Advances in the Theory of Nonlinear Analysis and its Applications **2** (2018) No. 3, 138–145.
https://doi.org/10.31197/atnaa.419517
Available online at www.atnaa.org
Research Article

Advances in the Theory of Nonlinear Analysis and its Applications

ISSN: 2587-2648

Peer-Reviewed Scientific Journal

Boundary Value Problems For Caputo-Hadamard Fractional Differential Equations

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Abstract

In this paper, we investigate the existence of solutions of a boundary value problem for Caputo-Hadamard fractional differential equations. Our analysis relies on classical fixed point theorems. Examples are given to illustrate our theoretical results.

Keywords: Fractional differential equation; Hadamard fractional derivative; Caputo-Hadamard fractional derivative; Hadamard fractional integral; fixed point.
2010 MSC: 26A33, 34A60

1. Introduction

The theory of fractional differential equations has received much attention over the past years and has become an important field of investigation due to its extensive applications in numerous branches of physics, chemistry, aerodynamics, electrodynamics of a complex medium, polymer rheology, etc. Fractional differential equations also serve as an excellent tool for the description of hereditary properties of various materials. As a consequence, the subject of fractional differential equations is gaining much importance and attention. For details, see [1, 4, 3, 5, 12, 17, 20, 28], and the references therein.

There has been a significant development in fractional differential equations in recent years; see the monographs of Kilbas *et al.* [22], Miller and Ross [25] and Samko *et al.* [27] and the papers of Delbosco and Rodino [13], Diethelm *et al.* [14], [15], [16], El-Sayed [17], Kilbas and Marzan [21], Mainardi [24] and Podlubny *et al.* [26].

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Received May 10, 2018, Accepted: August 14, 2018, Online: August 16, 2018.

There are several definitions of fractional derivatives, such as the definitions of Riemann-Liouville (1832), Riemann(1849), Caputo(1997), Grunwald-Letnikov(1867) and Hadamard(1891,[19]). The Caputo-Hadamard derivative is a new approach obtained from the Hadamard derivative by changing the order of its differential and integral parts. Despite the different requirements on the function itself, the main difference between the Caputo-Hadamard fractional derivative and the Hadamard fractional derivative is that the Caputo-Hadamard derivative of a constant is zero [20]. And the most important advantage of Caputo-Hadamard derivative is that it provided a new definition through which the integer order initial conditions can be defined for fractional.

Benchohra, Hamani and Ntouyas [5] studied the boundary value problem,

$$^{c}D^{\alpha}y(t) = f(t, y(t)), \text{ for a.e. } t \in [0, T], \quad 0 < \alpha \leq 1,$$

$$ay(0) + by(T) = c,$$

where $^{c}D^{\alpha}$ is the Caputo fractional derivative, $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a given function, and a, b and c are real constants such that $a + b \neq 0$.

Motivated by the work above, in this paper, we concentrate on the following boundary value problem,

$$^{c}_{H}D^{\alpha}y(t) = f(t, y(t)), \text{ for a.e. } t \in J = [1, T], \quad 0 < \alpha \leq 1, \tag{1.1}$$

$$ay(1) + by(T) = c, \tag{1.2}$$

where $^{c}_{H}D^{\alpha}$ is the Caputo-Hadamard fractional derivative, $f : [1, T] \times \mathbb{R} \to \mathbb{R}$ is a given function, and a, b and c are real constants such that $a + b \neq 0$.

This paper is organized as follows. In Section 2, we recall briefly some basic definitions and preliminary facts which will be used throughout subsequent sections. In Section 3, we shall provide sufficient conditions ensuring the existence of solutions for the problem (1.1)-(1.2) via applications of classical fixed point theorems. Finally in Section 4, we give examples to illustrate the theory presented in the previous sections.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that will be used in the remainder of this paper.

Let $C([1, T], \mathbb{R})$ be the Banach space of all continuous functions from $[1, T]$ into $\mathbb{R}$ with the norm

$$\|y\|_{\infty} = \sup\{|y(t)| : 1 \leq t \leq T\}.$$

Let the space

$$AC^{n}_{\delta}([a, b], \mathbb{R}) = \{h : [a, b] \to \mathbb{R} : \delta^{n-1}h(t) \in AC([a, b], \mathbb{R})\},$$

where $\delta = t\frac{d}{dt}$ is the Hadamard derivative and $AC([a, b], \mathbb{R})$ is the space of absolutely continuous functions on $[a, b]$.

Definition 2.1. *([22]). The Hadamard fractional integral of order r for a function $h : [1, +\infty) \to \mathbb{R}$ is defined as*

$$_{H}I^{r}h(t) = \frac{1}{\Gamma(r)} \int_{1}^{t} \left(\log \frac{t}{s}\right)^{r-1} \frac{h(s)}{s} ds, \; r > 0,$$

where Γ is the Gamma function.

