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Stability and Hopf Bifurcation for an SEIR Epidemic Model with Delay

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Abstract

In this paper, first a third degree transcendental polynomial is studied and the distribution of its zeros is established. Then the results are applied to study an SEIR model with a time delay. We show that, under some conditions, as the time delay increases, a stable endemic equilibrium will become unstable and periodic solution emerges by Hopf bifurcation. By finding the normal form of the system, the direction and the stability of the periodic solution are established. Numerical simulations are performed to demonstrate the theoretical results.

Keywords: Transcendental polynomial; SEIR model; Hopf bifurcation
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1. Introduction

SEIR epidemiological models have been extensively studied by many researchers for last few years. Typically, an SEIR model describes the dynamics of population that is divided into compartments which are susceptible, exposed, infectious, and recovered, respectively. An individual starts off susceptible, infected by infectious individual through direct contact to become exposed (not yet infectious), after a period of incubation time becomes infectious, and then finally recover with permanent immunity. The transfer diagram for

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SEIR models can be depicted in the following figure

$$\begin{array}{ccccccccc} \xrightarrow{\mu} & S & \xrightarrow{\lambda IS} & E & \xrightarrow{\epsilon E} & I & \xrightarrow{\gamma I} & R, \\ & \mu S \big\downarrow & & \mu E \big\downarrow & & \mu I \big\downarrow & & \mu R \big\downarrow \end{array}$$

where $S(t)$, $E(t)$, $I(t)$, and $R(t)$ represent the densities of the susceptible, the exposed (in the latent period), the infectious, and the recovered, respectively. The natural birth rate and death rate are assumed to be identical and denoted by μ. The disease is assumed not to cause extra death on the infected hosts so that the total population density is constant; i.e., $S(t) + E(t) + I(t) + R(t) = 1$. The incidence term λIS is of the standard mass-action form. The parameter $\epsilon > 0$ is the rate at which the exposed individuals become infectious, and $\gamma \geq 0$ is the rate at which the infectious individuals recover. For details of SEIR models and diseases they describe, see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10] and reference therein.

The transfer diagram leads to the following system of differential equations

$$\begin{cases} S'(t) = \mu - \lambda I(t)S(t) - \mu S(t) \\ E'(t) = \lambda I(t)S(t) - (\epsilon + \mu)E(t) \\ I'(t) = \epsilon E(t) - (\gamma + \mu)I(t) \\ R'(t) = \gamma I(t) - \mu R(t). \end{cases} \tag{1.1}$$

For system (1.1), let

$$R_0 = \frac{\lambda \epsilon}{(\mu + \epsilon)(\mu + \gamma)} \tag{1.2}$$

be the basic reproduction number. Then Li et al. [7] ($p = q = 0$) proved that if $R_0 \leq 1$, then $P_0 = (1, 0, 0, 0)$ is the only equilibrium and it is globally asymptotically stable, whereas if $R_0 > 1$, then P_0 is unstable and there exists a unique endemic equilibrium $P^* = (S^*, E^*, I^*, R^*)$ where $S^*, E^*, I^*, R^* > 0$ and it is globally asymptotically stable in the interior of the feasible region. This means that the reproduction number R_0 completely determines the global dynamics of system (1.1). If $R_0 \leq 1$, the disease dies out over time, whereas if $R_0 > 1$, the disease persists.

Various authors have previously studied SIR or SEIR models containing one or two time delays. Basically, the introduction of time delays will change the dynamics of the system, especially the stability property of the endemic equilibrium when $R_0 > 1$. The time delays may sometimes destabilize the unique endemic equilibrium if the time delay is large enough and periodic solutions can arise from it by Hopf bifurcation. In their series of papers, Hethcote along with his collaborators [11, 12, 13, 14, 15, 16] considered SIR and SEIR type models with constant time delays. Cooke and van den Driessche [17] investigated an SIS model with variable population size and a delay and an SEIRS model with two delays. Thieme [18, 19] studied SEIRS models with delays. More recently, Khan and Greenhalgh [20], Tchuenche *et al.* [21], Rost and Wu [22] also studied SIR or SEIR type models with different delays.

In this paper, assuming that the infectious individuals start recovering after a period of time $\tau \geq 0$, we introduce and study the following system with a time delay.

$$\begin{cases} S'(t) = \mu - \lambda I(t)S(t) - \mu S(t) \\ E'(t) = \lambda I(t)S(t) - (\epsilon + \mu)E(t) \\ I'(t) = \epsilon E(t) - (\gamma + \mu)I(t - \tau) \\ R'(t) = \gamma I(t - \tau) - \mu R(t). \end{cases} \tag{1.3}$$

Our interest is to find out the dynamical behavior for this new system and see how the introduction of the delay causes the behavior change from the old system (1.1) without any delays.

We first study the distribution of zeros of a third degree transcendental polynomial in Section 2. Then in Section 3, using the results we get from Section 2, we show that the introduction of the delay τ may or may not change the dynamics of the system (1.1) depending upon the regions where the parameter values lie in. When $R_0 > 1$, we show that under some conditions the unique endemic equilibrium P^* of system (1.3) becomes unstable as τ increases and periodic solutions arise from it by Hopf bifurcation. The normal form is derived in Section 4, and the stability and direction of the periodic solution are established. In Section 5, numerical simulations are provided in this section to illustrate our theoretical results. Finally, a discussion of our results is given in Section 6.

2. A third degree transcendental polynomial

In this section, we study the distribution of zeros of a third degree transcendental polynomial.

Consider the following third degree transcendental polynomial

$$\lambda^3 + a\lambda^2 + b\lambda + c + (d\lambda^2 + e\lambda + f)e^{-\lambda\tau} = 0, \tag{2.1}$$

where $a, b, c, d, e,$ and f are real numbers. Clearly, $iw(w > 0)$ is a root of equation (2.1) if and only if

$$-iw^3 - aw^2 + ibw + c + (-dw^2 + iew + f)(\cos w\tau - i\sin w\tau) = 0.$$

Separating the real and the imaginary parts, we have

$$aw^2 - c = (f - dw^2)\cos w\tau + ew\sin w\tau, \tag{2.2}$$

$$w^3 - bw = -(f - dw^2)\sin w\tau + ew\cos w\tau. \tag{2.3}$$

Adding up the squares of both equations (2.2) and (2.3) gives

$$w^6 + (a^2 - 2b - d^2)w^4 + (b^2 - 2ac - e^2 + 2df)w^2 + (c^2 - f^2) = 0. \tag{2.4}$$

Let $z = w^2$ and denote $p = a^2 - 2b - d^2, q = b^2 - 2ac - e^2 + 2df$, and $r = c^2 - f^2$. Then equation (2.4) can be rewritten as

$$z^3 + pz^2 + qz + r = 0. \tag{2.5}$$

The following result is due to Ruan and Wei [23, 24].

Lemma 2.1. *For equation (2.5), we have*

- (a) *If* $r < 0$, *then it has at least one positive root.*
- (b) *If* $r \geq 0$ *and* $\Delta = p^2 - 3q < 0$, *then it has no positive roots.*
- (c) *If* $r \geq 0$, *then it has positive roots if and only if*

$$\Delta \geq 0,\ \ \bar{z} = \frac{1}{3}(-p + \sqrt{\Delta}) > 0 \quad \text{and} \quad h(\bar{z}) \leq 0,$$

where $h(z) = z^3 + pz^2 + qz + r$.

