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The transmuted Weibull-G family of distributions

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Abstract

We introduce a new family of continuous distributions called the transmuted Weibull-G family of distributions which extends the transmuted class pioneered by Shaw and Buckley (2007). We study the mathematical properties of the new family. Some useful characterizations based on the ratio of two truncated moments as well as based on hazard function are presented. We estimate the model parameters by the maximum likelihood method. We assess the performance of the maximum likelihood estimators in terms of biases and mean squared errors by means of a simulation study.

Keywords: transmuted-G family, Weibull-G family, maximum likelihood, moments, order statistic, quantile function, moments of order statistics.

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1. Introduction

Several continuous univariate distributions have been extensively used for modeling data in many areas such as economics, engineering, biological studies and environmental sciences. However, applied areas such as finance, lifetime analysis and insurance clearly require extended forms of these distributions. So, several classes of distributions have been constructed by extending common families of continuous distributions. These generalized distributions give more flexibility by adding one "or more" parameters to the baseline model. They were pioneered by Gupta et al. [29] who proposed the exponentiated-G

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class, which consists of raising the cumulative distribution function (cdf) to a positive power parameter. Many other classes can be cited such as the Marshall-Olkin-G family by Marshall and Olkin [31], beta generalized-G family by Eugene et al. [25], the gamma-generated family by Zografos and Balakrishnan [45], Kumaraswamy G family by Cordeiro and de Castro [19], exponentiated generalized-G family by Cordeiro et al. [20,22], a new method for generating families of continuous distributions by Alzaatreh et al. [13], exponentiated T-X family of distributionsby Alzaghal et al. [12], the Lomax generator of distributions by Cordeiro et al. [23], beta Marshall-Olkin by Alizadeh et al. [8], Kumaraswamy odd log-logistic by Alizadeh et al. [9], beta odd log-logistic by Cordeiro et al. [17], Kumaraswamy Marshall-Olkin by Alizadeh et al. [6], transmuted exponentiated generalized-G family by Yousof et al. [42], generalized transmuted-G by Nofal et al. [36], generalized transmuted family by Alizadeh et al. [9], another generalized transmuted family by Merovci et al. [32], Kumaraswamy transmuted-G by Afify et al. [3], transmuted geometric-G by Afify et al. [2], beta transmuted-H by Afify et al. [4], the Zografos-Balakrishnan odd log-logistic family by Cordeiro et al. [18] and the type I half-logistic family by Cordeiro et al. [24], Burr X-G by Yousof et al. [43], exponentiated transmuted-G family by Merovci et al. [33], odd-Burr generalized family by Alizadeh et al. [7] the complementary generalized transmuted Poisson family by Alizadeh et al. [10], among others.

For an arbitrary baseline cdf $G(x)$, Shaw and Buckley [39] defined the transmuted-G (TG) family with cdf and probability density function (pdf) given by

$$F(x) = H(x;\psi)\left[1+\lambda-\lambda H(x;\psi)\right] \tag{1.1}$$

and

$$f(x) = h(x;\psi)\ \left[1+\lambda-2\lambda H(x;\psi)\right], \tag{1.2}$$

respectively, where $|\lambda| \leq 1$ is a shape parameters and $x \in \mathbb{R}$. The TG density is a mixture of the baseline density and the exponentiated-G (exp-G) density with power parameter two. For $\lambda = 0$, Equation (1.1) gives the baseline distribution. Let $h(x;\psi)$ and $H(x;\psi)$ denote the density and cumulative functions of the baseline model with parameter vector ψ and consider the Weibull cdf $F(x) = 1 - e^{-x^{\alpha}}$(for $x > 0$) with positive parameter α. Based on this density, Bourguignon et al. (2014) replaced the argument x by $H(x;\psi)/\overline{H}(x;\psi)$, where $\overline{H}(x;\psi) = 1 - H(x;\psi)$ and defined the cdf of their Weibull-G class by

$$H(x;\alpha) = \int_0^{\left\{\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right\}} \alpha t^{\alpha-1}\ \exp\left(-t^{\alpha}\right)dt = 1-\exp\left\{-\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\alpha}\right\}, \tag{1.3}$$

where $\psi = (\psi_k) = (\psi_1, \psi_2, ...)$ is a parameter vector. Based on the TG family and Weibull-G (WG) family, we construct a new generator by inserting (1.3) into (1.1). We have

$$F(x) = \left\{1-e^{-\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\alpha}}\right\}\left[1+\lambda e^{-\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\alpha}}\right], \tag{1.4}$$

where $G(x;\psi)$ is the baseline cdf, $\alpha > 0$ and $|\lambda| \leq 1$ are two additional shape parameters. The $TW - G(x;\alpha,\lambda,\psi)$ is a wider class of continuous distributions. It includes the TG family of distributions.

The rest of the paper is outlined as follows. In Section 2, we define the univariate extensions of the TW-G family. A useful mixture representation for the new pdf are derived in the same section. In Section 3, we derive some of its mathematical properties including asymptotics, probability weighted moments (PWMs), residual life and reversed residual life functions, stress-strength model, ordinary, incomplete moments and generating functions, Finally order statistics and their moments are introduced at the end of the section. Some characterizations results are provided in Section 4. Maximum likelihood estimation of the model parameters is addressed in Section 5. In Section 6, we define two special models and provide the plots of their pdf's and hazard rate functions (hrf's). In Section 7, simulation results to assess the performance of the proposed maximum likelihood estimation procedure are discussed. In Section 8, we provide the applications to real data to illustrate the importance of the new family. Finally, some concluding remarks are presented in Section 9.

2. The new family

The corresponding pdf is

$$f(x) = \alpha\, g(x;\psi) \frac{G(x;\psi)^{\alpha-1}}{\overline{G}(x;\psi)^{\alpha+1}} e^{-\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\alpha}} \left\{1-\lambda+2\lambda e^{-\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\alpha}}\right\}, x>0. \tag{2.1}$$

The hazard rate function for the new family can be expressed as

$$\tau(x) = \frac{\alpha\, g(x;\psi) \frac{G(x;\psi)^{\alpha-1}}{\overline{G}(x;\psi)^{\alpha+1}} e^{-\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\alpha}} \left\{1-\lambda+2\lambda e^{-\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\alpha}}\right\}}{1-\left\{1-e^{-\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\alpha}}\right\}\left[1+\lambda e^{-\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\alpha}}\right]}. \tag{2.2}$$