Definition 2.2. *([22]). For a function h given on the interval $[1,+\infty)$, and $n-1<r<n$, the Hadamard derivative of order r is defined by*

$$
\begin{aligned}
({}_HD^rh)(t) &= \frac{1}{\Gamma(n-r)}\left(\frac{d}{dt}\right)^n\int_1^t\left(\log\frac{t}{s}\right)^{n-r-1}\frac{h(s)}{s}ds,\\
&= \delta^n({}_HI^{n-r}h)(t),
\end{aligned}
$$

where $n=[r]+1$ and $[r]$ denotes the integer part of r, and $\log(\cdot)=\log_e(\cdot)$.

Definition 2.3. *([20])Let $r\geq 0$ and $n-1<r<n$, where $n=[r]+1$, and $h\in AC^n_\delta[1,+\infty)$. The Caputo-Hadamard fractional derivative of order r is defined by*

$$
\begin{aligned}
({}^c_HD^rh)(t) &= \frac{1}{\Gamma(n-r)}\int_1^t\left(\log\frac{t}{s}\right)^{n-r-1}\delta^nh(s)\frac{ds}{s}\\
&= {}_HI^{n-r}(\delta^nh)(t).
\end{aligned}
$$

Lemma 2.4. *([20]). Let $h\in AC^n_\delta[1,+\infty)$ and $r>0$. Then*

$$
{}_HI^r({}^c_HD^rh)(t)=h(t)-\sum_{i=0}^{n-1}\frac{\delta^iy(1)}{i!}(\log t)^i.
$$

The following result will be used in Subsection 3.3.

Theorem 2.5. *(Nonlinear alternative for single-valued maps) [18]*
Let E be a Banach space, C be a closed, convex subset of E, U be an open subset of C and $0\in U$. Suppose that $N:U\to C$ is a continuous, compact map (that is, $N(U)$ is a relatively compact subset of C). Then

(i) either N has a fixed point in U, or

(ii) there exist $x\in\partial U$ (the boundary of U in C) and $\lambda\in(0,1)$ for which $x=\lambda N(x)$.

3. Main Results

Let us start by defining what we meant by a solution of the problem (1.1)-(1.2).

Definition 3.1. *A function $y\in AC^1_\delta([1,T],\mathbb{R})$ is said to be a solution of (1.1)-(1.2) if y satisfies the equation ${}^c_HD^\alpha y(t)=f(t,y(t))$ on J, and the conditions (1.2).*

For the existence of solutions for the problem (1.1)-(1.2), we need the following auxiliary lemma.

Lemma 3.2. *Let $h:[1,+\infty)\to\mathbb{R}$ be a continuous function. A function y is a solution of the fractional integral equation*

$$
y(t)=\frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}h(s)\frac{d}{ds}-\frac{b}{\Gamma(\alpha)(a+b)}\int_1^T(\log\frac{T}{s})^{\alpha-1}h(s)\frac{d}{ds}+\frac{c}{(a+b)} \tag{3.1}
$$

if and only if y is a solution of the fractional boundary value problem,

$$
{}^c_HD^\alpha y(t)=h(t),\qquad 0<\alpha<1 \tag{3.2}
$$

$$
ay(1)+by(T)=c. \tag{3.3}
$$

Proof. Assume y satisfies (3.2). Then Lemma 2.4 implies that

$$y(t) =_H I^\alpha f(t) + c_1. \tag{3.4}$$

The boundary condition (3.3) implies that

$$ay(1) + by(T) = c_H I^\alpha f(T) + (a+b)y(1) = c,$$

so

$$c_1 = \frac{c - b\,_H I^\alpha f(T)}{(a+b)}.$$

Finally, we obtain the solution (3.1)

$$y(t) =_H I^\alpha f(t) - \frac{b}{(a+b)}\ _H I^\alpha f(T) + \frac{c}{(a+b)}.$$

Inversely, it is clear that if y satisfies equation (3.1), then equations (3.2)-(3.3) hold.

□

In the following subsections we prove existence, as well as existence and uniqueness results, for the boundary value problem (1.1)-(1.2) by using a variety of fixed point theorems.

3.1. Existence and uniqueness result via Banach's fixed point theorem

Theorem 3.3. *Assume the following hypothesis:*

(H1) There exists a constant $k > 0$ such that

$$|f(t,x) - f(t,y)| \le k|x-y| \quad \text{for a.e. } t \in J \text{ and each } x, y \in \mathbb{R}.$$

If

$$\left[1 + \frac{|b|}{|a+b|}\right] \frac{k(\log T)^\alpha}{\Gamma(\alpha+1)} < 1,$$

then the boundary value problem (1.1)-(1.2) has a unique solution on $[1,T]$,.