Suppose that equation (2.5) has positive roots. Without loss of generality, we assume that it has three positive roots, denoted by $z_1, z_2,$ and z_3, respectively. Then equation (2.4) has three positive roots,

$$w_1 = \sqrt{z_1}, \quad w_2 = \sqrt{z_2}, \quad \omega_3 = \sqrt{z_3}.$$

Solving equations (2.2) and (2.3) for $\sin w\tau$ and $cos\omega\tau$, we get

$$\sin\omega\tau = \frac{d\omega^5 + (ae - db - f)\omega^3 + (bf - ec)\omega}{e^2\omega^2 + (f - dw^2)^2} \equiv f_1(\omega),$$

$$\cos\omega\tau = \frac{(e-ad)\omega^4 + (dc+af-be)\omega^2 - cf}{e^2\omega^2 + (f-d\omega^2)^2} \equiv g_1(\omega).$$

Define

$$\tau_k^+ = \begin{cases} \frac{1}{\omega_k}(\arccos g_1(\omega_k)) & \text{if} \quad f_1(\omega_k) > 0, \\ \frac{1}{\omega_k}(2\pi - \arccos g_1(\omega_k)) & \text{if} \quad f_1(\omega_k) \le 0, \end{cases}$$

where $k = 1, 2, 3$. It follows that $\tau_k^+ > 0$ and equation (2.1) has a pair of purely imaginary roots $\pm i w_k$ when $\tau = \tau_k^+,\ k = 1, 2, 3$.

Let

$$\tau_0 = \tau_{k_0}^+ = \min\{\tau_k^+, k = 1, 2, 3\} > 0, \quad w_0 = w_{k_0}, \quad z_0 = z_{k_0}. \tag{2.6}$$

Lemma 2.2. *Let $p, q, r, \bar{z}, h(z)$, and τ_0 be defined above. Suppose that*

$$a + d > 0, \quad c + f > 0, \quad (a+d)(b+e) - (c+f) > 0. \tag{2.7}$$

Then the following results hold.

(a) *If $r \ge 0$ and $\Delta = p^2 - 3q < 0$, then all roots of equation (2.1) have negative real parts for all $\tau \ge 0$.*

(b) *If $r < 0$ or $r \ge 0, \Delta \ge 0, \bar{z} > 0$ and $h(\bar{z}) \le 0$, then all roots of equation (2.1) have negative real parts when $\tau \in [0, \tau_0)$, and it has a pair of purely imaginary roots $\pm i w_0$ and all other roots have negative real parts when $\tau = \tau_0$.*

Proof. When $\tau = 0$, equation (2.1) becomes

$$\lambda^3 + (a+d)\lambda^2 + (b+e)\lambda + (c+f) = 0. \tag{2.8}$$

By the Routh-Hurwitz Theorem, all roots of equation (2.8) have negative real parts if and only if (2.7) holds.

If $r \ge 0$ and $\Delta = p^2 - 3q < 0$, Lemma 2.1 (b) shows that equation (2.1) has no roots with zero real parts for all $\tau \ge 0$. By continuity, all zeros of equation (2.1) have negative real parts for all $\tau \ge 0$. When $r < 0$ or $r \ge 0, \Delta \ge 0, \bar{z} > 0$ and $h(\bar{z}) \le 0$, Lemma 2.1 (a) or (c) implies that if $\tau \in [0, \tau_0)$, equation (2.1) has no roots with zero real parts and τ_0 is the minimum value of τ such that equation (2.1) has purely imaginary roots. Claim (b) follows from condition (2.7), continuity and Lemma 1, completing the proof. □

Next we will try to establish the transversality condition for Hopf bifurcations. For $\tau \ge 0$, let

$$\lambda(\tau) = \alpha(\tau) + i w(\tau)$$

be the root of equation (2.1) satisfying

$$\alpha(\tau_0) = 0, \quad w(\tau_0) = w_0,$$

where τ_0 and w_0 are defined in (2.6). Differentiating both sides of equation (2.1) with respect to τ gives

$$\frac{d\lambda}{d\tau}[3\lambda^2 + 2a\lambda + b + (2d\lambda + e)e^{-\lambda\tau} - \tau(d\lambda^2 + e\lambda + f)e^{-\lambda\tau}] = \lambda(d\lambda^2 + e\lambda + f)e^{-\lambda\tau}.$$

Notice that

$$\lambda^3 + a\lambda^2 + b\lambda + c = -(d\lambda^2 + e\lambda + f)e^{-\lambda\tau},$$

we have

$$\begin{aligned} \left(\frac{d\lambda}{d\tau}\right)^{-1} &= -\frac{3\lambda^2 + 2a\lambda + b}{\lambda(\lambda^3 + a\lambda^2 + b\lambda + c)} + \frac{2d\lambda + e}{\lambda(d\lambda^2 + e\lambda + f)} - \frac{\tau}{\lambda} \\ &= -\frac{1}{\lambda}\left[\tau - \left(\frac{2d\lambda + e}{d\lambda^2 + e\lambda + f} - \frac{3\lambda^2 + 2a\lambda + b}{\lambda^3 + a\lambda^2 + b\lambda + c}\right)\right]. \end{aligned}$$

Substituting $\tau = \tau_0$ into the equality above, we get

$$\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\tau=\tau_0} = \frac{-1}{iw_0}\left[\tau_0 - \left(\frac{g'(iw_0)}{g(iw_0)} - \frac{f'(iw_0)}{f(iw_0)}\right)\right], \tag{2.9}$$

where $f(\lambda) = \lambda^3 + a\lambda^2 + b\lambda + c$ and $g(\lambda) = d\lambda^2 + e\lambda + f$.

Lemma 2.3. *Let condition (10) hold. If* $Re\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\tau=\tau_0} \neq 0$, *then* $\frac{d\alpha(\tau_0)}{d\tau} > 0$.

Proof. It is clear that

$$\text{Sign Re}\left(\frac{d\lambda}{d\tau}\right) = \text{Sign Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}.$$

It follows that $\text{Re}\left(\frac{d\lambda}{d\tau}\right)_{\tau=\tau_0} = \frac{d\alpha(\tau_0)}{d\tau} \neq 0$ when $\text{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\tau=\tau_0} \neq 0$. If $\frac{d\alpha(\tau_0)}{d\tau} < 0$, then $\frac{d\alpha}{d\tau} < 0$ for $\tau < \tau_0$ and close to τ_0. This implies that equation (2.1) has a root $\lambda(\tau) = \alpha(\tau) + iw(\tau)$ satisfying $\alpha(\tau) > 0$ for $\tau < \tau_0$ and close to τ_0, which contradicts Lemma 2.2, completing the proof. □

Remark 2.4. *From (2.9), we see that if* $Im\left(\frac{g'(iw_0)}{g(iw_0)} - \frac{f'(iw_0)}{f(iw_0)}\right) \neq 0$, *then* $Re\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\tau=\tau_0} \neq 0$, *and therefore* $\frac{d\alpha(\tau_0)}{d\tau} \neq 0$.

3. An SEIR epidemiological model with delay

In this section, we study the dynamics of the delay system (3). Notice that R does not appear in the first three equations, therefore we analyze the following subsystem

$$\begin{cases} S' = \mu - \beta IS - \mu S, \\ E' = \beta IS - (\epsilon + \mu)E, \\ I' = \epsilon E - \mu I - \gamma I(t-\tau). \end{cases} \tag{3.1}$$

The basic reproduction number R_0 for system (3.1) is defined by (2).

It can be shown [7] that if $R_0 > 1$, the system has two equilibria: the disease free equilibrium $P_0 = (1, 0, 0)$ and the endemic equilibrium $P^* = (S^*, E^*, I^*)$, where

$$\begin{aligned} S^* &= \frac{(\mu+\epsilon)(\mu+\gamma)}{\beta\epsilon} = \frac{1}{R_0}, \\ E^* &= \frac{\mu}{\mu+\epsilon}\left(1 - \frac{1}{R_0}\right), \\ I^* &= \frac{\mu\epsilon}{(\mu+\epsilon)(\mu+\gamma)}\left(1 - \frac{1}{R_0}\right). \end{aligned}$$

The disease-free equilibrium P_0 is unstable and P^* is asymptotically stable when $R_0 > 1$ and $\tau = 0$. The following theorem shows that the introduction of a delay τ will not change the instability of P_0.

Theorem 3.1. *If* $R_0 > 1$, *then* P_0 *is unstable for all* $\tau \geq 0$.

