + \gamma$. Let $p,\ q \in C_{rd}\left([a, \tau]_{\mathbb{T}}, \mathbb{R}^+\right)$ such that $\int_a^t (p(s))^{-1/(k-1)}\Delta s < \infty$. If $y : [a, \tau]_{\mathbb{T}} \to \mathbb{R}$ is delta differentiable with $y(a) = 0$, then*

$$\int_a^\tau q(t) |y^\sigma(t)|^\lambda \left|y^\Delta(t)\right|^\gamma \Delta t \le G_1(\lambda, \gamma, k) \left[\int_a^\tau p(t) \left|y^\Delta(t)\right|^k \Delta t\right]^{(\lambda+\gamma)/k}, \tag{1.11}$$

where

$$G_1(\lambda,\gamma,k) \quad : \quad = 2^{\lambda-1}\left(\frac{\gamma}{\lambda+\gamma}\right)^{\gamma/k}\left[\int_a^\tau \left(\frac{q^k(t)}{p^\gamma(t)}\right)^{\frac{1}{k-\gamma}}\left(\int_a^t p^{\frac{-1}{k-1}}(s)\Delta s\right)^{\frac{\lambda(k-1)}{(k-\gamma)}}\Delta t\right]^{\frac{k-\gamma}{k}}$$
$$+2^{\lambda-1}(\tau-a)^{\frac{k-\lambda-\gamma}{k}}\sup_{a\le t\le\tau}\left(\frac{\mu^\lambda(t)\,q(t)}{(p(t))^{\frac{\lambda+\gamma}{k}}}\right).$$

Proof. Since $y(a)=0$, we have that

$$(1.12)\qquad |y(t)| \le \int_a^t \left|y^\Delta(s)\right|\Delta s = \int_a^t (p(s))^{-1/k}(p(s))^{1/k}\left|y^\Delta(s)\right|\Delta s.$$

Applying Hölder inequality (1.6) with indices $k/(k-1)$ and k on the right hand side of (1.12), we get

$$(1.13)\qquad |y(t)| \le \left(\int_a^t (p(s))^{-1/(k-1)}\Delta s\right)^{(k-1)/k}\left(\int_a^t p(s)\left|y^\Delta(s)\right|^k\Delta s\right)^{1/k}.$$

This implies that

$$(1.14)\qquad |y(t)|^\lambda \le \left(\int_a^t (p(s))^{\frac{-1}{k-1}}\Delta s\right)^{\lambda(k-1)/k}\left(\int_a^t p(s)\left|y^\Delta(s)\right|^k\Delta s\right)^{\lambda/k}.$$

Now, let

$$(1.15)\qquad z(t) := \int_a^t p(s)\left|y^\Delta(s)\right|^k\Delta s.$$

This gives

$$(1.16)\qquad z^\Delta(t) = p(t)\left|y^\Delta(t)\right|^k > 0,$$

and hence

$$(1.17)\qquad \left|y^\Delta(t)\right|^\gamma = \left(\frac{z^\Delta(t)}{p(t)}\right)^{\gamma/k} \text{ and } \left|y^\Delta(t)\right|^{\lambda+\gamma} = \left(\frac{z^\Delta(t)}{p(t)}\right)^{(\lambda+\gamma)/k}.$$

Now, since $y^\sigma = y+\mu y^\Delta$ by applying the inequality

$$(1.18)\qquad |a+b|^\lambda \le 2^{\lambda-1}\left(|a|^\lambda+|b|^\lambda\right), \quad \text{for } \lambda\ge 1,$$

we have that

$$(1.19)\qquad |y^\sigma|^\lambda = \left|y+\mu y^\Delta\right|^\lambda \le 2^{\lambda-1}\left(|y|^\lambda+\mu^\lambda\left|y^\Delta\right|^\lambda\right).$$

Since q is a positive function on $[a,\tau]_{\mathbb{T}}$, we have from (1.14), (1.17) and (1.19) that

$$\begin{aligned}
& q(t)\,|y^\sigma(t)|^\lambda\left|y^\Delta(t)\right|^\gamma \\
\le\ & 2^{\lambda-1}\left[q(t)\,|y(t)|^\lambda\left|y^\Delta(t)\right|^\gamma+\mu^\lambda(t)\,q(t)\left|y^\Delta(t)\right|^{\lambda+\gamma}\right] \\
\le\ & 2^{\lambda-1}q(t)(p(t))^{\frac{-\gamma}{k}}\left(\int_a^t (p(s))^{\frac{-1}{k-1}}\Delta s\right)^{\lambda(k-1)/k}\times(z(t))^{\lambda/k}\left(z^\Delta(t)\right)^{\gamma/k} \\
& +2^{\lambda-1}\mu^\lambda(t)\,q(t)\left(\frac{z^\Delta(t)}{p(t)}\right)^{(\lambda+\gamma)/k}.
\end{aligned}$$

This implies that

$$
\begin{aligned}
& \int_a^\tau q(t)\left|y^\sigma(t)\right|^\lambda\left|y^\Delta(t)\right|^\gamma \Delta t \\
\leq\; & 2^{\lambda-1}\int_a^\tau q(t)(p(t))^{\frac{-\gamma}{k}}\left(\int_a^t (p(s))^{\frac{-1}{k-1}}\Delta s\right)^{\lambda(k-1)/k} \times (z(t))^{\lambda/k}\left(z^\Delta(t)\right)^{\gamma/k}\Delta t \\
& +2^{\lambda-1}\int_a^\tau\left(\frac{\mu^\lambda(t)\,q(t)}{(p(t))^{\frac{\lambda+\gamma}{k}}}\right)\left(z^\Delta(t)\right)^{(\lambda+\gamma)/k}\Delta t \\
\leq\; & 2^{\lambda-1}\int_a^\tau q(t)(p(t))^{\frac{-\gamma}{k}}\left(\int_a^t (p(s))^{\frac{-1}{k-1}}\Delta s\right)^{\lambda(k-1)/k} \times (z(t))^{\lambda/k}\left(z^\Delta(t)\right)^{\gamma/k}\Delta t \\
& +2^{\lambda-1}\sup_{a\leq t\leq\tau}\left(\frac{\mu^\lambda(t)\,q(t)}{(p(t))^{\frac{\lambda+\gamma}{k}}}\right)\int_a^\tau\left(z^\Delta(t)\right)^{\frac{\lambda+\gamma}{k}}\Delta t.
\end{aligned}
$$