For simulation of this family, if $U \sim u(0,1)$, when $\lambda = 0$, then

$$X_U = G^{-1}\left\{\frac{[-\log(1-U)]^{1/\alpha}}{1+[-\log(1-U)]^{1/\alpha}}\right\}$$

and for $\lambda \neq 0$, we have

$$X_U = G^{-1}\left\{\frac{\left\{-\log\left[\frac{\lambda-1+\sqrt{(1+\lambda)^2-4\,\lambda\, U}}{2\,\lambda}\right]\right\}^{1/\alpha}}{1+\left\{-\log\left[\frac{\lambda-1+\sqrt{(1+\lambda)^2-4\,\lambda\, U}}{2\,\lambda}\right]\right\}^{1/\alpha}}\right\}$$

has cdf (1.4). Below is a simple motivation for the development of TW-G family of distributions. Suppose "$T_1$and T_2" are two independent random variables from cdf in (1.3). Define

$$X = \begin{cases} T_{1:2} & \text{with probability } \frac{1}{2}(\lambda+1)\,; \\ T_{2:2} & \text{with probability } \frac{1}{2}(1-\lambda)\,, \end{cases}$$

where $T_{1:2} = min\{T_1, T_2\}$ and $T_{2:2} = max\{T_1, T_2\}$. Then the cdf of X is given by (1.4). The TW-G family of distributions appears to be more felxible and could be used for modeling various types of data. For illustration propose we provide pdf and hrf of some special models of this family in figures 1,2. It can be seen that the hazard rate can take increasing, decreasing, upside down and bathtub shapes. Therefore, this family of distributions could be used to model diverse nature of data sets. Furthermore, the basic motivations for using the TW-G family in practice are the following:

i. to make the kurtosis more flexible compared to the baseline model;
ii. to produce a skewness for symmetrical distributions;
iii. to construct heavy-tailed distributions for modeling real data;
iv. to generate distributions with symmetric, left-skewed, right-skewed or reversed-J shape;

v. to define special models with all types of the hrf;
vi. to provide consistently better fits than other generated models under the same underlying distribution.

The cdf of the TW-G family in (1.4) can be expressed as

$$F(x) = 1 + (\lambda - 1)\, e^{-\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\alpha}} - \lambda e^{-2\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\alpha}} \tag{2.3}$$

and after some algebra, we get

$$\begin{aligned} F(x) &= 1 + (\lambda - 1) \sum_{i=0}^{\infty} \frac{(-1)^i \left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\alpha i}}{i!} - \lambda \sum_{i=0}^{\infty} \frac{2^i (-1)^i \left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\alpha i}}{i!} \\ &= 1 + (\lambda - 1) \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \frac{G(x;\psi)^{\alpha i}}{\overline{G}(x;\psi)^{\alpha i}} - \lambda \sum_{i=0}^{\infty} \frac{2^i (-1)^i}{i!} \frac{G(x;\psi)^{\alpha i}}{\overline{G}(x;\psi)^{\alpha i}} \\ &= 1 + (\lambda - 1) \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}}{i!} \binom{-\alpha i}{j} G(x;\psi)^{\alpha i + j} - \lambda \sum_{i,j=0}^{\infty} \frac{2^i (-1)^{i+j}}{i!} \binom{-\alpha i}{j} G(x;\psi)^{\alpha i + j} \\ &= 1 + \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}}{i!} \binom{-\alpha i}{j} \left(\lambda - 1 - \lambda \times 2^i\right) G(x;\psi)^{\alpha i + j} \end{aligned}$$

then

$$F(x) = 1 + \sum_{i,j=0}^{\infty} w_{i,j} G(x;\psi)^{\alpha i + j} = 1 + \sum_{i,j=0}^{\infty} w_{i,j}\ \mathbf{\Pi}_{\alpha i + j}(x), \tag{2.4}$$

where

$$w_{i,j} = \frac{(-1)^{i+j}}{i!} \binom{-\alpha i}{j} \left(\lambda - 1 - \lambda \times 2^i\right)$$

and $\mathbf{\Pi}_{\delta}(x) = G(x;\psi)^{\delta}$ is the cdf of the exp-G distribution with power parameter δ. The corresponding TW-G density function is obtained by differentiating (2.4)

$$f(x) = \sum_{i,j=0}^{\infty} w_{i,j}\ \pi_{\alpha i + j}(x), \tag{2.5}$$

where $\pi_{\delta}(x) = \delta g(x;\psi)\, G(x;\psi)^{\delta - 1}$ is the pdf of the exp-G distribution with power parameter δ.

3. Mathematical properties

Here, we investigate mathematical properties of the TW-G family of distributions including Asymptotes, ordinary and incomplete moments, generating function, probability weighted moments and entropies. Established algebraic expansions to determine some structural properties of the TW-G family of distributions can be more efficient than computing those directly by numerical integration of its density function.

3.1. Asymptotics. Let $c = \inf\{x | G(x;\psi) > 0\}$, Then the asymptotics of cdf, pdf and hrf as $x \to c$ are given by

$$F(x) \sim (1 + \lambda)\, G(x;\psi)^{\alpha} \qquad \text{as} \qquad x \to c,$$

$$f(x) \sim \alpha (1 + \lambda)\, g(x;\psi) G(x;\psi)^{\alpha - 1} \qquad \text{as} \qquad x \to c,$$

$$h(x) \sim \alpha (1 + \lambda)\, g(x;\psi) G(x;\psi)^{\alpha - 1} \qquad \text{as} \qquad x \to c.$$

The asymptotics of cdf, pdf and hrf when $x \to \infty$ are given by

$$1 - F(x) \sim e^{-\overline{G}(x;\psi)^{-\alpha}} \qquad \text{as} \qquad x \to \infty,$$

$$f(x) \sim \alpha g(x;\psi)\overline{G}(x;\psi)^{-\alpha-1} e^{-\overline{G}(x;\psi)^{-\alpha}} \qquad \text{as} \qquad x \to \infty,$$

$$h(x) \sim \alpha g(x;\psi)\overline{G}(x;\psi)^{-\alpha-1} \qquad \text{as} \qquad x \to \infty.$$

3.2. Probability weighted moments. The PWMs are expectations of certain functions of a random variable and they can be defined for any random variable whose ordinary moments exist. The PWMs method can generally be used for estimating parameters of a distribution whose inverse form cannot be expressed explicitly. The (s,r)th PWMs of X following the TW-G family, say $\rho_{s,r}$, is formally defined by

$$\rho_{s,r} = E\{X^s F(X)^r\} = \int_{-\infty}^{\infty} x^s F(x)^r f(x)\, dx.$$

Using equations (2.1) and (2.2), we can write

$$f(x) F(x)^r = \sum_{i,j=0}^{\infty} m_{i,j}\ \pi_{\alpha(i+1)+j}(x),$$

where $m_{i,j} = \sum_{k,h=0}^{\infty} \frac{(-1)^{k+h+i+j}(h+1)^i\left[(1+\lambda)\binom{r+k}{h} - 2\lambda\binom{r+k+1}{h}\right]}{i!\alpha^{-1}\lambda^{-k}(1+\lambda)^{k-r}[\alpha(i+1)+j]} \binom{r}{k}\binom{-[\alpha(i+1)+1]}{j}$.
Then, the (s,r)th PWMs of X can be expressed as

$$\rho_{s,r} = \sum_{i,j=0}^{\infty} m_{i,j} E\left(Y^s_{\alpha(i+1)+j}\right).$$