Proof. The characteristic equation associated with the system (3.1) at P_0 is

$$(\lambda+\mu)[\lambda^2 + (2\mu+\epsilon)\lambda + \mu(\mu+\epsilon) - \beta\epsilon + \gamma(\lambda+\mu+\epsilon)e^{-\lambda\tau}] = 0$$

and $\lambda = -\mu$ is a root. The other roots are determined by the equation

$$\lambda^2 + (2\mu+\epsilon)\lambda + \mu(\mu+\epsilon) - \beta\epsilon + \gamma(\lambda+\mu+\epsilon)e^{-\lambda\tau} = 0.$$

Define

$$F(\lambda) = \lambda^2 + (2\mu+\epsilon)\lambda + \mu(\mu+\epsilon) - \beta\epsilon + \gamma(\lambda+\mu+\epsilon)e^{-\lambda\tau}.$$

Obviously, $F(0) = (\mu+\epsilon)(\mu+\gamma) - \beta\epsilon < 0$ if $R_0 > 1$, and $F(\lambda) \to \infty$ as $\lambda \to \infty$. Thus, there always exists a positive root regardless of the delay, that proves P_0 is unstable. □

We now study the stability of the endemic equilibrium P^* when $R_0 > 1$.

Let $u_1 = S - S^*, u_2 = E - E^*$, and $u_3 = I - I^*$. Then system (3.1) becomes

$$\begin{cases} u_1' = -(\mu + \beta I^*)u_1 - \beta S^* u_2 - \beta u_1 u_3, \\ u_2' = \beta I^* u_1 - (\mu + \epsilon)u_2 + \beta S^* u_3 + \beta u_1 u_3, \\ u_3' = \epsilon u_2 - \mu u_3 - \gamma u_3(t-\tau). \end{cases} \tag{3.2}$$

The linearization of the system at $(0, 0, 0)$ is

$$\begin{cases} u_1' = -(\mu + \beta I^*)u_1 - \beta S^* u_3, \\ u_2' = \beta I^* u_1 - (\mu + \epsilon)u_2 + \beta S^* u_3, \\ u_3' = 3\epsilon u_2 - \mu u_3 - \gamma u_3(t-\tau). \end{cases} \tag{3.3}$$

The characteristic equation associated with the system (3.3) at $(0, 0, 0)$ is

$$\lambda^3 + a\lambda^2 + b\lambda + c + (d\lambda^2 + e\lambda + f)e^{-\lambda\tau} = 0 \tag{3.4}$$

where

$$\begin{aligned} a &= 3\mu + \epsilon + \beta I^*, \\ b &= (2\mu + \epsilon)(\mu + \beta I^*) - \gamma(\mu + \epsilon), \\ c &= \mu(\mu + \epsilon)(\beta I^* - \gamma), \\ d &= \gamma, \\ e &= \gamma(2\mu + \epsilon + \beta I^*), \\ f &= \gamma(\mu + \epsilon)(\mu + \beta I^*). \end{aligned}$$

Consequently, We have

$$\begin{aligned} a + d &= 3\mu + \epsilon + \beta I^* + \gamma > 0, \\ b + e &= (2\mu + \epsilon + \gamma)(\mu + \beta I^*) > 0, \\ c + f &= (\mu + \epsilon)(\mu + \gamma)\beta I^* > 0, \end{aligned}$$

and obviously one can see that

$$(a + d)(b + e) - (c + f) = (3\mu + \epsilon + \beta I^* + \gamma)(2\mu + \epsilon + \gamma)(\mu + \beta I^*) - (\mu + \epsilon)(\mu + \gamma)\beta I^* > 0.$$

That is, condition (10) always holds when $R_0 > 1$.

We can also get

$$\begin{aligned} p &= a^2 - 2b - d^2 = (\mu + \beta I^*)^2 + (2\mu + \epsilon)^2 + 2\gamma(\mu + \epsilon) - \gamma^2, \\ q &= b^2 - 2ac - e^2 + 2df \\ &= [(2\mu + \epsilon)^2 - \gamma^2](\mu + \beta I^*)^2 \\ &\quad - 2(\mu + \epsilon)[\gamma(2\mu + \epsilon)(\mu + \beta I^*) + \mu(3\mu + \epsilon + \beta I^*)(\beta I^* - \gamma)] \end{aligned}$$

and

$$r = c^2 - f^2 = \beta I^*(\mu + \epsilon)^2(\mu + \gamma)[(\mu - \gamma)\beta I^* - 2\mu\gamma].$$

Applying Lemmas from the previous sections we obtain the following theorem.

Theorem 3.2. *Let p, q, r be defined above. Then*

(a) *If $r \geq 0$ and $\Delta = p^2 - 3q < 0$, then all roots of equation (3.4) have negative real parts for all $\tau \geq 0$. Therefore, the equilibrium point $P^* = (S^*, E^*, I^*)$ is locally asymptotically stable for all $\tau \geq 0$.*

(b) *If $r < 0$ or $r \geq 0, \Delta \geq 0, \bar{z} > 0$, and $h(\bar{z}) \leq 0$, then there exists a $\tau_0 > 0$ defined by (9) such that all roots of equation (3.4) have negative real parts for $\tau \in [0, \tau_0)$, the equilibrium P^* is locally asymptotically stable. When $\tau = \tau_0$, equation (3.4) has a pair of purely imaginary roots $\pm i\omega_0$ and all other roots have negative real parts. In addition, if $Re\left(\frac{d\lambda}{d\tau}\right)_{\tau=\tau_0} \neq 0$, then the system (3.1) undergoes a Hopf bifurcation as τ passes the critical value τ_0.*

4. Normal form of Hopf bifurcation

From Section 3, we know that the characteristic equation of linearized system (3.1) at the endemic equilibrium point $P^* = (S^*, E^*, I^*)$ has a pair of purely imaginary roots $\pm i\omega_0$ if $\tau = \tau_0$ under some conditions from part (b) of Theorem 3.2. As the delay τ passes the critical values τ_0, Hopf bifurcation occurs and periodic solution emerges. In this section, using center manifold reduction, we study the direction and the stability of the bifurcating periodic solutions. We first normalize the delay in system (3.2) by rescaling $t \to t/\tau$ to get the following system

$$\begin{cases} u_1' = \tau[-(\mu+\beta I^*)u_1 - \beta S^* u_3 - \beta u_1 u_3], \\ u_2' = \tau[\beta I^* u_1 - (\mu+\epsilon)u_2 + \beta S^* u_3 + \beta u_1 u_3], \\ u_3' = \tau[\epsilon u_2 - \mu u_3 - \gamma u_3(t-1)]. \end{cases} \tag{4.1}$$

Let $\tau = \tau_0 + \xi$. Then ξ is the bifurcation parameter for system (4.1) which can be rewritten as

$$\begin{cases} u_1' = (\tau_0+\xi)[-(\mu+\beta I^*)u_1 - \beta S^* u_3 - \beta u_1 u_3], \\ u_2' = (\tau_0+\xi)[\beta I^* u_1 - (\mu+\epsilon)u_2 + \beta S^* u_3 + \beta u_1 u_3], \\ u_3' = (\tau_0+\xi)[\epsilon u_2 - \mu u_3 - \gamma u_3(t-1)]. \end{cases} \tag{4.2}$$

The linearization of system (4.2) at $(0,0,0)$ is

$$\begin{cases} u_1' = \tau_0[-(\mu+\beta I^*)u_1 - \beta S^* u_3], \\ u_2' = \tau_0[\beta I^* u_1 - (\mu+\epsilon)u_2 + \beta S^* u_3], \\ u_3' = \tau_0[\epsilon u_2 - \mu u_3 - \gamma u_3(t-1)]. \end{cases} \tag{4.3}$$

Let

$$\eta(\theta) = \mathbb{A}\delta(\theta) + \mathbb{B}\delta(\theta+1)$$

where

$$\mathbb{A} = \tau_0 \begin{pmatrix} -(\mu+\beta)I^* & 0 & -\beta S^* \\ \beta I^* & -(\mu+\epsilon) & \beta S^* \\ 0 & \epsilon & -\mu \end{pmatrix}, \ \mathbb{B} = \tau_0 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\gamma \end{pmatrix}.$$