Applying Hölder's inequality (1.6) with indices $k/(k-\gamma)$ and k/γ on the first integral on the right hand side and with indices $k/(\lambda+\gamma)$ and $k/(k-\lambda-\gamma)$ on the second integral, we have

$$
\begin{aligned}
& \int_a^\tau q(t)\left|y^\sigma(t)\right|^\lambda\left|y^\Delta(t)\right|^\gamma \Delta t \\
\leq\; & 2^{\lambda-1}\left[\int_a^\tau \frac{q^{\frac{k}{k-\gamma}}(t)}{p^{\frac{\gamma}{k-\gamma}}(t)}\left(\int_a^t p^{\frac{-1}{k-1}}(s)\Delta s\right)^{\frac{\lambda(k-1)}{k-\gamma}}\Delta t\right]^{\frac{k-\gamma}{k}} \\
& \times\left[\int_a^\tau (z(t))^{\lambda/\gamma} z^\Delta(t)\Delta t\right]^{\frac{\gamma}{k}} \\
& +2^{\lambda-1}(\tau-a)^{\frac{k-\lambda-\gamma}{k}}\sup_{a\leq t\leq\tau}\left(\frac{\mu^\lambda(t)\,q(t)}{(p(t))^{\frac{\lambda+\gamma}{k}}}\right)\left(\int_a^\tau z^\Delta(t)\Delta t\right)^{\frac{\lambda+\gamma}{k}}.
\end{aligned}
\tag{1.20}
$$

From the chain rule (1.4), and the fact that $z^\Delta(t)>0$, we see that

$$
(z(t))^{\lambda/\gamma} z^\Delta(t) \leq \frac{\gamma}{\lambda+\gamma}\left(z^{(\lambda+\gamma)/\gamma}(t)\right)^\Delta. \tag{1.21}
$$

Substituting (1.21) into (1.20), we have

$$\begin{aligned}
&\int_a^\tau q(t)\left|y^\sigma(t)\right|^\lambda \left|y^\Delta(t)\right|^\gamma \Delta t \\
\leq\ & 2^{\lambda-1}\left[\int_a^\tau \frac{q^{\frac{k}{k-\gamma}}(t)}{p^{\frac{\gamma}{k-\gamma}}(t)}\left(\int_a^t p^{\frac{-1}{k-1}}(s)\Delta s\right)^{\frac{\lambda(k-1)}{k-\gamma}}\Delta t\right]^{\frac{k-\gamma}{k}} \\
&\times\left(\frac{\gamma}{\lambda+\gamma}\right)^{\gamma/k}\left(\int_\alpha^\tau \left(z^{(\lambda+\gamma)/\gamma}(t)\right)^\Delta \Delta t\right)^{\gamma/k} \\
&+2^{\lambda-1}(\tau-a)^{\frac{k-\lambda-\gamma}{k}}\sup_{a\leq t\leq\tau}\left(\frac{\mu^\lambda(t)\,q(t)}{(p(t))^{\frac{\lambda+\gamma}{k}}}\right)(z(\tau))^{\frac{\lambda+\gamma}{k}} \\
=\ & 2^{\lambda-1}\left[\int_a^\tau \frac{q^{\frac{k}{k-\gamma}}(t)}{p^{\frac{\gamma}{k-\gamma}}(t)}\left(\int_a^t p^{\frac{-1}{k-1}}(s)\Delta s\right)^{\frac{\lambda(k-1)}{k-\gamma}}\Delta t\right]^{\frac{k-\gamma}{k}} \\
&\times\left(\frac{\gamma}{\lambda+\gamma}\right)^{\gamma/k}(z(\tau))^{\frac{\lambda+\gamma}{k}} \\
&+2^{\lambda-1}(\tau-a)^{\frac{k-\lambda-\gamma}{k}}\sup_{a\leq t\leq\tau}\left(\frac{\mu^\lambda(t)\,q(t)}{(p(t))^{\frac{\lambda+\gamma}{k}}}\right)(z(\tau))^{\frac{\lambda+\gamma}{k}}.
\end{aligned}$$

From (1.15) and the last inequality, we have

$$\int_a^\tau q(t)\left|y^\sigma(t)\right|^\lambda \left|y^\Delta(t)\right|^\gamma \Delta t \leq G_1(\lambda,\gamma,k)\left[\int_a^\tau p(t)\left|y^\Delta(t)\right|^k \Delta t\right]^{(\lambda+\gamma)/k},$$

which is the desired inequality (1.11). The proof is complete. □

Here, we only state the following theorem, since its proof is similar to that in Theorem 2.1 with $[a,\,\tau]_{\mathbb{T}}$ replaced by $[\tau,\,b]_{\mathbb{T}}$.

1.2. Theorem. *Assume that $\mathbb{T}$ be a time scale with $\tau,\,b\in\mathbb{T}$, $k>1$, $\lambda\geq 1$, $0<\gamma<k$ and $k\geq\lambda+\gamma$. Let $p,\,q\in C_{rd}\left([\tau,b]_{\mathbb{T}},\mathbb{R}^+\right)$ such that $\int_t^b (p(s))^{-1/(k-1)}\Delta s<\infty$. If $y:[\tau,b]_{\mathbb{T}}\to\mathbb{R}$ is delta differentiable with $y(b)=0$, then*

$$\int_\tau^b q(t)\left|y^\sigma(t)\right|^\lambda \left|y^\Delta(t)\right|^\gamma \Delta t \leq G_2(\lambda,\gamma,k)\left[\int_\tau^b p(t)\left|y^\Delta(t)\right|^k \Delta t\right]^{(\lambda+\gamma)/k}, \tag{1.22}$$

where

$$\begin{aligned}
G_2(\lambda,\gamma,k)\ :\ &= 2^{\lambda-1}\left(\frac{\gamma}{\lambda+\gamma}\right)^{\gamma/k} \\
&\times\left[\int_\tau^b\left(\frac{q^k(t)}{p^\gamma(t)}\right)^{\frac{1}{k-\gamma}}\left(\int_t^b (p(s))^{\frac{-1}{(k-1)}}\Delta s\right)^{\frac{\lambda(k-1)}{(k-\gamma)}}\Delta t\right]^{\frac{k-\gamma}{k}} \\
&+2^{\lambda-1}(b-\tau)^{\frac{k-\lambda-\gamma}{k}}\sup_{\tau\leq t\leq b}\left(\frac{\mu^\lambda(t)\,q(t)}{(p(t))^{\frac{\lambda+\gamma}{k}}}\right).
\end{aligned}$$

In the following, we assume that there exists $\tau\in(a,b)$ which is the unique solution of the equation

$$G(\lambda,\gamma,k)=G_1(\lambda,\gamma,k)=G_2(\lambda,\gamma,k)<\infty. \tag{1.23}$$

Combining (1.11) and (1.22), we have the following result.