3.3. Residual life and reversed residual life functions. The nth moment of the residual life, say $m_n(t) = E[(X-t)^n \mid X > t]$, $n = 1, 2, \ldots$, uniquely determined $F(x)$. The nth moment of the residual life of X is given by $m_n(t) = \frac{1}{R(t)} \int_t^{\infty} (x - t)^n dF(x)$.Therefore,

$$m_n(t) = \frac{1}{R(t)} \sum_{i,j=0}^{\infty} w^{\star}_{i,j} \int_t^{\infty} x^r \pi_{\alpha i+j}(x) dx,$$

where $w^{\star}_{i,j} = w_{i,j} \sum_{r=0}^{n} \binom{n}{r} (-t)^{n-r}$. Another interesting function is the mean residual life (MRL) function or the life expectation at age t defined by $m_1(t) = E[(X-t) \mid X > t]$, which represents the expected additional life length for a unit which is alive at age t. The MRL of X can be obtained by setting $n = 1$ in the last equation. The nth moment of the reversed residual life, say $M_n(t) = E[(t-X)^n \mid X \le t]$ for $t > 0$ and $n = 1, 2, \ldots$ uniquely determines $F(x)$. We obtain $M_n(t) = \frac{1}{F(t)} \int_0^t (t-x)^n dF(x)$.Then, the nth moment of the reversed residual life of X becomes

$$M_n(t) = \frac{1}{F(t)} \sum_{i,j=0}^{\infty} w^{\star\star}_{i,j} \int_0^t x^r \pi_k(x) dx,$$

where $w^{\star\star}_{i,j} = w_{i,j} \sum_{r=0}^{n} (-1)^r \binom{n}{r} t^{n-r}$. The mean inactivity time (MIT) or mean waiting time (MWT), also called the mean reversed residual life function, is given by $M_1(t) = E[(t-X) \mid X \le t]$, and it represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in $(0, t)$. The MIT of the TW-G family of distributions can be obtained easily by setting $n = 1$ in the above equation.

3.4. Stress-strength model. Stress-strength model is the most widely approach used for reliability estimation. This model is used in many applications in physics and engineering such as strength failure and system collapse. In stress-strength modeling, $\mathbf{R} = \Pr(X_2 < X_1)$ is a measure of reliability of the system when it is subjected to random stress X_2 and has strength X_1. The system fails if and only if the applied stress is greater than its strength and the component will function satisfactorily whenever $X_1 > X_2$. $\mathbf{R}$ can be considered as a measure of system performance and naturally arise in electrical and electronic systems. Other interpretation can be that, the reliability of the system is the probability that the system is strong enough to overcome the stress imposed on it. Let X_1 and X_2 be two independent random variables with TW-G$(\lambda_1, \alpha_1, \psi)$ and TW-G$(\lambda_1, \alpha_1, \psi)$ distributions. The reliability is defined by

$$\mathbf{R} = \int_0^\infty f_1\left(x; \lambda_1, \alpha_1, \psi\right) F_2\left(x; \lambda_1, \alpha_1, \psi\right) dx.$$

Then, we can write

$$\mathbf{R} = \sum_{i,j=0}^\infty a_{i,j} \int_0^\infty \pi_{\alpha_1 i+j}\left(x\right) dx + \sum_{i,j,h,k=0}^\infty b_{i,j,k,h} \int_0^\infty \pi_{\alpha_1 i+j+\alpha_2 h+k}\left(x\right) dx$$

where

$$a_{i,j} = \frac{(-1)^{i+j}}{i!} \binom{-\alpha_1 i}{j} \left(\lambda_1 - 1 - \lambda_1 \times 2^i\right),$$

and

$$b_{i,j,k,h} = \frac{(-1)^{i+j+h+k}\left(\lambda_1 - 1 - \lambda_1 \times 2^i\right)\binom{-\alpha_1 i}{j}\binom{-\alpha_2 h}{k}}{i!\left(\alpha_1 i + j\right)\left[\alpha_1 i + j + \alpha_2 h + k\right]\left(\lambda_2 - 1 - \lambda_2 \times 2^h\right)^{-1}}.$$

Thus, the reliability, $\mathbf{R}$, can be expressed as

$$\mathbf{R} = \sum_{i,j=0}^\infty a_{i,j} + \sum_{i,j,h,k=0}^\infty b_{i,j,h,k}$$

3.5. Moments, incomplete moments and generating function. The rth ordinary moment of X is given by $\mu_r' = E(X^r) = \int_{-\infty}^\infty x^r f\left(x\right) dx$. Then we obtain

$$\mu_r' = \sum_{i,j=0}^\infty w_{i,j} E(Y_{\alpha i+j}^r). \tag{3.1}$$

Henceforth, Y_δ denotes the exp-G distribution with power parameter (δ). Setting $r = 1$ in (3.1), we have the mean of X. The last integration can be computed numerically for most parent distributions. The skewness and kurtosis measures can be calculated from the ordinary moments using well-known relationships. The nth central moment of X, say M_n, follows as $M_n = E(X - \mu)^n = \sum_{h=0}^n (-1)^h \binom{n}{h} \left(\mu_1'\right)^n \mu_{n-h}'$.The cumulants (κ_n) of X follow recursively from $\kappa_n = \mu_n' - \sum_{r=0}^{n-1} \binom{n-1}{r-1} \kappa_r \mu_{n-r}'$, where $\kappa_1 = \mu_1'$, $\kappa_2 = \mu_2' - \mu_1'^2$, $\kappa_3 = \mu_3' - 3\mu_2'\mu_1' + \mu_1'^3$, etc. The skewness and kurtosis measures also can be calculated from the ordinary moments using well-known relationships. The main applications of the first incomplete moment refer to the mean deviations and the Bonferroni and Lorenz curves. These curves are very useful in economics, reliability, demography, insurance and medicine. The rth incomplete moment, say $\varphi_r\left(t\right)$, of X can be expressed from (3.1) as

$$\varphi_r\left(t\right) = \int_{-\infty}^t x^r f\left(x\right) dx = \sum_{i,j=0}^\infty w_{i,j} \int_{-\infty}^t x^r \pi_{\alpha i+j}(x) dx. \tag{3.2}$$