Let $C = C([-1,0], \mathbb{C}^3)$ and define a linear operator $\mathcal{L}$ on C as

$$\mathcal{L}\varphi = \int_{-1}^{0} d\eta(\theta)\varphi(\theta) = \mathbb{A}\varphi(0) + \mathbb{B}\varphi(-1), \ \forall \varphi \in C.$$

Then system (4.2) can be transformed into

$$\dot{U}(t) = \mathcal{L}U_t + F(U_t) + O(|\xi|\|U\|^2),$$

where $U = (u_1, u_2, u_3)^T$, $U_t = U(t+\theta)$, $\theta \in [-1,0]$, and $F(U_t) = (F^1, F^2, F^3)^T$ where

$$\begin{aligned} F^1 &= \xi[-(\mu+\beta I^*)u_1 - \beta S^* u_3] - \tau_0\beta u_1 u_3, \\ F^2 &= \xi[\beta I^* u_1 - (\mu+\epsilon)u_2 + \beta S^* u_3] + \beta u_1 u_3, \\ F^3 &= \xi[\epsilon u_2 - \mu u_3 - \gamma u_3(t-1)]. \end{aligned}$$

Write

$$F(\varphi) = \frac{1}{2}F_2(\varphi) + \frac{1}{3!}F_3(\varphi) + \text{h.o.t.}$$

where $F_k(\varphi)$ represents the terms with degree k of φ and "h.o.t." high order terms. Clearly $F_3(\varphi) = 0$.

Take the enlarged space of C

$$BC = \{\varphi : [-1,0] \to \mathbb{C}^2 : \ \varphi \text{ is continuous on } [-1,0), \ \exists \lim_{\theta\to 0^-} \varphi(\theta) \in \mathbb{C}^2\}.$$

Then the elements of BC can be expressed as $\psi = \varphi + X_0\nu$, $\varphi \in C, \nu \in \mathbb{C}^2$ and

$$X_0(\theta) = \begin{cases} 0, \ -1 \le \theta < 0, \\ I, \ \theta = 0, \end{cases}$$

where I is the identity matrix on C and the norm of BC is $|\varphi + X_0\nu| = |\varphi|_\infty + |\nu|$. Let $C^1 = C^1([-1,0), \mathbb{C}^2)$. Then the infinitesimal generator $\mathcal{A} : C^1 \to BC$ associated with $\mathcal{L}$ is given by

$$\begin{aligned} \mathcal{A}\varphi &= \dot{\varphi} + X_0[\mathcal{L}\varphi - \dot{\varphi}(0)] \\ &= \begin{cases} \dot{\varphi}, & \text{if } -1 \le \theta < 0, \\ \int_{-1}^0 d\eta(t)\varphi(t), & \text{if } \theta = 0, \end{cases} \end{aligned}$$

and its adjoint

$$\mathcal{A}^*\psi = \begin{cases} -\dot{\psi}, & \text{if } 0 < s \le 1, \\ \int_{-1}^0 \psi(-t)d\eta(t), & \text{if } s = 0, \end{cases}$$

for $\forall\psi \in C^{1*}$, where $C^{1*} = C^1((0,1], \mathbb{C}^{2*})$. Let $C' = C((0,1], \mathbb{C}^{2*})$ and define a bilinear inner product between C and C' by

$$\begin{aligned} \langle \psi, \varphi \rangle &= \bar{\psi}(0)\varphi(0) - \int_{-1}^0 \int_0^\theta \bar{\psi}(\xi - \theta)d\eta(\theta)\varphi(\xi)d\xi \\ &= \bar{\psi}(0)\varphi(0) + \int_{-1}^0 \psi(\xi+1)\mathbb{B}\varphi(\xi)d\xi. \end{aligned}$$

From Section 3, we know that $\pm i\tau_0\omega_0$ are eigenvalues of $\mathcal{A}$ and $\mathcal{A}^*$. Now we compute eigenvectors of $\mathcal{A}$ associated with $i\tau_0\omega_0$ and eigenvectors of $\mathcal{A}^*$ associated with $-i\tau_0\omega_0$. For simplicity and convenience, we will use ω, instead of ω_0, in all the following calculations. Let $q(\theta) = (\rho, \sigma, 1, 0)^T e^{i\tau_0\omega\theta}$ be an eigenvector of $\mathcal{A}$ associated with $i\tau_0\omega$. Then $\mathcal{A}q(\theta) = i\tau_0\omega q(\theta)$. It follows from the definition of $\mathcal{A}$ that

$$\begin{pmatrix} \mu + i\tau_0\omega & 0 & \beta \\ \frac{\mu((\gamma+\mu)(\epsilon+\mu)-\beta\epsilon)}{\beta\epsilon} & \epsilon + \mu + i\tau_0\omega & \frac{(-\gamma-\mu)(\epsilon+\mu)}{\epsilon} \\ 0 & -\epsilon & e^{-i\tau_0\omega}\gamma + \mu + i\tau_0\omega \end{pmatrix} q(0) = 0.$$

We can obviously choose $q(\theta) = (\rho, \sigma, 1)^T e^{i\tau_0\omega\theta}$ where

$$\rho = \frac{i(\gamma+\mu)^2(\mu+\epsilon)^2}{\epsilon(\omega(\gamma+\mu)(\mu+\epsilon) - i\beta\mu\epsilon)}, \sigma = \frac{(\gamma+\mu)^2(\omega - i\mu)(\mu+\epsilon)^2}{\epsilon(\mu + i\omega + \epsilon)(\omega(\gamma+\mu)(\mu+\epsilon) - i\beta\mu\epsilon)}.$$

Similarly, we can find an eigenvector $p(s)$ of $\mathcal{A}^*$ associated with $-i\tau_0\omega$

$$p(s) = \frac{1}{D}(\delta, \nu, 1)e^{i\tau_0\omega s}$$

where

$$\delta = -\frac{\mu\epsilon\left(-\beta\epsilon + \gamma\mu + \gamma\epsilon + \mu^2 + \mu\epsilon\right)}{(\mu - i\omega + \epsilon)\left(\beta\mu\epsilon - i\gamma\mu\omega - i\gamma\omega\epsilon - i\mu^2\omega - i\mu\omega\epsilon\right)}, \nu = \frac{\epsilon}{\mu - i\omega + \epsilon}$$

and D being a constant to be determined such that $\langle \bar{p}(s), q(\theta) \rangle = 1$. In fact,

$$D = \delta\bar{\rho} + \nu\bar{\sigma} - \gamma\tau_0 e^{i\tau_0\omega} + 1.$$

Let P be spanned by $q, \bar{q}$ and P^* by $p, \bar{p}$. Then C can be decomposed as

$$C = P \oplus Q \text{ where } Q = \{\varphi \in C : \langle \psi, \varphi \rangle = 0, \forall\psi \in P^*\}.$$

Let $Q^1 = Q \cap C^1$. Let $\Phi(\theta) = (q(\theta), \bar{q}(\theta))$ and $\Psi(s) = (\bar{p}(s), p(s))^T$. Then $\dot{\Phi} = \Phi J$ and $\dot{\Psi} = -J\Psi$ where $J = \mathrm{diag}(i\tau_0\omega, -i\tau_0\omega, 0)$. Let $u = \Phi x + y$, namely

$$\begin{aligned} u_1(\theta) &= e^{i\tau_0\omega\theta}\rho x_1 + e^{-i\tau_0\omega\theta}\bar{\rho}x_2 + y_1(\theta), \\ u_2(\theta) &= e^{i\tau_0\omega\theta}\sigma x_1 + e^{-i\tau_0\omega\theta}\sigma x_2 + y_2(\theta), \\ u_3(\theta) &= e^{i\tau_0\omega\theta}x_1 + e^{-i\tau_0\omega\theta}x_2 + y_3(\theta). \end{aligned}$$

Then system (4.3) can be decomposed as

$$\begin{cases} \frac{dx}{dt} = Jx + \Psi(0)F(\Phi x + y), \\ \frac{dy}{dt} = A_{Q^1}y + (I-\pi)X_0F(\Phi x + y). \end{cases}$$