1.3. Theorem. *Assume that $\mathbb{T}$ be a time scale with a, $b \in \mathbb{T}$, $k > 1$, $\lambda \geq 1$, $0 < \gamma < k$ and $k \geq \lambda + \gamma$. Let p, $q \in C_{rd}\left([a,b]_{\mathbb{T}}, \mathbb{R}^{+}\right)$ such that $\int_a^b (p(t))^{-1/(k-1)} \Delta t < \infty$. If $y : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is delta differentiable with $y(a) = y(b) = 0$, then*

$$\int_a^b q(t) \left|y^{\sigma}(t)\right|^{\lambda} \left|y^{\Delta}(t)\right|^{\gamma} \Delta t \leq 2^{(k-\lambda-\gamma)/k} G(\lambda, \gamma, k) \left[\int_a^b p(t) \left|y^{\Delta}(t)\right|^{k} \Delta t\right]^{(\lambda+\gamma)/k}, \tag{1.24}$$

where $G(\lambda, \gamma, k)$ is defined as in (1.23).

1.4. Remark. As a special case when we take $k = \lambda + \gamma$, we see that inequality (1.11) becomes

$$\int_a^{\tau} q(t) \left|y^{\sigma}(t)\right|^{\lambda} \left|y^{\Delta}(t)\right|^{\gamma} \Delta t \leq G_1(\lambda, \gamma) \int_a^{\tau} p(t) \left|y^{\Delta}(t)\right|^{\lambda+\gamma} \Delta t, \tag{1.25}$$

where

$$\begin{aligned} G_1(\lambda, \gamma) \quad : \quad &= 2^{\lambda-1} \left(\frac{\gamma}{\lambda+\gamma}\right)^{\gamma/k} \\ &\times \left[\int_a^{\tau} \left(\frac{q^{\lambda+\gamma}(t)}{p^{\gamma}(t)}\right)^{\frac{1}{\lambda}} \left(\int_a^t p^{\frac{-1}{\lambda+\gamma-1}}(s) \Delta s\right)^{\lambda+\gamma-1} \Delta t\right]^{\frac{\lambda}{\lambda+\gamma}} \\ &+ 2^{\lambda-1} \sup_{a \leq t \leq \tau} \left(\mu^{\lambda}(t) \frac{q(t)}{p(t)}\right). \end{aligned}$$

If we put $p = q$ in (1.11), we obtain the following result.

1.5. Corollary. *Assume that $\mathbb{T}$ be a time scale with a, $\tau \in \mathbb{T}$, $k > 1$, $\lambda \geq 1$, $0 < \gamma < k$ and $k \geq \lambda + \gamma$. Let $p \in C_{rd}\left([a,\tau]_{\mathbb{T}}, \mathbb{R}^{+}\right)$ such that $\int_a^t (p(s))^{-1/(k-1)} \Delta s < \infty$. If $y : [a,\tau]_{\mathbb{T}} \to \mathbb{R}$ is delta differentiable with $y(a) = 0$, then*

$$\int_a^{\tau} p(t) \left|y^{\sigma}(t)\right|^{\lambda} \left|y^{\Delta}(t)\right|^{\gamma} \Delta t \leq G_1^{*}(\lambda, \gamma, k) \left[\int_a^{\tau} p(t) \left|y^{\Delta}(t)\right|^{k} \Delta t\right]^{(\lambda+\gamma)/k}, \tag{1.26}$$

where

$$\begin{aligned} G_1^{*}(\lambda, \gamma, k) \quad : \quad &= 2^{\lambda-1} \left(\frac{\gamma}{\lambda+\gamma}\right)^{\gamma/k} \\ &\times \left[\int_a^{\tau} p(t) \left(\int_a^t (p(s))^{\frac{-1}{k-1}} \Delta s\right)^{\frac{\lambda(k-1)}{(k-\gamma)}} \Delta t\right]^{\frac{k-\gamma}{k}} \\ &+ 2^{\lambda-1} (\tau - a)^{\frac{k-\lambda-\gamma}{k}} \sup_{a \leq t \leq \tau} \left(\mu^{\lambda}(t) (p(t))^{\frac{k-\lambda-\gamma}{k}}\right). \end{aligned}$$

1.6. Remark. Note that when $\mathbb{T} = \mathbb{R}$, $p = 1$, $k = 2$ and $\lambda = \gamma = 1$, we have the following result

$$\int_a^{\tau} |y(t)| \left|y'(t)\right| dt \leq \left(\frac{\tau - a}{2}\right) \int_a^{\tau} \left|y'(t)\right|^2 dt. \tag{1.27}$$

2. Lyapunov's type inequalities

In this section, we present some new Lyapunov type inequalities on time scales for the second order half-linear dynamic equations with a damping term

$$\left(r(t)\left(y^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} + p(t)\left(y^{\Delta}(t)\right)^{\gamma} + q(t)\left(y^{\sigma}(t)\right)^{\gamma} = 0, \quad t \in [a,b]_{\mathbb{T}}, \tag{2.1}$$

where $\gamma \geq 1$ is a quotient of odd positive integers, $r(t)$, $p(t)$ and $q(t)$ are rd-continuous functions defined on $\mathbb{T}$ with $r(t) > 0$. In particular, we employ dynamic inequalities of

Opial's type proved in Section 2, to obtain lower bounds for the spacing $b-a$, where y is a solution of (2.1) satisfying $y(a)=y^{\Delta}(b)=0$.