The mean deviations about the mean $[\delta_1 = E(|X - \mu_1'|)]$ and about the median $[\delta_2 = E\left(|X - M|\right)]$ of X are given by $\delta_1 = 2\mu_1^{'} F(\mu_1') - 2\varphi_1(\mu_1')$ and $\delta_2 = \mu_1' - 2\varphi_1\left(M\right)$, respectively, where $\mu_1' = E\left(X\right)$, $M = Median(X) = Q(0.5)$ is the median, $F(\mu_1')$ is easily calculated from (1.4) and $\varphi_1\left(t\right)$ is the first incomplete moment given by (11) with $r = 1$. A general equation for $\varphi_1\left(t\right)$ can be derived from (3.2) as $\varphi_1\left(t\right) = \sum_{i,j=0}^{\infty} w_{i,j}\, I_{\alpha i+j}\left(x\right),$ where $I_\delta\left(x\right) = \int_{-\infty}^{t} x\, \pi_\delta\left(x\right) dx$ is the first incomplete moment of the exp-G distribution. The moment generating function (mgf) $M_X\left(t\right) = E\left(e^{t\,X}\right)$ of X can be derived from equation (2.5) as

$$M_X\left(t\right) = \sum_{i,j=0}^{\infty} w_{i,j}\, M_{\alpha i+j}\left(t\right),$$

where $M_\delta\left(t\right)$ is the mgf of Y_δ. Hence, $M_X\left(t\right)$ can be determined from the exp-G generating function.

3.6. Order statistics. Let $X_1, \ldots, X_n$ be a random sample from the TW-G family of distributions and let $X_{1:n}, \ldots, X_{n:n}$ be the corresponding order statistics. The pdf of the ith order statistic, $X_{i:n}$, can be written as

$$f_{i:n}\left(x\right) = \frac{f\left(x\right)}{\mathrm{B}\left(i, n-i+1\right)} \sum_{j=0}^{n-i} \left(-1\right)^j \binom{n-i}{j} F^{j+i-1}\left(x\right), \tag{3.3}$$

where $B(\cdot,\cdot)$ is the beta function. Substituting (1.4) and (2.1) in equation (3.3) and using a power series expansion,we arrive at

$$f\left(x\right) F\left(x\right)^{j+i-1} = \sum_{m,w=0}^{\infty} t_{m,w}\ \pi_{\alpha(m+1)+w}(x),$$

where $t_{m,w} = \sum_{k,h=0}^{\infty} \frac{(-1)^{k+h+m+w}(h+1)^m\left[(1+\lambda)\binom{j+i+k-1}{h} - 2\lambda\binom{j+i+k}{h}\right]}{i!\alpha^{-1}\lambda^{-k}(1+\lambda)^{k-(j+i-1)}[\alpha(m+1)+w]} \binom{j+i-1}{k}\binom{-[\alpha(m+1)+1]}{w}$. The pdf of $X_{i:n}$ can be expressed as

$$f_{i:n}\left(x\right) = \sum_{j=0}^{n-i} \frac{\left(-1\right)^j \binom{n-i}{j}}{\mathrm{B}\left(i, n-i+1\right)} \sum_{m,w=0}^{\infty} t_{m,w}\ \pi_{\alpha(m+1)+w}(x).$$

Then, the density function of the TW-G order statistics is a mixture of exp-G densities. Based on the last equation, we note that the properties of $X_{i:n}$ follow from those properties of Y_{k+1}. For example, the moments of $X_{i:n}$ can be expressed as

$$E\left(X_{i:n}^q\right) = \sum_{m,w=0}^{\infty} \sum_{j=0}^{n-i} \frac{\left(-1\right)^j \binom{n-i}{j} t_{m,w}}{\mathrm{B}\left(i, n-i+1\right)} E\left(Y_{\alpha(m+1)+w}^q\right). \tag{3.4}$$

4. Characterizations

This section deals with various characterizations of TW-G family of distributions. These characterizations are based on: (i) the ratio of two truncated moments and (ii) the hazard function. We present our characterizations (i) and (ii) in two subsections.

4.1. Characterizations based on ratio of two truncated moments. In this subsection we present characterizations of TW-G distribution in terms of a simple relationship between two truncated moments. This characterization result employs a theorem due to Glänzel (1987), see Theorem 1 in Appendix A. Note that the result holds also when the interval H is not closed. Moreover, it could be also applied when the cdf F does not have a closed form. As shown in Glänzel (1990), this characterization is stable in the sense of weak convergence.

4.1.1. Proposition. Let $X : \Omega \to \mathbb{R}$ be a continuous random variable and let $q_1(x) = \left\{(1-\lambda) + 2\lambda e^{-\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\alpha}}\right\}^{-1}$ and $q_2(x) = q_1(x)\, e^{-\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\alpha}}$ for $x \in \mathbb{R}$. The random variable X has pdf (2.1) if and only if the function ξ defined in Theorem 1 has the form

$$\xi(x) = \frac{1}{2} e^{-\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\alpha}}, \quad x \in \mathbb{R}.$$

Proof. Let X be a random variable with pdf (2.1), then

$$(1 - F(x))\, E\left[q_1(x) \mid X \geq x\right] = e^{-\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\alpha}}, \quad x \in \mathbb{R},$$

and

$$(1 - F(x))\, E\left[q_2(x) \mid X \geq x\right] = \frac{1}{2} e^{-\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\alpha}}, \quad x \in \mathbb{R},$$

and finally

$$\xi(x)\, q_1(x) - q_2(x) = -\frac{1}{2} q_1(x)\, e^{-\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\alpha}} < 0, \quad x \in \mathbb{R}.$$

Conversely, if ξ is given as above, then

$$s'(x) = \frac{\xi'(x)\, q_1(x)}{\xi(x)\, q_1(x) - q_2(x)} = \frac{\alpha g(x;\psi)\, G(x;\psi)^{\alpha-1}}{\overline{G}(x;\psi)} \quad x \in \mathbb{R},$$

and hence

$$s(x) = \left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\alpha}, \quad x \in \mathbb{R}.$$

Now, in view of Theorem 1, X has density (2.1).

4.1.1. Corollary. Let $X : \Omega \to \mathbb{R}$ be a continuous random variable and let $q_1(x)$ be as in Proposition 4.1.1. Then X has pdf (2.2) if and only if there exist functions q_2 and ξ defined in Theorem 1 satisfying the differential equation

$$\frac{\xi'(x)\, q_1(x)}{\xi(x)\, q_1(x) - q_2(x)} = \frac{\alpha g(x;\psi)\, G(x;\psi)^{\alpha-1}}{\overline{G}(x;\psi)^{\alpha+1}} \quad x \in \mathbb{R}.$$

The general solution of the differential equation in Corollary 4.1.1 is

$$\xi(x) = e^{\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\alpha}} \left[-\int \frac{\alpha g(x;\psi)\, G(x;\psi)^{\alpha-1}}{\overline{G}(x;\psi)^{\alpha+1}} e^{-\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\alpha}} (q_1(x))^{-1} q_2(x) + D\right],$$

where D is a constant. Note that a set of functions satisfying the above differential equation is given in Proposition 4.1.1 with $D = 0$. However, it should be also noted that there are other triplets (q_1, q_2, ξ) satisfying the conditions of Theorem 1.