This can be rewritten as

$$\begin{cases} \frac{dx}{dt} = Jx + \frac{1}{2}f_2^1(x,y,\kappa) + \frac{1}{3!}f_3^1(x,y,\kappa) + \text{h.o.t.}, \\ \frac{dy}{dt} = A_{Q^1}y + \frac{1}{2}f_2^2(x,y,\kappa) + \frac{1}{3!}f_3^2(x,y,\kappa) + \text{h.o.t.} \end{cases} \tag{4.4}$$

where

$$f_j^1(x,y,\kappa) = \Psi(0)F_j(\Phi x + y), \;\; f_j^2(x,y,\kappa) = (I-\pi)X_0F_j(\Phi x + y).$$

Let $x = (x_1, x_2)$. Then on the center manifold, system (4.4) can be transformed as the following normal form:

$$\frac{dx}{dt} = Jx + \frac{1}{2}g_2^1(x,0,\kappa) + \frac{1}{3!}g_3^1(x,0,\kappa) + \text{h.o.t.} \tag{4.5}$$

where $g_j^1(x,0,\kappa)$ is a homogeneous polynomial of degree j in (x,κ) and

$$g_2^1(x,0,\kappa) = \mathrm{Proj}_{S_2}f_2^1(x,0,\kappa), \;\; g_3^1(x,0,\kappa) = \mathrm{Proj}_{S_3}\tilde{f}_3^1(x,0,0) + \mathcal{O}(\kappa^2|x|).$$

Here

$$S_2 = \mathrm{Span}\left\{\begin{pmatrix}\kappa x_1 \\ 0\end{pmatrix}, \begin{pmatrix}0 \\ \kappa x_2\end{pmatrix}\right\}, \;\; S_3 = \mathrm{Span}\left\{\begin{pmatrix}x_1^2x_2 \\ 0\end{pmatrix}, \begin{pmatrix}0 \\ x_1x_2^2\end{pmatrix}\right\}.$$

Let us compute $g_2^1(x,0,\kappa)$ first. Since

$$\frac{1}{2}f_2^1(x,0,\kappa) = \begin{pmatrix} a_1\kappa x_1 + a_2\kappa x_2 + +a_{20}x_1^2 + a_{11}x_1x_2 + a_{02}x_2^2 \\ \bar{a}_2\kappa x_1 + \bar{a}_1\kappa x_2 + \bar{a}_{02}x_1^2 + a_{11}x_1x_2 + \bar{a}_{20}x_2^2 \end{pmatrix},$$

where

$$\begin{aligned} &a_1 = \\ &[e^{-i\tau_0\omega}(-\bar{\delta}e^{i\tau_0\omega}\left(\mu\left(\epsilon^2(\beta\rho+\mu)+\mu^3+2\mu^2\epsilon\right)+\gamma^2(\mu+\epsilon)^2+2\gamma\mu(\mu+\epsilon)^2\right) \\ &+\bar{\nu}e^{i\tau_0\omega}(\mu\left(\epsilon^2(\beta\rho+\mu(-\rho)-2\mu\sigma+\mu)+\mu^3-\sigma\epsilon^3-\mu^2\epsilon(\rho+\sigma-2)\right) \\ &+\gamma^2(\mu+\epsilon)^2-\gamma(\mu+\epsilon)\left(-2\mu^2+\sigma\epsilon^2+\mu\epsilon(\rho+\sigma-2)\right))-\epsilon(\gamma+\mu)(\mu+\epsilon) \\ &\left(\gamma+\mu e^{i\tau_0\omega}\right))][\epsilon\bar{D}(\gamma+\mu)(\mu+\epsilon)], \\ &a_2 = \\ &[-\bar{\delta}\left(\beta\mu\epsilon^2\bar{\rho}+(\gamma+\mu)^2(\mu+\epsilon)^2\right)+\bar{\nu}((\gamma+\mu)(\mu+\epsilon)^2\left(-\epsilon\bar{\sigma}+\gamma+\mu\right) \\ &-\mu\epsilon\bar{\rho}((\gamma+\mu)(\mu+\epsilon)-\beta\epsilon))-\epsilon(\gamma+\mu)(\mu+\epsilon)\left(\mu+\gamma e^{i\tau_0\omega}\right)] \\ &[\epsilon\bar{D}(\gamma+\mu)(\mu+\epsilon)], \\ &a_{20} = \frac{\beta\rho\tau_0\left(\bar{\nu}-\bar{\delta}\right)}{\bar{D}}, a_{11} = -\frac{\beta\tau_0\left(\bar{\rho}+\rho\right)\left(\bar{\delta}-\bar{\nu}\right)}{\bar{D}}, a_{02} = \frac{\beta\tau_0\bar{\rho}\left(\bar{\nu}-\bar{\delta}\right)}{\bar{D}}, \end{aligned}$$

we obtain

$$\frac{1}{2}g_2^1(x,0,\mu) \;=\; \frac{1}{2}\mathrm{Proj}_{S_2}f_2^1(x,0,\mu) = \begin{pmatrix} a_1\mu x_1 \\ \bar{a}_1\mu x_2 \end{pmatrix}.$$

Next we compute $\frac{1}{3!}g_3^1(x,0,\mu)$. Note that

$$\begin{aligned}\frac{1}{3!}g_3^1(x,0,\mu) \;=\; & \frac{1}{3!}\mathrm{Proj}_{S_3}f_3^1(x,0,0) + \frac{1}{4}\mathrm{Proj}_{S_3}[(D_xf_2^1)(x,0,0)U_2^1(x,0) \\ & +(D_yf_2^1)(x,0,0)U_2^2(x,0)] + \mathcal{O}(\mu^2|x|).\end{aligned}$$

Step 1. Compute $\frac{1}{3!}\mathrm{Proj}_{S_3}f_3^1(x,0,0)$. Since $f_3^1(x,0,0)=0$, we have $a_{21}=0$.

Step 2. Compute $\mathrm{Proj}_{S_3}[D_xf_2^1(x,0,0)U_2^1(x,0)]$. The elements of the canonical basis of $V_2^2(\mathbb{C}^2)$ are

$$\begin{pmatrix} x_1^2 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} x_2^2 \\ 0 \end{pmatrix}, \begin{pmatrix} \mu x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} \mu x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} \mu^2 \\ 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 \\ x_1^2 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1x_2 \end{pmatrix}, \begin{pmatrix} 0 \\ x_2^2 \end{pmatrix}, \begin{pmatrix} 0 \\ \mu x_1 \end{pmatrix}, \begin{pmatrix} 0 \\ \mu x_2 \end{pmatrix}, \begin{pmatrix} 0 \\ \mu^2 \end{pmatrix},$$

whose images under $\frac{1}{i\omega_+}M_2^1$ are, respectively

$$\begin{pmatrix} x_1^2 \\ 0 \end{pmatrix}, -\begin{pmatrix} x_1x_2 \\ 0 \end{pmatrix}, -3\begin{pmatrix} x_2^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, -2\begin{pmatrix} \mu x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} \mu^2 \\ 0 \end{pmatrix},$$
$$3\begin{pmatrix} 0 \\ x_1^2 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1x_2 \end{pmatrix}, -\begin{pmatrix} 0 \\ x_2^2 \end{pmatrix}, 2\begin{pmatrix} 0 \\ \mu x_1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \mu^2 \end{pmatrix}.$$