When $\mathbb{T}=\mathbb{R}$, we derive results due to Brown and Hinton [7], Harris and Kong [10] for differential equations and when $\mathbb{T}=\mathbb{Z}$ we present some new result for second-order half-linear difference equations.

We say that a solution y of (2.1) has a generalized zero at t if $y(t)=0$, and has a generalized zero in $(t,\sigma(t))$ if $y(t)y^{\sigma}(t)<0$ and $\mu(t)>0$. Equation (2.1) is disconjugate on the interval $[t_0,b]_{\mathbb{T}}$, if there is no nontrivial solution of (2.1) with two (or more) generalized zeros in $[t_0,b]_{\mathbb{T}}$. The solution $y(t)$ of (2.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. Equation (2.1) is said to be oscillatory if all its solutions are oscillatory.

We say that (2.1) is right disfocal (left disfocal) on $[a,b]_{\mathbb{T}}$ if the solutions of (2.1) such that $y^{\Delta}(a)=0$ $\left(y^{\Delta}(b)=0\right)$ have no generalized zeros in $[a,b]_{\mathbb{T}}$. For the equation (2.1) the point $b>a$ is called a right focal point of a if the solution of (2.1) with initial conditions $y(a)\neq 0$, $y^{\Delta}(a)=0$ satisfies $y(b)=0$. A left focal point is defined similarly. For more details of oscillation theory on dynamic equations on time scales, we refer the reader to the book [19].

We note that, equation (2.1) in its general form covers several different types of differential and difference equations depending on the choice of the time scale $\mathbb{T}$. For example when $\mathbb{T}=\mathbb{R}$, we have $\sigma(t)=t$, $\mu(t)=0$, $y^{\Delta}(t)=y^{'}(t)$ and (2.1) becomes the second-order differential equation

$$\left(r(t)\left(y^{'}(t)\right)^{\gamma}\right)^{'}+p(t)\left(y^{'}(t)\right)^{\gamma}+q(t)\,y^{\gamma}(t)=0. \tag{2.2}$$

When $\mathbb{T}=\mathbb{Z}$, we have $\sigma(t)=t+1$, $\mu(t)=1$, $y^{\Delta}(t)=\Delta y(t)=y(t+1)-y(t)$ and (2.1) becomes the second-order difference equation

$$\Delta\left(r(t)\left(\Delta y(t)\right)^{\gamma}\right)+p(t)\left(\Delta y(t)\right)^{\gamma}+q(t)\,y^{\gamma}(t+1)=0. \tag{2.3}$$

When $\mathbb{T}=q^{\mathbb{N}_0}=\{q^t : t\in\mathbb{N}_0,\ q>1\}$, we have $\sigma(t)=qt$, $\mu(t)=(q-1)\,t$, $y^{\Delta}(t)=\Delta_q y(t)=\frac{y(qt)-y(t)}{(q-1)t}$ and (2.1) becomes the second-order $q-$difference equation

$$\Delta_q\left(r(t)\left(\Delta_q y(t)\right)^{\gamma}\right)+p(t)\left(\Delta_q y(t)\right)^{\gamma}+q(t)\,y^{\gamma}(qt)=0. \tag{2.4}$$

The well-known Lyapunov inequality for Hill's equation

$$y^{''}(t)+q(t)\,y(t)=0,\quad t\in[a,b], \tag{2.5}$$

states that if a, b, $a<b$, are consecutive zeros of a nontrivial solution y of this equation, then

$$\int_a^b |q(t)|\,dt>\frac{4}{b-a}. \tag{2.6}$$

This was later strengthened with $|q(t)|$ replaced by $q^{+}(t)$ by Wintner [26] and thereafter by some other authors, where $q^{+}(t)=\max\{q(t),0\}$ is the nonnegative part of $q(t)$ and $q(t)$ is a real-valued continuous measurable function on $[a,b]$. The constant 4 is the best possible (see [11, Theorem 5.1]). In fact, the best Lyapunov type inequality for (2.5) is

$$\int_a^b (b-t)(t-a)\,q^{+}(t)\,dt>b-a, \tag{2.7}$$

which is due to Hartman [11].

There are several generalizations and extensions of Lyapunov's result. Hartman and Wintner [12] proved that if y is a solution of the linear differential equation

$$y^{''}(t)+p(t)\,y^{'}(t)+q(t)\,y(t)=0,\quad t\in[a,b], \tag{2.8}$$

such that $y(a) = y(b) = 0$ and p, q are a real-valued continuous measurable function on $[a, b]$, then

$$\int_a^b (t-a)(b-t)\, q^+(t)\, dt + \max\left\{\int_a^b (t-a)\,|p(t)|\, dt, \int_a^b (b-t)\,|p(t)|\, dt\right\} > b-a. \tag{2.9}$$

Hartman [11] showed that if y is a solution of the equation

$$\left(r(t)\, y'(t)\right)' + q(t)\, y(t) = 0, \quad t \in [a, b], \tag{2.10}$$

such that $y(a) = y(b) = 0$ and $r(t) > 0$, then

$$\int_a^b q^+(t)\, dt > \frac{4}{\int_a^b r^{-1}(t)\, dt}. \tag{2.11}$$

Cohn [8] and Kwong [15] established that if y is a solution of (2.5) with $y(a) = y'(c) = 0$, then

$$\int_a^c (t-a)\, q(t)\, dt > 1.$$

If $y(b) = y'(c) = 0$, then

$$\int_c^b (b-t)\, q(t)\, dt > 1.$$

Harris and Kong [10] proved that if y is a solution of (2.5) with $y(a) = y'(b) = 0$, then

$$(b-a) \sup_{a \le t \le b} \left|\int_t^b q(s)\, ds\right| > 1. \tag{2.12}$$

If $y(b) = y'(a) = 0$, then

$$(b-a) \sup_{a \le t \le b} \left|\int_a^t q(s)\, ds\right| > 1. \tag{2.13}$$

Brown and Hinton [7] established that if y is a solution of (2.5) with $y(a) = y'(b) = 0$, then

$$2\int_a^b Q^2(t)\,(t-a)\, dt > 1, \quad \text{where} \quad Q(t) = \int_t^b q(s)\, ds. \tag{2.14}$$

If $y(b) = y'(a) = 0$, then

$$2\int_a^b Q^2(t)\,(b-t)\, dt > 1, \quad \text{where} \quad Q(t) = \int_a^t q(s)\, ds. \tag{2.15}$$