4.2. Characterization based on hazard function. It is known that the hazard function, h_F, of a twice differentiable distribution function, F, satisfies the first order differential equation

$$\frac{f'(x)}{f(x)} = \frac{h_F'(x)}{h_F(x)} - h_F(x).$$

For many univariate continuous distributions, this is the only characterization available in terms of the hazard function. The following characterization establish a non-trivial characterization of TW-G distribution in terms of the hazard function, which is not of the above trivial form.

4.2.1. Proposition. Let $X : \Omega \to \mathbb{R}$ be a continuous random variable. The pdf of X is (2.1) if and only if its hazard function $h_F(x)$ satisfies the differential equation

$$\begin{aligned} & h_F'(x) - g'(x;\psi)\,(g(x;\psi))^{-1}\,h_F(x) \\ = \ & \alpha g(x;\psi)\,\frac{d}{dx}\left\{\frac{G(x;\psi)^{\alpha-1}}{\overline{G}(x;\psi)^{\alpha+1}}\left[1+\frac{\lambda e^{-\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\alpha}}}{(1-\lambda)+\lambda e^{-\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\alpha}}}\right]\right\}, \end{aligned}$$

$x \in \mathbb{R}$, with the initial condition $h_F(0) = 0$ for $\alpha > 1$.
Proof. If X has pdf (2.1), then clearly the above differential equation holds. Now, if the differential equation holds, then

$$\frac{d}{dx}\left\{(g(x;\psi))^{-1}\,h_F(x)\right\} = \alpha\frac{d}{dx}\left\{\frac{G(x;\psi)^{\alpha-1}}{\overline{G}(x;\psi)^{\alpha+1}}\left[1+\frac{\lambda e^{-\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\alpha}}}{(1-\lambda)+\lambda e^{-\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\alpha}}}\right]\right\},$$

or

$$h_F(x) = \alpha g(x;\psi)\left\{\frac{G(x;\psi)^{\alpha-1}}{\overline{G}(x;\psi)^{\alpha+1}}\left[1+\frac{\lambda e^{-\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\alpha}}}{(1-\lambda)+\lambda e^{-\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\alpha}}}\right]\right\},$$

which is the hazard function of the TW-G distribution.

5. Estimation

Several approaches for parameter estimation have been proposed in the literature, however, the maximum likelihood method is the most commonly employed. The maximum likelihood estimators (MLEs) enjoy desirable properties and can be used for constructing confidence intervals and regions and also in test statistics. The normal approximation for these estimators in large samples can be easily handled either analytically or numerically. So, we consider the estimation of the unknown parameters of this family from complete samples only by maximum likelihood. Let $x_1, \ldots, x_n$ be a random sample from the TW-G distribution with parameters λ, α and ψ. Let $\Theta =(\lambda, \alpha, \psi)^{\intercal}$ be the $p \times 1$ parameter vector. For determining the MLE of Θ, we have the log-likelihood function

$$\begin{aligned} \ell \ = \ & \ell(\Theta) = n\log\alpha + \sum_{i=1}^{n}\log g(x_i;\psi) + (\alpha-1)\sum_{i=1}^{n}\log G(x_i;\psi) \\ & -(\alpha+1)\sum_{i=1}^{n}\log\overline{G}(x_i;\psi) - \sum_{i=1}^{n}s_i + \sum_{i=1}^{n}\log\left\{1-\lambda+2\lambda e^{-s_i}\right\}, \end{aligned}$$

where $s_i = \left[\frac{G(x_i;\psi)}{\overline{G}(x_i;\psi)}\right]^{\alpha}$. The components of the score vector, $\mathbf{U}(\Theta) = \frac{\partial\ell}{\partial\Theta} = \left(\frac{\partial\ell}{\partial\lambda}, \frac{\partial\ell}{\partial\alpha}, \frac{\partial\ell}{\partial\psi}\right)^{\intercal}$, are

$$U_\lambda = \sum_{i=1}^{n}\frac{2e^{-s_i}-1}{1-\lambda+2\lambda e^{-s_i}},$$

$$U_\alpha = \frac{n}{\alpha} + \sum_{i=1}^{n}\log G(x_i;\psi) - \sum_{i=1}^{n}\log\overline{G}(x_i;\psi) - \sum_{i=1}^{n}p_i + \sum_{i=1}^{n}\frac{-2\lambda p_i e^{-s_i}}{1-\lambda+2\lambda e^{-s_i}}$$

and

$$U_\psi = \sum_{i=1}^{n}\frac{g'(x_i;\psi)}{g(x_i;\psi)} + (\alpha-1)\sum_{i=1}^{n}\frac{G'(x_i;\psi)}{G(x_i;\psi)} + (\alpha+1)\sum_{i=1}^{n}\frac{G'(x_i;\psi)}{\overline{G}(x_i;\psi)} - \sum_{i=1}^{n} q_i + \sum_{i=1}^{n}\frac{-2\lambda q_i e^{-s_i}}{1-\lambda+2\lambda e^{-s_i}},$$

where

$$g'(x_i;\psi) = \frac{\partial g(x_i;\psi)}{\partial\psi},\ \ p_i = \left[\frac{G(x_i;\psi)}{\overline{G}(x_i;\psi)}\right]^{\alpha}\log\left[\frac{G(x_i;\psi)}{\overline{G}(x_i;\psi)}\right],$$

$$q_i = \alpha\left[\frac{G(x_i;\psi)}{\overline{G}(x_i;\psi)}\right]^{\alpha-1}\frac{G'(x_i;\psi)}{\left[\overline{G}(x_i;\psi)\right]^2} \text{ and } G'(x_i;\psi) = \frac{\partial G(x_i;\psi)}{\partial\psi}.$$

Setting the nonlinear system of equations $U_\lambda = U_\alpha = 0$ and $U_\psi = \mathbf{0}$ and solving them simultaneously yields the MLE $\widehat{\Theta} = (\widehat{\lambda},\widehat{\alpha},\widehat{\psi})^\intercal$. To solve these equations, it is usually more convenient to use nonlinear optimization methods such as the quasi-Newton algorithm to numerically maximize ℓ. For interval estimation of the parameters, we obtain the $p\times p$ observed information matrix $J(\Theta) = \{\frac{\partial^2\ell}{\partial r\,\partial s}\}$ (for $r,s = \lambda,\alpha,\psi$), whose elements can be computed numerically. Under standard regularity conditions when $n\to\infty$, the distribution of $\widehat{\Theta}$ can be approximated by a multivariate normal $N_p(0, J(\widehat{\Theta})^{-1})$ distribution to construct approximate confidence intervals for the parameters. Here, $J(\widehat{\Theta})$ is the total observed information matrix evaluated at $\widehat{\Theta}$. The method of the re-sampling bootstrap can be used for correcting the biases of the MLEs of the model parameters. Good interval estimates may also be obtained using the bootstrap percentile method.