Hence

$$U_2^1(x,0) = \frac{1}{i\omega}\begin{pmatrix} a_{20}x_1^2 - a_{11}x_1x_2 - \frac{1}{3}a_{02}x_2^2 \\ \frac{1}{3}\bar{a}_{02}x_1^2 + \bar{a}_{11}x_1x_2 - \bar{a}_{20}x_2^2 \end{pmatrix},$$

and

$$\frac{1}{2}\mathrm{Proj}_{S_3}[D_xf_2^1(x,0,0)U_2^1(x,0)] = \begin{pmatrix} C_1x_1^2x_2 \\ \bar{C}_1x_1x_2^2 \end{pmatrix}$$

where

$$C_1 = \frac{2i\beta^2\tau_0^2\left(\bar{\delta}-\bar{\nu}\right)\left(3d\rho\left(\bar{\rho}+\rho\right)\left(\bar{\delta}-\bar{\nu}\right) - \bar{d}\left(8\rho\bar{\rho}+3\bar{\rho}^2+3\rho^2\right)\left(\delta-\nu\right)\right)}{3D\omega\bar{D}^2}.$$

Step 3. Compute $\mathrm{Proj}_{S_3}[(D_yf_2^1)(x,0,0)U_2^2(x,0)]$, where $U_2^2(x,0)$ is a homogeneous polynomial of order 2 in (x_1,x_2) with coefficient in Q^1. Let

$$h(x) = U_2^2(x,0) = h_{20}x_1^2 + h_{11}x_1x_2 + h_{02}x_2^2.$$

The coefficients $h_{jk} = (h_{jk}^{(1)}, h_{jk}^{(2)}, h_{jk}^{(3)})^T$ are determined by $M_2^2h(x) = f_2^2(x,0,0)$ or

$$D_xh(x)Bx - A_{Q^1}(h(x)) = (I-\pi)X_0F_2(\Phi x,0)$$

which is equivalent to

$$\begin{aligned}\dot{h}(x) - D_xh(x)Bx &= \Phi\Psi(0)F_2(\Phi x,0),\\ \dot{h}(x)(0) - \mathcal{L}h(x) &= F_2(\Phi x,0),\end{aligned}$$

where $\dot{h}$ denotes the derivative of $h(x)(\theta)$ with respect to θ. Note that

$$F_2(\Phi x,0) = A_{20}x_1^2 + A_{11}x_1x_2 + A_{02}x_2^2$$

where

$$A_{20}=(0,2\beta\rho\tau_0,0,0)^T,\ A_{11}=(0,2\beta\tau_0(\rho+\bar{\rho},0,0)^T,\ A_{02}=(0,2\beta\bar{\rho}\tau_0,0,0)^T.$$

Comparing the coefficients of x_1^2, x_1x_2, x_2^2 of these equations, it is not hard to verify that $\bar{h}_{02}=h_{20}, \bar{h}_{11}=h_{11}$ and that h_{20}, h_{11} satisfy the following equations

$$\begin{cases} \dot{h}_{20}-2i\omega h_{20}=\Phi\Psi(0)A_{20}, \\ \dot{h}_{20}(0)-\mathcal{L}h_{20}=A_{20}, \end{cases} \tag{4.6}$$

and

$$\begin{cases} \dot{h}_{11}=\Phi\Psi(0)A_{11}, \\ \dot{h}_{11}(0)-\mathcal{L}h_{11}=A_{11}, \end{cases} \tag{4.7}$$

respectively. From this, we deduce

$$\mathrm{Proj}_{S_3}[(D_yf_2^1)h](x,0,0)=\begin{pmatrix} C_2x_1^2x_2 \\ \bar{C}_2x_1x_2^2 \end{pmatrix},$$

where

$$C_2 \quad = \quad -\frac{\beta\tau_0(\bar{\delta}-\bar{\nu})\left(h_{20}^{(3)}(0)\bar{\rho}+h_{11}^{(3)}(0)\rho+h_{11}^{(1)}(0)+h_{20}^{(1)}(0)\right)}{\bar{D}},$$

where h_{20}, h_{11} are determined by system (4.6) and system (4.7). After long but basic calculations, we obtain

$$\begin{aligned}
&h_{11}^{(1)}(0)= \\
&[2\beta\left(\bar{\rho}+\rho\right)(\gamma+\mu)(\mu+\epsilon)(\epsilon\left(D\sigma\left(\bar{\delta}-\bar{\nu}\right)+\bar{D}\left(\bar{\sigma}(\delta-\nu)+D\right)\right) \\
&-e^{-i\tau_0\omega}(\mu+\epsilon)\left(\bar{D}(\delta-\nu)e^{i\tau_0\omega}\left(-1+\gamma\tau_0e^{i\tau_0\omega}\right)-D\left(\bar{\delta}-\bar{\nu}\right)\left(-\gamma\tau_0+e^{i\tau_0\omega}\right)\right))] \\
&/[|D|^2\mu\epsilon((\gamma+\mu)(\mu+\epsilon)-\beta\epsilon)],
\end{aligned}$$

$$\begin{aligned}
&h_{20}^{(1)}(0)= \\
&[2\beta\rho(\gamma+\mu)e^{-i\tau_0\omega}(\mu+\epsilon)(D\left(\bar{\delta}-\bar{\nu}\right)(\gamma^2\left(-1+e^{2i\tau_0\omega}\right)(\mu+\epsilon)(\mu+2i\omega+\epsilon) \\
&+\gamma(\mu\left(\mu^2\left(-1+e^{2i\tau_0\omega}\right)-2i\mu\omega\left(-e^{2i\tau_0\omega}+e^{3i\tau_0\omega}+1\right)+4\omega^2e^{3i\tau_0\omega}\right) \\
&+\epsilon^2\left(\mu\left(-1+e^{2i\tau_0\omega}\right)-2i\omega e^{i\tau_0\omega}\left(-\rho+(\rho+\sigma+1)e^{2i\tau_0\omega}\right)\right)+2\epsilon(\mu^2\left(-1+e^{2i\tau_0\omega}\right) \\
&-i\mu\omega\left((\rho+\sigma+2)e^{3i\tau_0\omega}+\rho\left(-e^{i\tau_0\omega}\right)-e^{2i\tau_0\omega}+1\right)+2\omega^2e^{i\tau_0\omega}\left(-\rho+e^{2i\tau_0\omega}\right))) \\
&-2i\omega e^{3i\tau_0\omega}(\mu^2(\mu+2i\omega)+\epsilon^2(\mu\sigma+\mu-2i\rho\omega)+\epsilon(\mu^2(\sigma+2) \\
&-2i\mu(2\rho-1)\omega+4\rho\omega^2)))+\bar{D}e^{i\tau_0\omega}(-2\omega\epsilon\bar{\rho}(\delta-\nu)(\gamma(i\mu\left(-1+e^{2i\tau_0\omega}\right)+2\omega \\
&+i\epsilon\left(-1+e^{2i\tau_0\omega}\right))+2\omega e^{2i\tau_0\omega}(2\mu+2i\omega+\epsilon))-i(2\omega\epsilon\bar{\sigma}(\gamma+\mu)(\delta-\nu)e^{2i\tau_0\omega}(\mu \\
&+\epsilon)+(\mu+2i\omega+\epsilon)(-i\gamma(\gamma+\mu)(\delta-\nu)e^{i\tau_0\omega}(\mu+\epsilon)+i\gamma(\gamma+\mu)(\delta-\nu)e^{3i\tau_0\omega}(\mu \\
&+\epsilon)+2\omega e^{2i\tau_0\omega}(\gamma(\delta-\nu)(\mu+\epsilon)+\mu(\delta-\nu)(\mu+\epsilon)+D\epsilon(\mu+2i\omega))+2\gamma D\omega\epsilon))))] \\
&/[|D|^2\omega\epsilon(i\beta\mu\epsilon(\mu+2i\omega+\epsilon)\left(\gamma+(\mu+2i\omega)e^{2i\tau_0\omega}\right)+(\gamma+\mu)(\mu+\epsilon)(2\omega \\
&-i\mu)e^{2i\tau_0\omega}\left(\gamma(\mu+\epsilon)+\mu^2-2i\mu\omega+4\omega^2+\epsilon(\mu-2i\omega)\right)-2\gamma\omega(\mu+2i\omega+\epsilon)))],
\end{aligned}$$