Yang [27] showed that if y is a solution of the second-order half-linear equation

$$\left(r(t)\, \varphi\left(y'(t)\right)\right)' + q(t)\, \varphi(y(t)) = 0, \tag{2.16}$$

such that $y(a) = y(b) = 0$ and $\varphi(u) = |u|^{\gamma-1} u$, $\gamma > 0$, then

$$\int_a^b q^+(t)\, dt > 2^{\gamma-1}\left(\int_a^b r^{-1/\gamma}(t)\, dt\right)^{-\gamma}. \tag{2.17}$$

Inequalities similar to (2.17) for half-linear equations were obtained by Došlý and Řehák [9], Lee et al. [16] and Tiryaki [25]. Saker [21] proved that if y is a nontrivial solution of (2.2) with $y(a) = y'(b) = 0$, then

$$K_1(\gamma, p, r) + K_1(\gamma, Q, r) \geq 1 \quad \text{with} \quad Q(t) = \int_t^b q(s)\,ds. \tag{2.18}$$

If $y(b) = y'(a) = 0$, then

$$K_2(\gamma, p, r) + K_2(\gamma, Q, r) \geq 1 \quad \text{with} \quad Q(t) = \int_a^t q(s)\,ds, \tag{2.19}$$

where

$$\begin{cases} K_1(\gamma, p, r) := \left(\frac{\gamma}{\gamma+1}\right)^{\frac{\gamma}{\gamma+1}} \left(\int_a^b \frac{p^{\gamma+1}(t)}{r^{\gamma}(t)} \left(\int_a^t r^{-1/\gamma}(s)\,ds\right)^{\gamma} dt\right)^{\frac{1}{\gamma+1}}, \\ K_1(\gamma, Q, r) := (\gamma+1)^{\frac{\gamma}{\gamma+1}} \left(\int_a^b \frac{Q^{\frac{\gamma+1}{\gamma}}(t)}{r^{\frac{1}{\gamma}}(t)} \left(\int_a^t r^{-1/\gamma}(s)\,ds\right)^{\gamma} dt\right)^{\frac{\gamma}{\gamma+1}}, \\ K_2(\gamma, p, r) := \left(\frac{\gamma}{\gamma+1}\right)^{\frac{\gamma}{\gamma+1}} \left(\int_a^b \frac{p^{\gamma+1}(t)}{r^{\gamma}(t)} \left(\int_t^b r^{-1/\gamma}(s)\,ds\right)^{\gamma} dt\right)^{\frac{1}{\gamma+1}}, \\ K_2(\gamma, Q, r) := (\gamma+1)^{\frac{\gamma}{\gamma+1}} \left(\int_a^b \frac{Q^{\frac{\gamma+1}{\gamma}}(t)}{r^{\frac{1}{\gamma}}(t)} \left(\int_t^b r^{-1/\gamma}(s)\,ds\right)^{\gamma} dt\right)^{\frac{\gamma}{\gamma+1}}. \end{cases}$$

In [3] the authors considered the second-order dynamic equation

$$y^{\Delta\Delta}(t) + q(t)\, y^{\sigma}(t) = 0, \tag{2.20}$$

where q is a positive rd-continuous function defined on $\mathbb{T}$ and showed that if y is a solution of (2.20) with $y(a) = y(b) = 0$, then

$$\int_a^b q(t)\,\Delta t > \frac{4}{b-a}. \tag{2.21}$$

This result was proved by employing the quadratic functional equation

$$F(x) := \int_a^b \left[\left(y^{\Delta}(t)\right)^2 - q(t)\,(y^{\sigma})^2\right] \Delta t = 0.$$

In [14] the authors established that if $q \in C_{rd}\left([a,b]_{\mathbb{T}}, \mathbb{R}\right)$ and y is a solution of (2.20) with $y(a) = y^{\Delta\sigma}(b) = 0$, then

$$\left(2\int_a^{\sigma(b)} Q^2(u)\,(\sigma(u) - a)\,\Delta u\right)^{\frac{1}{2}} \geq 1, \quad \text{where} \quad Q(u) = \int_u^{\sigma(b)} q(t)\,\Delta t. \tag{2.22}$$

In [22] the author proved new Lyapunov type inequalities for the dynamic equation

$$\left(r(t)\, y^{\Delta}(t)\right)^{\Delta} + q(t)\, y^{\sigma}(t) = 0, \quad t \in [a,b]_{\mathbb{T}}, \tag{2.23}$$

where r, q are rd-continuous functions satisfying

$$\int_a^b \frac{\Delta t}{r(t)} < \infty, \quad \text{and} \quad \int_a^b q(t)\,\Delta t < \infty.$$

In [20] the author considered the second-order half-linear dynamic equation

$$\left(r(t)\,\varphi\left(y^{\Delta}(t)\right)\right)^{\Delta} + q(t)\,\varphi\left(y^{\sigma}(t)\right) = 0, \quad t \in [a,b]_{\mathbb{T}}, \tag{2.24}$$

where $\varphi(u) = |u|^{\gamma-1} u$, $\gamma \geq 1$ and r, q are real rd–continuous positive functions, and he showed that if y is a nontrivial solution of (2.24) with $y(a) = y(b) = 0$ and y has a maximum at a point $c \in (a,b)$, then

$$\left(\int_a^b \frac{\Delta t}{r^\gamma(t)}\right)^\gamma \int_a^b q(t)\,\Delta t \geq 2^{\gamma+1}. \tag{2.25}$$