6. Special TW-G models

The pdf (2.1) will be most tractable when $g(x)$ and $G(x)$ have simple analytic forms. In this section, we provide two special models of the TW-G family. These special models generalize some well-known distributions reported in the literature. They correspond to the baseline Weibull (W) and Lindley (L) distributions and illustrate the flexibility of the new family.

6.1. The TW-W distribution. Consider the pdf $g(x) = ba^b x^{b-1}\mathrm{e}^{-(\mathrm{ax})^\mathrm{b}}$ and cdf $G(x) = 1-\mathrm{e}^{-(\mathrm{ax})^\mathrm{b}}$ of the W distribution with scale $a>0$ and shape $b>0$ parameters. Inserting these functions in (2.1), the pdf of the TW-W model (for $x>0$) is given by

$$f(x) = \alpha b a^b x^{b-1} e^{-(ax)^b}\frac{\left(1-e^{-(ax)^b}\right)^{\alpha-1}}{e^{-(\alpha+1)(ax)^b}} e^{-\left[\frac{1-e^{-(ax)^b}}{e^{-(ax)^b}}\right]^{\alpha}} \times\left\{1-\lambda+2\lambda e^{-\left[\frac{1-e^{-(ax)^b}}{e^{-(ax)^b}}\right]^{\alpha}}\right\}. \tag{6.1}$$

A random variable having pdf (6.1) is denoted by $X\sim$TW-W(α,λ,a,b). For $b=1$, we have the TW-exponential distribution. The TW-W density and hrf plots for selected parameter values are displayed in Figure 1.

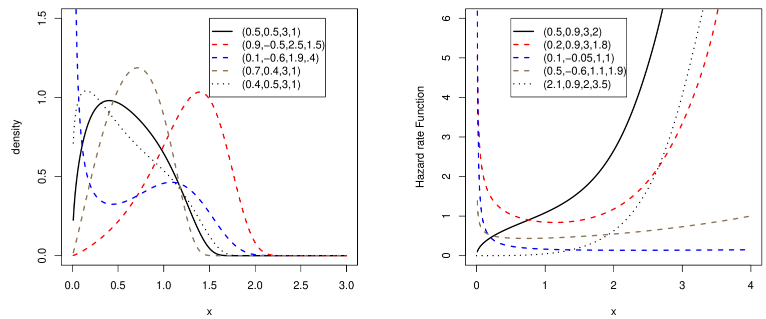

Figure 1. Transmuted Weibull-Weibull distribution: pdf (left figure), hrf (right figure).

6.2. The TW-L distribution. Consider the pdf $g(x) = \frac{a^2}{1+a}(1+x)\mathrm{e}^{-ax}$ and cdf $G(x) = 1 - \frac{1+a+ax}{1+a}\mathrm{e}^{-ax}$ (for $x > 0$) of the L distribution with positive shape parameter a. The pdf of the TW-L model is given by

$$\begin{aligned} f(x) = {} & \frac{\alpha\, a^2\,(1+x)\,e^{\alpha\,a\,x}\,[1-(1+\frac{a\,x}{1+a})\,e^{-a\,x}]^{\alpha-1}}{(1+a)\,(1+\frac{a\,x}{1+a})^{\alpha+1}} \\ & \times e^{-[\frac{(1+a)e^{a\,x}}{1+a+a\,x}-1]^{\alpha}} \left\{1-\lambda+2\lambda\,e^{-[\frac{(1+a)e^{a\,x}}{1+a+a\,x}-1]^{\alpha}}\right\}. \end{aligned}$$

The TW-L density and hrf plots for some parameter values are displayed in Figure 2.

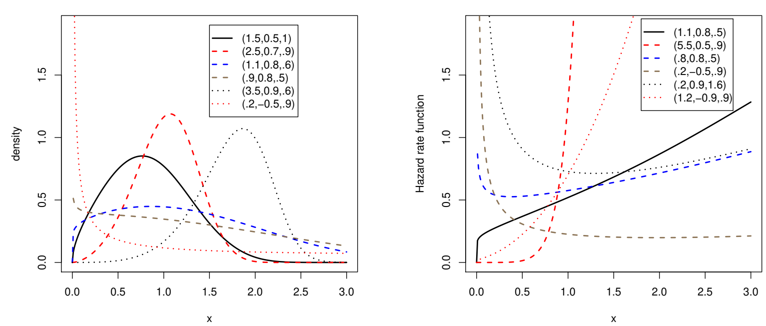

Figure 2. Transmuted Weibull-Lindley distribution: pdf (left figure), hrf (right figure).

7. Simulation study

In this section, we investigate the performance of the maximum likelihood estimators presented in Section 5 for Transmuted Weibull-Weibull distribution with respect to sample size n. The evaluation is based on a simulation study:

1. Generate 5000 samples of size n from $TW-W$ distribution. The inversion method was used to generate samples (See last paragraph of Page 1673).
2. Calculate the maximum likelihood estimates for the five thousand samples, say $\left(\hat{\alpha}_i, \hat{\lambda}_i, \hat{a}_i, \hat{b}_i\right)$ for $i = 1, 2, ..., 5000$.
3. Compute the biases and mean squared errors given by

$$Bias(n) = \frac{1}{5000}\sum_{i=1}^{5000}\left(\hat{h}_i - h\right)$$

and

$$MSE(n) = \frac{1}{5000}\sum_{i=1}^{5000}\left(\hat{h}_i - h\right)^2,$$

for $h = \alpha, \lambda, a, b$.

We repeat these steps for $n = 10, 15, 20, ..., 320$ with $\alpha = 0.1$, $\lambda = -0.6$,$a = 1.9$ and $b = 0.4$ (a special case of Figure 1), so computing $Bias_\alpha(n)$, $Bias_\lambda(n)$, $Bias_a(n)$, $Bias_b(n)$ and $MSE_\alpha(n)$, $MSE_\lambda(n)$, $MSE_a(n)$, $MSE_b(n)$ for $n = 10, 15, 20, ..., 320$. Figures 3 and 4 show how the biases and mean squared errors change with respect to n.