$$\begin{aligned}
&h_{11}^{(3)}(0)= \\
&[2\beta\epsilon\left(\bar{\rho}+\rho\right)e^{-i\tau_0\omega}(\bar{D}e^{i\tau_0\omega}(-\bar{\rho}(\delta-\nu)(\beta\epsilon-(\gamma+\mu)(\mu+\epsilon))+\beta\epsilon\bar{\sigma}(\nu-\delta)
\end{aligned}$$

$$-(\mu+\epsilon)\left(D(\gamma+\mu)-\beta(\delta-\nu)\left(-1+\gamma\tau_0 e^{i\tau_0\omega}\right)\right)-D\left(\bar{\delta}-\bar{\nu}\right)\left(-\beta\gamma\tau_0(\mu+\epsilon)\right.$$
$$+e^{i\tau_0\omega}(\beta(\mu+\epsilon(\rho+\sigma+1))+\rho(-\gamma-\mu)(\mu+\epsilon))))]/[|D|^2(\gamma+\mu)(\mu+\epsilon)((\gamma$$
$$+\mu)(\mu+\epsilon)-\beta\epsilon)],$$

$$h_{20}^{(3)}(0)=$$
$$2e^{-i\tau\omega}\beta\rho(e^{2i\tau\omega}\bar{d}(-4e^{i\tau\omega}\delta\omega^2\mu^3+4e^{i\tau\omega}\nu\omega^2\mu^3-2ie^{2i\tau\omega}\gamma\delta\omega\mu^3+2i\gamma\delta\omega\mu^3$$
$$+2iDe^{i\tau\omega}\epsilon\omega\mu^3+2ie^{2i\tau\omega}\gamma\nu\omega\mu^3-2i\gamma\nu\omega\mu^3-8ie^{i\tau\omega}\delta\omega^3\mu^2+8ie^{i\tau\omega}\nu\omega^3\mu^2$$
$$-4e^{i\tau\omega}\gamma\delta\omega^2\mu^2+4e^{2i\tau\omega}\gamma\delta\omega^2\mu^2-4\gamma\delta\omega^2\mu^2-4de^{i\tau\omega}\epsilon\omega^2\mu^2-8e^{i\tau\omega}\delta\epsilon\omega^2\mu^2$$
$$+4e^{i\tau\omega}\gamma\nu\omega^2\mu^2-4e^{2i\tau\omega}\gamma\nu\omega^2\mu^2+4\gamma\nu\omega^2\mu^2+8e^{i\tau\omega}\epsilon\nu\omega^2\mu^2-e^{2i\tau\omega}\beta\gamma\delta\epsilon\mu^2$$
$$+\beta\gamma\delta\epsilon\mu^2+e^{2i\tau\omega}\beta\gamma\epsilon\nu\mu^2-\beta\gamma\epsilon\nu\mu^2+2ide^{i\tau\omega}\epsilon^2\omega\mu^2-2ie^{2i\tau\omega}\gamma^2\delta\omega\mu^2$$
$$+2i\gamma^2\delta\omega\mu^2+2ide^{i\tau\omega}\gamma\epsilon\omega\mu^2+2ie^{i\tau\omega}\beta\delta\epsilon\omega\mu^2-4ie^{2i\tau\omega}\gamma\delta\epsilon\omega\mu^2+4i\gamma\delta\epsilon\omega\mu^2$$
$$+2ie^{2i\tau\omega}\gamma^2\nu\omega\mu^2-2i\gamma^2\nu\omega\mu^2-2ie^{i\tau\omega}\beta\epsilon\nu\omega\mu^2+4ie^{2i\tau\omega}\gamma\epsilon\nu\omega\mu^2-4i\gamma\epsilon\nu\omega\mu^2$$
$$-8ie^{i\tau\omega}\gamma\delta\omega^3\mu-8ie^{i\tau\omega}\delta\epsilon\omega^3\mu+8ie^{i\tau\omega}\gamma\nu\omega^3\mu+8ie^{i\tau\omega}\epsilon\nu\omega^3\mu-e^{2i\tau\omega}\beta\gamma\delta\epsilon^2\mu$$
$$+\beta\gamma\delta\epsilon^2\mu-4De^{i\tau\omega}\epsilon^2\omega^2\mu-4e^{i\tau\omega}\delta\epsilon^2\omega^2\mu+4e^{2i\tau\omega}\gamma^2\delta\omega^2\mu-4\gamma^2\delta\omega^2\mu$$
$$-4De^{i\tau\omega}\gamma\epsilon\omega^2\mu-4e^{i\tau\omega}\beta\delta\epsilon\omega^2\mu-8e^{i\tau\omega}\gamma\delta\epsilon\omega^2\mu+4e^{2i\tau\omega}\gamma\delta\epsilon\omega^2\mu-4\gamma\delta\epsilon\omega^2\mu$$
$$-4e^{2i\tau\omega}\gamma^2\nu\omega^2\mu+4\gamma^2\nu\omega^2\mu+4e^{i\tau\omega}\epsilon^2\nu\omega^2\mu+4e^{i\tau\omega}\beta\epsilon\nu\omega^2\mu+8e^{i\tau\omega}\gamma\epsilon\nu\omega^2\mu$$
$$-4e^{2i\tau\omega}\gamma\epsilon\nu\omega^2\mu+4\gamma\epsilon\nu\omega^2\mu+e^{2i\tau\omega}\beta\gamma\epsilon^2\nu\mu-\beta\gamma\epsilon^2\nu\mu+2ide^{i\tau\omega}\gamma\epsilon^2\omega\mu$$
$$+2ie^{i\tau\omega}\beta\delta\epsilon^2\omega\mu-2ie^{2i\tau\omega}\gamma\delta\epsilon^2\omega\mu+2i\gamma\delta\epsilon^2\omega\mu-4ie^{2i\tau\omega}\gamma^2\delta\epsilon\omega\mu+4i\gamma^2\delta\epsilon\omega\mu$$
$$-2ie^{2i\tau\omega}\beta\gamma\delta\epsilon\omega\mu+2i\beta\gamma\delta\epsilon\omega\mu-2ie^{i\tau\omega}\beta\epsilon^2\nu\omega\mu+2ie^{2i\tau\omega}\gamma\epsilon^2\nu\omega\mu-2i\gamma\epsilon^2\nu\omega\mu$$
$$+4ie^{2i\tau\omega}\gamma^2\epsilon\nu\omega\mu-4i\gamma^2\epsilon\nu\omega\mu+2ie^{2i\tau\omega}\beta\gamma\epsilon\nu\omega\mu-2i\beta\gamma\epsilon\nu\omega\mu-2ie^{i\tau\omega}\epsilon((\gamma$$
$$+\mu)(\epsilon+\mu)-\beta\epsilon)(\delta-\nu)\omega\bar{\rho}\mu-8ie^{i\tau\omega}\gamma\delta\epsilon\omega^3+8ie^{i\tau\omega}\gamma\epsilon\nu\omega^3-4de^{i\tau\omega}\gamma\epsilon^2\omega^2$$
$$-4e^{i\tau\omega}\gamma\delta\epsilon^2\omega^2+4e^{2i\tau\omega}\gamma^2\delta\epsilon\omega^2-4\gamma^2\delta\epsilon\omega^2+4e^{i\tau\omega}\gamma\epsilon^2\nu\omega^2-4e^{2i\tau\omega}\gamma^2\epsilon\nu\omega^2$$
$$+4\gamma^2\epsilon\nu\omega^2-2ie^{2i\tau\omega}\gamma^2\delta\epsilon^2\omega+2i\gamma^2\delta\epsilon^2\omega+2ie^{2i\tau\omega}\gamma^2\epsilon^2\nu\omega-2i\gamma^2\epsilon^2\nu\omega$$
$$-2e^{i\tau\omega}\epsilon(\delta-\nu)\omega(2(\gamma+\mu)(\epsilon+\mu)\omega-i\beta\epsilon\mu)\bar{\sigma})-D(\beta\epsilon\mu(\left(-1+e^{2i\tau\omega}\right)\gamma(\epsilon$$
$$+\mu+2i\omega)-2ie^{3i\tau\omega}(\mu+\epsilon(\rho+\sigma+1)+2i\omega)\omega)+2(\gamma+\mu)(\epsilon+\mu)\omega(i(-1$$
$$+e^{2i\tau\omega})\gamma(\epsilon+\mu+2i\omega)+e^{3i\tau\omega}(2(\mu+2i\omega)\omega+\epsilon(i\mu\rho+2(\sigma+1)\omega))))(\bar{\delta}$$
$$-\bar{\nu}))]/[|D|^2\omega(i\beta\epsilon\mu(\gamma+e^{2i\tau\omega}(\mu+2i\omega))(\epsilon+\mu+2i\omega)+(\gamma+\mu)(\epsilon+\mu)(e^{2i\tau\omega}(2\omega$$
$$-i\mu)\left(\mu^2-2i\omega\mu+4\omega^2+\gamma(\epsilon+\mu)+\epsilon(\mu-2i\omega)\right)-2\gamma(\epsilon+\mu+2i\omega)\omega))].$$