In [23] the author considered the second-order half-linear dynamic equation

$$\left(r(t)\left(y^\Delta(t)\right)^\gamma\right)^\Delta + p(t)\left(y^\Delta(t)\right)^\gamma + q(t)\left(y^\sigma(t)\right)^\gamma = 0, \quad t \in [a,b]_{\mathbb{T}}, \tag{2.26}$$

where $\gamma \geq 1$ is a quotient of odd positive integers, $r(t)$, $p(t)$ and $q(t)$ are rd-continuous functions defined on $\mathbb{T}$ with $r(t) > 0$ and $\mu(t)|p(t)| \leq r(t)/c$, $c \geq 1$. In particular, he proved that if y is a nontrivial solution of (2.26) with $y(a) = y^\Delta(b) = 0$, then

$$\begin{aligned} &2^{2\gamma-2}\Lambda(b) + \frac{2^{3\gamma-2}}{(\gamma+1)^{\frac{1}{\gamma+1}}}\left(\int_a^b \frac{|Q(t)|^{\frac{\gamma+1}{\gamma}}}{r^{\frac{1}{\gamma}}(t)}\left(R_a(t)\right)^\gamma \Delta t\right)^{\frac{\gamma}{\gamma+1}} \\ &+\left(\frac{\gamma}{\gamma+1}\right)^{\frac{\gamma}{\gamma+1}}\left(\int_a^b \frac{|p(t)|^{\gamma+1}}{r^\gamma(t)}\left(R_a(t)\right)^\gamma \Delta t\right)^{\frac{1}{\gamma+1}} \geq 1 - \frac{1}{c}, \end{aligned} \tag{2.27}$$

where

$$\Lambda(b) := \sup_{a \leq t \leq b} \mu^\gamma(t)\frac{|Q(t)|}{r(t)} \quad \text{with} \quad Q(t) = \int_t^b q(s)\,\Delta s,$$

and

$$R_a(t) := \int_a^t \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}.$$

In the following, we will apply the dynamic Opial-type inequalities (1.10) and (1.11) on the second-order half-linear dynamic equation (2.1) to obtain some new Lyapunov-type inequalities. To simplify the presentation of the results, we let

$$\begin{aligned} K_1(a,b,\gamma) \quad &: \quad = \left[\int_a^b |Q(t)|^{\frac{\gamma+1}{\gamma}}\left(R_a^{\gamma+1}\right)^\Delta(t)\Delta t\right]^{\frac{\gamma}{\gamma+1}}, \\ G_1(a,b,\gamma) \quad &: \quad = \sup_{a \leq t \leq b}\left(\mu(t)\frac{|p(t)|}{r(t)}\right) + \left(\frac{\gamma}{\gamma+1}\right)^{\frac{\gamma}{\gamma+1}}\left[\int_a^b \frac{|p(t)|^{\gamma+1}}{r^\gamma(t)}\left(R_a(t)\right)^\gamma \Delta t\right]^{\frac{1}{\gamma+1}}, \end{aligned}$$

where

$$Q(t) = \int_t^b q(s)\Delta s, \quad \text{and} \quad R_a(t) := \int_a^t \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}.$$

2.1. Theorem. *Assume that y is a nontrivial solution of (2.1). If $y(a) = y^\Delta(b) = 0$, then*

$$K_1(a,b,\gamma) + G_1(a,b,\gamma) \geq 1. \tag{2.28}$$

Proof. Without loss of generality we may assume that $y(t) \geq 0$ in $[a,b]_{\mathbb{T}}$. Multiplying (2.1) by y^σ and integrating by parts, we have

$$\begin{aligned} \int_a^b \left(r(t)\left(y^\Delta(t)\right)^\gamma\right)^\Delta y^\sigma \Delta t \quad &= \quad r(t)\left(y^\Delta(t)\right)^\gamma y(t)\Big|_a^b - \int_a^b r(t)\left(y^\Delta(t)\right)^{\gamma+1}\Delta t \\ &= \quad -\int_a^b q(t)\left(y^\sigma(t)\right)^{\gamma+1}\Delta t \\ &\quad\quad -\int_a^b p(t)\left(y^\Delta(t)\right)^\gamma y^\sigma(t)\,\Delta t. \end{aligned}$$

Using the assumptions that $y(a) = y^{\Delta}(b) = 0$ and $Q(t) = \int_t^b q(s)\Delta s$, we get

$$\int_a^b r(t)\left(y^{\Delta}(t)\right)^{\gamma+1}\Delta t = \int_a^b p(t)\left(y^{\Delta}(t)\right)^{\gamma} y^{\sigma}(t)\,\Delta t - \int_a^b Q^{\Delta}(t)\left(y^{\sigma}(t)\right)^{\gamma+1}\Delta t.$$

Integrating by parts the term $\int_a^b Q^{\Delta}(t)\left(y^{\sigma}(t)\right)^{\gamma+1}\Delta t$ and using the fact that $y(a) = 0 = Q(b)$, we obtain

$$\int_a^b r(t)\left(y^{\Delta}(t)\right)^{\gamma+1}\Delta t = \int_a^b p(t)\left(y^{\Delta}(t)\right)^{\gamma} y^{\sigma}(t)\,\Delta t + \int_a^b Q(t)\left(y^{\gamma+1}(t)\right)^{\Delta}\Delta t.$$

This implies that

$$\begin{aligned}(2.29)\qquad \int_a^b r(t)\left|y^{\Delta}(t)\right|^{\gamma+1}\Delta t &\le \int_a^b |p(t)|\,|y^{\sigma}(t)|\left|y^{\Delta}(t)\right|^{\gamma}\Delta t \\ &\quad + \int_a^b |Q(t)|\left|\left(y^{\gamma+1}(t)\right)^{\Delta}\right|\Delta t.\end{aligned}$$

Applying the inequality (1.10) on the integral $\int_a^b |Q(t)|\left|\left(y^{\gamma+1}(t)\right)^{\Delta}\right|\Delta t$, with $q(t) = |Q(t)|$, $p(t) = r(t)$, $\lambda = \gamma+1$ and $k = \gamma+1$, we see that

$$(2.30)\qquad \int_a^b |Q(t)|\left|\left(y^{\gamma+1}(t)\right)^{\Delta}\right|\Delta t \le K_1(a,b,\gamma)\int_a^b r(t)\left|y^{\Delta}(t)\right|^{\gamma+1}\Delta t,$$

where

$$K_1(a,b,\gamma) = \left\{\int_a^b |Q(t)|^{\frac{\gamma+1}{\gamma}}\left(R_a^{\gamma+1}\right)^{\Delta}(t)\Delta t\right\}^{\frac{\gamma}{\gamma+1}} \quad \text{with} \quad R_a(t) = \int_a^t \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}.$$