The following observations can be made:
1. the biases for λ, a and b are generally positive,
2. the biases for α have both sign,
3. the biases for each parameter generally approach zero as $n \to \infty$,
4. the biases appear smallest for α,
5. the mean squared errors for each parameter generally decrease to zero as $n \to \infty$;
6. the mean squared errors appear smallest for all parameters for n large enough ($n \geq 200$).

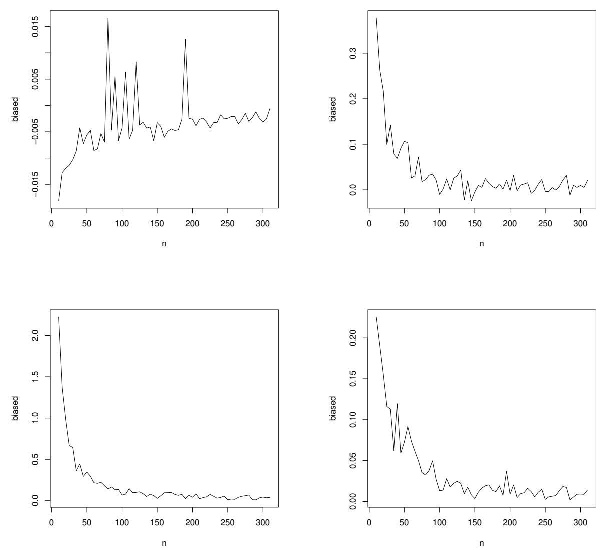

Figure 3. $Bias_\alpha(n)$ (top left), $Bias_\lambda(n)$ (top right), $Bias_a(n)$ (bottom left) and $Bias_b(n)$ (bottom right) versus $n = 10, 15, 20, ..., 320$.

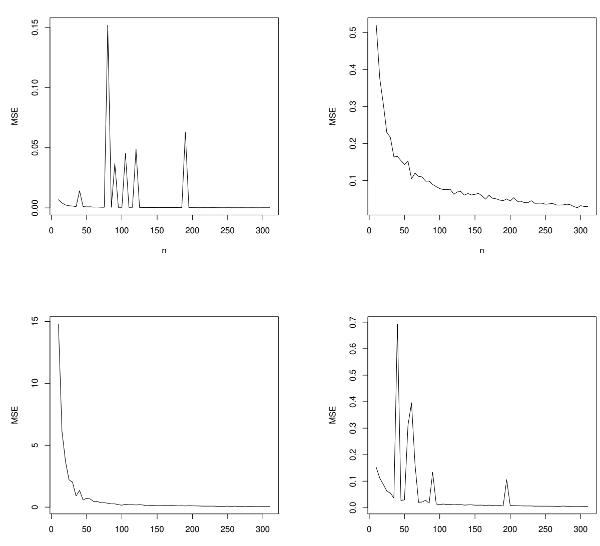

Figure 4. $MSE_\alpha(n)$ (top left), $MSE_\lambda(n)$ (top right), $MSE_a(n)$ (bottom left) and $MSE_b(n)$ (bottom right) versus $n = 10, 15, 20, ..., 320$.

8. Applications

We present two applications based on two real data sets to demonstrate the flexibility of the TW-W and TW-L distributions. We compare TW-W with Kw-Weibull (Kw-W) (Cordiero et al., 2010), Beta-Weibull (BW) (Lee et al., 2007), Beta- Exponentiated Weibull (BEW)(Cordeiro et al., 2013), Kw-Exponential Weibull (Kw-EW)(Cordeiro et al., 2016) and Beta-Modified Weibull (BMW)(Silva et al., 2010) distributions. Also, we compare TW-L with Exponentiated power Lindley (EPL)(Warahena-Liyanage and Pararai, 2014), extended Lindley (EXL)(Bakouch et al., 2012), extended power Lindley (EPL)(Alkarni, 2015) and generalized Lindley (GL)(Zakerzadeh and Dolati, 2009) distributions.

The first data set is given by Murthy et al. (2004) on failure times for a particular model aircraft windshield. The data set consists of 84 observations and was also analyzed by Ramos et al. (2013).

The second data set is the fracture toughness of Alumina (Al2O3) (in the units of MPa m1/2), Nadarajah and Kotz (2008). These data are 5.5, 5, 4.9, 6.4, 5.1, 5.2, 5.2, 5, 4.7, 4, 4.5, 4.2, 4.1, 4.56, 5.01, 4.7, 3.13, 3.12, 2.68, 2.77, 2.7, 2.36, 4.38, 5.73, 4.35, 6.81, 1.91, 2.66, 2.61, 1.68, 2.04, 2.08, 2.13, 3.8, 3.73, 3.71, 3.28, 3.9, 4, 3.8, 4.1, 3.9, 4.05, 4, 3.95, 4, 4.5, 4.5, 4.2, 4.55, 4.65, 4.1, 4.25, 4.3, 4.5, 4.7, 5.15, 4.3, 4.5, 4.9, 5, 5.35, 5.15, 5.25, 5.8, 5.85 ,5.9, 5.75, 6.25, 6.05, 5.9, 3.6, 4.1, 4.5, 5.3, 4.85, 5.3, 5.45, 5.1, 5.3, 5.2, 5.3, 5.25, 4.75, 4.5, 4.2, 4, 4.15, 4.25, 4.3, 3.75, 3.95, 3.51, 4.13, 5.4, 5, 2.1, 4.6, 3.2, 2.5, 4.1, 3.5, 3.2, 3.3, 4.6, 4.3, 4.3, 4.5, 5.5, 4.6, 4.9, 4.3, 3, 3.4, 3.7, 4.4, 4.9, 4.9, 5.

The MLE of parameters, $-$maximized log-likelihood function, Akaike information criterion (AIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC), Consistent Akaike information criterion (CAIC) statistics are determined by fitting mentioned distributions using the two data sets.

In general, the smaller values of these statistics show the better fit to the data sets. The MLEs are computed using the "optim" function in R statistical program. The estimated parameters based on MLE procedure reports in Tables 1 and 2, whereas the values of goodness-of-fit statistics are given in Tables 3 and 4.

In the applications, the information about the hazard shape can help in selecting a particular model. For this aim, a device called the total time on test (TTT) plot (Aarset, 1987) is useful. The TTT plot is obtained by plotting

$$G(r/n) = \left[\left(\sum_{i=1}^{r} y_{(i)}\right) + (n-r)y_{(r)}\right] / \sum_{i=1}^{n} y_{(i)},$$

where $r = 1, ..., n$ and $y_{(i)}$ $(i = 1, ..., n)$ are the order statistics of the sample, against r/n. If the shape is a straight diagonal the hazard is constant. It is convex shape for decreasing hazards and concave shape for increasing hazards.