Collecting the results above, we obtain

$$\frac{1}{3!}g_3^1(x,0,\kappa) = \begin{pmatrix} b_{21}x_1^2x_2 \\ \bar{b}_{21}x_1x_2^2 \end{pmatrix} + \mathcal{O}(\kappa^2|x|),$$

where $b_{21}=a_{21}+\frac{1}{4}(C_1+C_2)$. Let $x_1=w_1+iw_2, x_2=w_1-iw_2$ and $w_1=r\cos\zeta, w_2=r\sin\zeta$. Then (4.5) can be further written as

$$\begin{cases} \frac{dr}{dt} = K_1\kappa r+K_2r^3+\text{h.o.t.}, \\ \frac{d\zeta}{dt} = \tau_0\omega+\text{h.o.t.}, \end{cases}$$

where $K_1=\mathrm{Re}[a_1]$ and $K_2=\mathrm{Re}[b_{21}]$. Hence the first Lyapunov coefficient is $l_1(\kappa)=K_2+\mathcal{O}(\kappa)$, see [25, 26].

Theorem 4.1. *Let K_1 and K_2 be given above. Then*

(a) *the bifurcating periodic solution is stable if $K_2<0$ and unstable if $K_2>0$;*
(b) *the Hopf bifurcation is supercritical if $K_1K_2<0$ and subcritical if $K_1K_2>0$.*

5. Numerical simulations

In this section, we give an example to verify the results we obtained in Sections 3 and 4. Let $\mu = 0.03,\ \beta = 1,\ \epsilon = 0.5,\ \gamma = 0.2$. Then $R_0 = 4.10172 > 1$, and $P^* = (0.2438, 0.0428, 0.093)$. Then we have

$$a = 0.59,\ b = -0.0892,\ c = 0.008163,\ d = 0.2,\ e = 0.1,\ f = 0.00318$$

and

$$p = 0.0.4865,\ q = -0.0104037,\ r = 0.0000565222.$$

It is easy to see that $\Delta = p^2 - 3q = 0.267893 > 0$ and $\bar{z} = 0.0103614 > 0$ and hence $h(\bar{z}) = 2.06762 * 10^{-6} > 0, \tau_0 = 4.54569065648764 25$.

If we take $\tau = 4 < \tau_0$, according to (b) of Theorem 3.2, the equilibrium point P^* is asymptotically stable and solutions converge to P^*. See Figure 1 for stability of P^*.

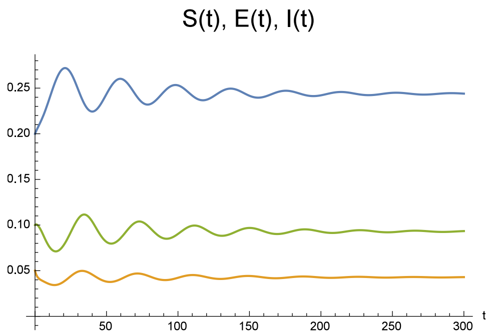
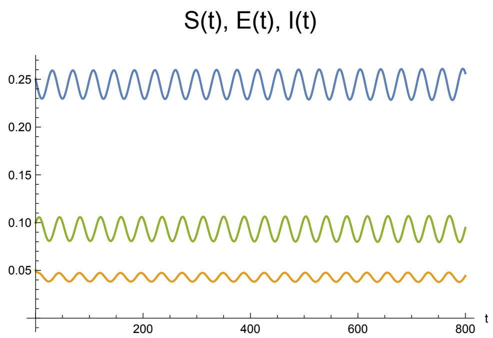


Figure 1: $\tau = 4$. Solution converges to P^*.

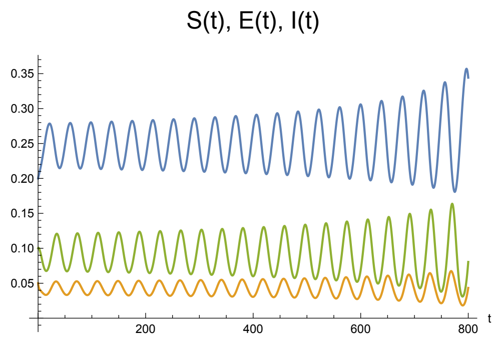
If we take $\tau = 4.5457906564876425 = \tau_0$, according to (b) of Theorem 3.2, then the equilibrium point P^* is unstable and there is a limit cycle bifurcating from it. Moreover using the algorithm in Section 4, we have

$$a_1 = 0.00575658 + 0.246617i, b_{21} = 47.9 - 124.77i$$

from which we can obtain that $K_1 = 0.00575658 > 0, K_2 = 47.9 > 0$ in Theorem 4.1. Therefore the bifurcating limit cycle is unstable and subcritical. See Figure 2 for the periodic solution and the instability of it. Figure 3 shows a three dimensional unstable periodic solution.



(a) The existence of periodic solution.



(b) The instability of the periodic solution.

Figure 2: $\tau = 4.5457906564876425$, The existence of periodic solution and its instability.

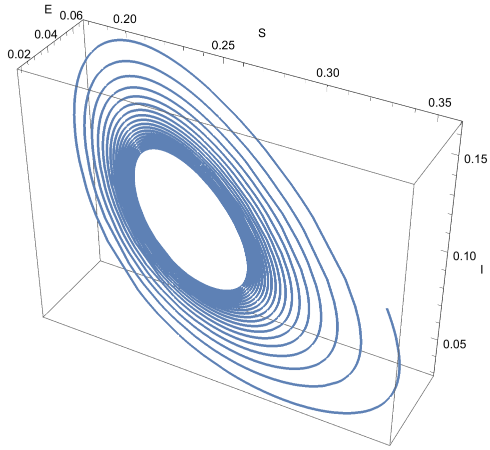


Figure 3: $\tau = 4.5457906564876425$. The instability of the periodic solution.

6. Discussion

In this paper, we first studied the distribution of zeros of a degree three transcendental equation in the form of

$$\lambda^3 + a\lambda^2 + b\lambda + c + (a_1\lambda^2 + e\lambda + c_1)e^{-\lambda\tau} = 0.$$

The conditions are established under which the the equation may have pure imaginary roots. Then we applied these results to an SEIR model with a time delay. We show that indeed the introduction of a time delay may or may not change the dynamics of the system totally depending upon the regions where the system parameters lie in. Using the same basic reproduction number R_0 derived in [7], we obtained that if $R_0 > 1$, the delay system has two equilibria: disease free equilibrium $P_0 = (1, 0, 0)$ and the endemic equilibrium $P^* = (S^*, E^*, I^*)$ where $S^*, E^*, I^* > 0$. P_0 is always unstable. We show that under some conditions the endemic equilibrium P^* is locally asymptotically stable for all delays. We found the parameter regions in which P^* will only locally asymptotically stable for the delay τ being less than a critical value $\tau_0 > 0$ and unstable if $\tau > \tau_0$. A Hopf bifurcation occurs and periodic solutions bifurcate from P^* as τ passes through the critical value τ_0. By finding the normal form using the center manifold theory we are able to determine the direction and the stability of the periodic solution bifurcated from the Hopf bifurcation.

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