Applying the inequality (1.11) on the integral $\int_a^b |p(t)|\,|y^{\sigma}(t)|\left|y^{\Delta}(t)\right|^{\gamma}\Delta t$, with $q(t) = |p(t)|$, $p(t) = r(t)$, $\lambda = 1$ and $k = \gamma+1$, we see that

$$(2.31)\qquad \int_a^b |p(t)|\,|y^{\sigma}(t)|\left|y^{\Delta}(t)\right|^{\gamma}\Delta t \le G_1(a,b,\gamma)\int_a^b r(t)\left|y^{\Delta}(t)\right|^{\gamma+1}\Delta t,$$

where

$$\begin{aligned}G_1(a,b,\gamma) &= \left(\frac{\gamma}{\gamma+1}\right)^{\frac{\gamma}{\gamma+1}}\left[\int_a^b \frac{|p(t)|^{\gamma+1}}{r^{\gamma}(t)}\left(R_a(t)\right)^{\gamma}\Delta t\right]^{\frac{1}{\gamma+1}} \\ &\quad + \sup_{a\le t\le b}\left(\mu(t)\frac{|p(t)|}{r(t)}\right).\end{aligned}$$

Substituting (2.30) and (2.31) into (2.29) and cancelling the term $\int_a^b r(t)\left|y^{\Delta}(t)\right|^{\gamma+1}\Delta t$, we have

$$1 \le K_1(a,b,\gamma) + G_1(a,b,\gamma),$$

which is the desired inequality (2.28). The proof is complete. □

2.2. Remark. The result (2.28) is usually connected with the disfocality of (2.1), i.e., if

$$K_1(a,b,\gamma) + G_1(a,b,\gamma) < 1,$$

then (2.1) is right disfocal in $[a,b]_{\mathbb{T}}$. This means that there is no nontrivial solution of (2.1) in $[a,b]_{\mathbb{T}}$ satisfying $y(a) = y^{\Delta}(b) = 0$.

2.3. Remark. Note that inequality (2.28) give an improvement of the inequality (2.27) due to Saker [23] by removing the additional constant c.

If we using the assumption that $|p(t)|\,\mu(t) \le r(t)/c$, $c \ge 1$, we also present an improvement of the inequality (2.27).

2.4. Corollary. *Assume that y is a nontrivial solution of (2.1). If $y(a) = y^{\Delta}(b) = 0$, then*

$$K_1(a,b,\gamma) + G_1^*(a,b,\gamma) \geq 1 - \frac{1}{c}, \tag{2.32}$$

where

$$G_1^*(a,b,\gamma) := \left(\frac{\gamma}{\gamma+1}\right)^{\frac{\gamma}{\gamma+1}} \left[\int_a^b \frac{|p(t)|^{\gamma+1}}{r^{\gamma}(t)} (R_a(t))^{\gamma} \Delta t\right]^{\frac{1}{\gamma+1}}.$$

If we put $r(t) = 1$ in Theorem 3.1, we have the following result.

2.5. Corollary. *Assume that y is a nontrivial solution of (2.1). If $y(a) = y^{\Delta}(b) = 0$, then*

$$\left[\int_a^b |Q(t)|^{\frac{\gamma+1}{\gamma}} \left((t-a)^{\gamma+1}\right)^{\Delta}(t)\Delta t\right]^{\frac{\gamma}{\gamma+1}} + \left(\frac{\gamma}{\gamma+1}\right)^{\frac{\gamma}{\gamma+1}} \left[\int_a^b |p(t)|^{\gamma+1} (t-a)^{\gamma} \Delta t\right]^{\frac{1}{\gamma+1}} + \sup_{a\leq t\leq b} \mu(t)|p(t)| \geq 1.$$

On a time scale $\mathbb{T}$, we note from the chain rule (1.4) that

$$\begin{aligned}\left((t-a)^{\gamma+1}\right)^{\Delta} &= (\gamma+1)\int_0^1 [h(\sigma(t)-a) + (1-h)(t-a)]^{\gamma} dh \\ &\geq (\gamma+1)\int_0^1 [h(t-a) + (1-h)(t-a)]^{\gamma} dh \\ &= (\gamma+1)(t-a)^{\gamma}.\end{aligned}$$

This implies that

$$\int_a^b (t-a)^{\gamma} \Delta t \leq \frac{(b-a)^{\gamma+1}}{\gamma+1}. \tag{2.33}$$

Using the maximum of $|Q(t)|$ and $|p(t)|$ on $[a,b]_{\mathbb{T}}$ and substituting (2.33) into the results of Corollary 3.2, we get the following result.

2.6. Corollary. *Assume that y is a nontrivial solution of (2.1). If $y(a) = y^{\Delta}(b) = 0$, then*

$$\sup_{a\leq t\leq b} \mu(t)|p(t)| + (b-a)^{\gamma} \max_{a\leq t\leq b}\left|\int_t^b q(s)\Delta s\right| + \frac{\gamma^{\frac{\gamma}{\gamma+1}}}{\gamma+1}(b-a)\max_{a\leq t\leq b}|p(t)| \geq 1.$$

If we put $p(t) = 0$, then the result in Corollary 3.3 reduces to the following result for the second-order half-linear dynamic equation

$$\left(\left(y^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} + q(t)(y^{\sigma}(t))^{\gamma} = 0, \quad t \in [a,b]_{\mathbb{T}}. \tag{2.34}$$

2.7. Corollary. *Assume that y is a nontrivial solution of (2.34). If $y(a) = y^{\Delta}(b) = 0$, then*

$$(b-a)^{\gamma} \max_{a\leq t\leq b}\left|\int_t^b q(s)\Delta s\right| \geq 1.$$

If we put $p(t) = 0$ and $\gamma = 1$ in Theorem 3.1, we obtain the following result.

2.8. Corollary. *Assume that y is a nontrivial solution of*

$$\left(r(t)y^{\Delta}(t)\right)^{\Delta} + q(t)y^{\sigma}(t) = 0, \quad \textit{for} \quad t \in [a,b]_{\mathbb{T}}. \tag{2.35}$$

If $y(a) = y^{\Delta}(b) = 0$, *then*

$$\left[\int_a^b |Q(t)|^2 \left(R_a^2\right)^{\Delta}(t)\Delta t\right]^{\frac{1}{2}} \geq 1.$$

With $r(t) = 1$ in Corollary 3.5, we have the following result.