The TTT plot for both data sets presented in Figure 6. These figures indicates that first and second data set has increasing hazard rate functions.

In both real data sets, the results show that the TW-W and TW-L distribution yields a better fit than other generalizations of Weibull and Lindley distributions. Figure 5 shows the fitted pdf on histogram of both data sets.

Table 1. Parameters estimates and standard deviation in parenthesis for first data set

Model	Estimates	−Log Likelihood
TW-W	0.10, 1.00, 21.85, 4.21	102.602
(α, λ, a, b)	$(9e^{-3})$, (0.12), (1.57), (0.04)	
Kw-W	17.47, 594.78, 0.29, 0.60	107.756
(a, b, β, c)	(0.33), (69.14), (0.01), (0.02)	
BW	1.28, 39.78, 0.07, 2.34	107.705
(a, b, β, c)	(0.10), (4.08), $(3e^{-3})$, (0.07)	
BEW	11.86, 3.86, 0.02, 1.22, 0.16	106.523
$(a, b, \alpha, c, \lambda)$	(0.77), (0.24), $(1e^{-3})$, (0.66), $(6e^{-3})$	
Kw-EW	0.24, 0.01, 2.44, 3.17, 4.89	107.752
$(a, b, \alpha, c, \lambda)$	(0.01), $(2e^{-3})$, (0.18), (0.19), (0.56)	
BMW	4.84, 0.11, 1.03, 0.51, 0.65	106.632
$(a, b, \alpha, \lambda, \gamma)$	(1.50), (0.01), (0.02), (0.01), (0.02)	

Table 2. Parameters estimates and standard deviation in parenthesis for second data set

Model	Estimates	−Log Likelihood
TW-L$(\alpha, \lambda, \theta)$	3.00(0.21), 0.78(0.11), 0.27$(4e^{-3})$	168.250
EPL(θ, α, β)	0.01$(6e^{-4})$, 3.43(0.04), 0.88(0.08)	169.948
EXL(θ, α, β)	0.21$(3e^{-3})$, -0.01(0.08), 4.99(0.34)	168.887
EXPL(θ, α, β)	0.01$(6e^{-4})$3.44(0.04), 0.07(0.05)	169.381
GL(θ, α, β)	3.64(0.08), 15.05(0.36), 8.03(12.30)	177.271

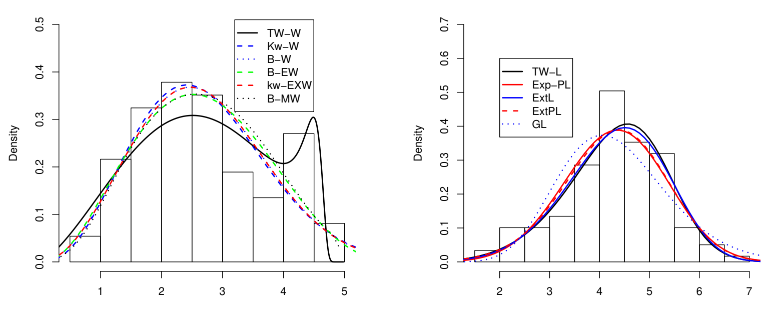

Figure 5. Fitted pdfs on histogram: (left) example 1, (right) example 2

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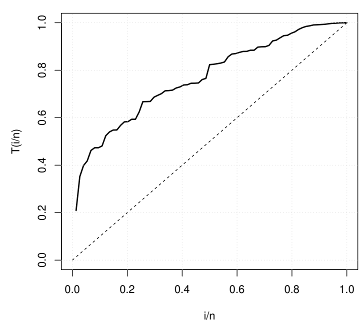
Table 3. Formal goodness of fit statistics for first data set

Model	Goodness of fit criteria			
	AIC	*BIC*	*HQIC*	*CAIC*
TW-W	213.20	224.32	211.46	213.55
Kw-W	223.51	234.63	221.77	223.86
BW	223.41	234.52	221.66	223.76
BEW	223.04	236.94	220.86	223.57
Kw-EW	225.50	239.40	223.32	226.03
BMW	223.26	237.15	221.08	223.79

Table 4. Formal goodness of fit statistics for second data set

Model	Goodness of fit criteria			
	AIC	*BIC*	*HQIC*	*CAIC*
TW-L	342.50	350.83	342.70	341.19
EPL	345.89	354.23	346.10	344.59
EXL	343.77	352.11	343.98	342.46
EXPL	348.76	353.10	348.97	343.45
GL	364.54	368.88	364.75	359.23



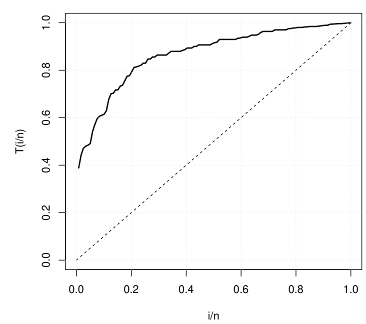


Figure 6. TTT-plot for the first data set (left figure) and for the second data set (right figure).

9. Conclusions

Recently, there has been a great interest among the specialists, statisticians and practitioners to generate new extended families from classic ones. In this paper, we present a new class of distributions called the transmuted Weibull-G (TW-G) family of distributions, which extends the transmuted family by adding one extra shape parameter. The mathematical properties of this new family including explicit expansions for the ordinary and incomplete moments, generating function, mean deviations, order statistics, probability weighted moments are provided. Characterizations based on the ratio of two truncated moments as well as based on hazard function are presented. The model parameters are estimated by the maximum likelihood estimation method and the observed information matrix is determined. It is shown, by means of two real data sets, that special cases of the TW-G class can give better fit than other models generated by the well-known families.

Appendix A

Theorem. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a given probability space and let $H = [d, e]$ be an interval for some $d < e$ $(d = -\infty,\ e = \infty$ might as well be allowed$)$. Let $X : \Omega \to H$ be a continuous random variable with the distribution function F and let q_1 and q_2 be two real functions defined on H such that

$$\mathbf{E}\left[q_2(X) \mid X \geq x\right] = \mathbf{E}\left[q_1(X) \mid X \geq x\right]\xi(x), \quad x \in H,$$

is defined with some real function ξ. Assume that $q_1,\ q_2 \in C^1(H)$, $\xi \in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\xi q_1 = q_2$ has no real solution in the interior of H. Then F is uniquely determined by the functions q_1, q_2 and ξ , particularly

$$F(x) = \int_a^x C\left|\frac{\xi'(u)}{\xi(u)\,q_1(u) - q_2(u)}\right| \exp(-s(u))\ du\ ,$$

where the function s is a solution of the differential equation $s' = \frac{\xi'\ q_1}{\xi\ q_1\ -\ q_2}$ and C is the normalization constant, such that $\int_H dF = 1$.

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