2.9. Corollary. *Assume that* y *is a nontrivial solution of*

$$y^{\Delta\Delta}(t) + q(t)\, y^{\sigma}(t) = 0, \quad \text{for} \quad t \in [a,b]_{\mathbb{T}}. \tag{2.36}$$

If $y(a) = y^{\Delta}(b) = 0$, *then*

$$\left[\int_a^b |Q(t)|^2 \left((t-a)^2\right)^{\Delta}(t)\Delta t\right]^{\frac{1}{2}} \geq 1.$$

2.10. Remark. As a special case of Theorem 3.1 when $\mathbb{T} = \mathbb{R}$, we get the inequality (2.18) due to Saker [21].

2.11. Remark. As a special case of Corollary 3.4 when $\mathbb{T} = \mathbb{R}$, $\gamma = 1$, we get the inequality (2.12) due to Harris and Kong [10].

2.12. Remark. As a special case of Corollary 3.6 when $\mathbb{T} = \mathbb{R}$, we get the inequality (2.14) due to Brown and Hinton [7].

As a special case when $\mathbb{T} = \mathbb{Z}$, we see that $\mu(n) = 1$, $\sigma(n) = n+1$ and then the result in Corollary 3.3 reduces to the following result for the second order half-linear difference equation

$$\Delta\left((\Delta y(n))^{\gamma}\right) + p(n)\,(\Delta y(n))^{\gamma} + q(n)\,(y(n+1))^{\gamma} = 0, \quad a \leq n \leq b, \tag{2.37}$$

where $\gamma \geq 1$ is a quotient of odd positive integers.

2.13. Corollary. *Assume that* y *is a nontrivial solution of (2.37). If* $y(a) = \Delta y(b) = 0$, *then*

$$\sup_{a \leq n \leq b} |p(n)| + (b-a)^{\gamma} \max_{a \leq n \leq b} \left|\sum_{s=n}^{b-1} q(s)\Delta s\right| + \frac{\gamma^{\frac{\gamma}{\gamma+1}}}{\gamma+1}(b-a) \max_{a \leq n \leq b} |p(n)| \geq 1.$$

References

[1] Agarwal, R., Bohner, M., O'Regan, D., Osman, M., Saker, S., A general dynamic inequality of Opial type, Appl. Math. Infor. Sci. 10, 1–5, 2016.

[2] Agarwal, R. P., O'Regan, D., Saker, S. H., *Dynamic Inequalities on Time Scales,* Springer, 2014.

[3] Bohner, M., Clark, S., Ridenhour, J., Lyapunov inequalities for time scales, J. Ineq. Appl. 7, 61–77, 2002.

[4] Bohner, M., Kaymakçalan, B., Opial Inequalities on time scales, Ann. Polon. Math. 77 (1), 11–20, 2001.

[5] Bohner, M., Peterson, A., *Dynamic Equations on Time Scales*: An Introduction with Applications, Birkhäuser, Boston, 2001.

[6] Bohner, M., Peterson, A., *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2003.

[7] Brown, R. C., Hinton, B. D., Opial's inequality and oscillation of 2^{nd} order equations, Proc. Amer. Math. Soc. 125 (4), 1123–1129, 1997.

[8] Cohn, J. H. E., Consecutive zeros of solutions of ordinary second order differential equations, J. London Math. Soc. 2 (5), 465–468, 1972.

[9] Došlý, O., Řehák, P., *Half-Linear Differential Equations*, Elsevier Ltd, Heidelberg, 2005.

[10] Harris, B. J., Kong, Q., On the oscillation of differential equations with an oscillatory coefficient, Trans. Amer. Math. Soc. 347 (5), 1831-1839, 1995.

[11] Hartman, P., *Ordinary Differential Equations*, Wiley, New York, 1964 and Birkhuser, Boston, 1982.

[12] Hartman, P., Wintner, A., On an oscillation criterion of de la Vallée Poussin, The Quarterly J. Math. 13, 330–332, 1955.

[13] Hilger, S., Analysis on measure chain–a unified approach to continuous and discrete calculus, Results Math. 18, 18–56, 1990.

[14] Karpuz, B., Kaymakçalan, B., Öcalan, Ö., A generalization of Opial's inequality and applications to second order dynamic equations, Diff. Eqns. Dyn. Syst. 18 (1–2), 11–18, 2010.

[15] Kwong, M. K., On Lyapunov's inequality for disfocality, J. Math. Anal. Appl. 83, 486-494, 1981.

[16] Lee, C. F., Yeh, C. C., Hong, C. H., Agarwal, R. P., Lyapunov and Wirtinger inequalities, Appl. Math. Lett. 17, 847–853, 2004.

[17] Olech, Z., Asimple proof of a certain result of Z. Opial, Ann. Polon. Math. 8, 61–63, 1960.

[18] Opial, Z., Sur uné inegalité, Ann. Polon. Math. 8, 29–32, 1960.

[19] Saker, S. H.,*Oscillation Theory of Dynamic Equations on Time Scales: Second and Third Orders,* Lambert Academic Publishing, Germany, 2010.

[20] Saker, S. H., Lyapunov inequalities for half-linear dynamic equations on time scales and disconjugacy, Dyn. Cont. Discr. Impuls. Syst. Ser. B Appl. Algorithms 18, 149–161, 2011.

[21] Saker, S. H., Lyapunov type inequalities for a second order differential equation with a damping term, Ann. Polon. Math. 103 (1), 37–57, 2012.

[22] Saker, S. H., Opial's type inequalities on time scales and some applications, Ann. Polon. Math. 104, 243–260, 2012.

[23] Saker, S. H., Applications of Opial inequalities on time scales on dynamic equations with damping terms, Math. Comp. Model.58, 1777–1790, 2013.

[24] Saker, S. H., Osman, M. M., O'Regan, D., Agarwal, R. P., Some new Opial dynamic inequalities with weighted functions on time scales, Math. Ineq. Appl. 18 (3), 1171–1187, 2015.

[25] Tiryaki, A., Recent development of Lyapunov-type inequalities, Advances in Dyn. Syst. Appl. 5 (2), 231–248, 2010.

[26] Wintner, A., On the nonexistence of conjugate points, Amer. J. Math. 73, 368–380, 1951.

[27] Yang, X., On inequalities of Lyapunov type, Appl. Math. Comp. 134, 293–300, 2003.