Proof. Transform the problem (1.1)-(1.2) into a fixed point problem for the operator N defined by

$$\begin{aligned} N(y(t)) &= \tfrac{1}{\Gamma(\alpha)} \int_1^t (\log \tfrac{t}{s})^{\alpha-1} f(s, y(s)) \tfrac{d}{ds} \\ &- \tfrac{b}{(a+b)\Gamma(\alpha)} \int_1^T (\log \tfrac{T}{s})^{\alpha-1} f(s, y(s)) \tfrac{d}{ds} + \tfrac{c}{(a+b)}. \end{aligned} \tag{3.5}$$

Applying the Banach contraction mapping principle, we shall show that N is a contraction.

Let $x, y \in AC^1_\delta([1,T], \mathbb{R})$. Then for each $t \in J$ we have

$$\begin{aligned} |(Nx)(t) - (Ny)(t)| &\le \tfrac{1}{\Gamma(\alpha)} \int_1^t (\log \tfrac{t}{s})^{\alpha-1} |f(s,x(s)) - f(s,y(s))| \tfrac{d}{ds} \\ &+ \tfrac{|b|}{\Gamma(\alpha)|a+b|} \int_1^T (\log \tfrac{T}{s})^{\alpha-1} |f(s,x(s)) - f(s,y(s))| \tfrac{d}{ds} \\ &\le \tfrac{1}{\Gamma(\alpha)} \int_1^t (\log \tfrac{t}{s})^{\alpha-1} |f(s,x(s)) - f(s,y(s))| \tfrac{d}{ds} \\ &+ \tfrac{|b|}{\Gamma(\alpha)|a+b|} \int_1^T (\log \tfrac{T}{s})^{\alpha-1} |f(s,x(s)) - f(s,y(s))| \tfrac{d}{ds} \\ &\le \left[1 + \tfrac{|b|}{|a+b|}\right] \tfrac{k(\log T)^\alpha}{\Gamma(\alpha+1)} \|x-y\|_\infty. \end{aligned}$$

Thus

$$\|N(x) - N(y)\|_\infty \le \left[1 + \frac{|b|}{|a+b|}\right] \frac{k(\log T)^\alpha}{\Gamma(\alpha+1)} \|x-y\|_\infty.$$

We deduce that N has a unique fixed point which is the unique solution of the problem (1.1)-(1.2).

□

3.2. Existence result via Schaefer's fixed point theorem

Theorem 3.4. *Assume the hypotheses:*

(H2) The function $f : [1,T] \times \mathbb{R} \to \mathbb{R}$ is continuous,

(H3) There exists a constant $M > 0$ such that

$$|f(t,u)| \le M \quad \text{for a.e. } t \in J \text{ and each } u \in \mathbb{R}.$$

Then the boundary value problem (1.1)-(1.2) has at least one solution on $[1,T]$.

Proof. We shall use Schaefer's fixed point theorem to prove that N defined by (3.5) has a fixed point. The proof will be given in several steps.

Step 1: *N is continuous.*

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $AC^1_\delta(J,\mathbb{R})$. Then, for each $t \in J$,

$$\begin{array}{rcl} |(Ny_n)(t) - (Ny)(t)| & \le & \frac{1}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{s})^{\alpha-1} |f(s,y_n(s)) - f(s,y(s))| \frac{d}{ds} \\ & + & \frac{|b|}{\Gamma(\alpha)|a+b|} \int_1^T (\log \frac{T}{s})^{\alpha-1} |f(s,y_n(s)) - f(s,y(s))| \frac{d}{ds} \\ & \le & [1 + \frac{|b|}{\Gamma(\alpha)|a+b|}] \frac{(\log T)^\alpha}{\Gamma(\alpha+1)} \|f(\cdot,y_n(\cdot)) - f(\cdot,y(\cdot))\|_\infty. \end{array}$$

Since f is continuous, we have $\|N(y_n) - N(y)\|_\infty \to 0$ as $n \to \infty$.

Step 2: *N maps bounded sets into bounded sets in $C([1,T],\mathbb{R})$.*
Indeed, it is enough to show that for any $\mu^* > 0$, there exists a positive constant l such that for each $y \in B_{\mu^*} := \{y \in C([1,T],\mathbb{R}) : \|y\|_\infty \le \mu^*\}$, we have $\|N(y)\|_\infty < l$. In fact, we have

$$\begin{array}{rcl} |N(y)(t)| & \le & \frac{1}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{s})^{\alpha-1} |f(s,y(s))| \frac{d}{ds} \\ & + & \frac{|b|}{\Gamma(\alpha)|a+b|} \int_1^T (\log \frac{T}{s})^{\alpha-1} |f(s,y(s))| \frac{d}{ds} + \frac{|c|}{|a+b|}. \end{array}$$

Thus

$$\|N(y)\|_\infty \quad \le \quad \frac{M(\log T)^\alpha}{\Gamma(\alpha+1)} [1 + \frac{|b|}{|a+b|}] + \frac{|c|}{|a+b|} := l.$$

Step 3: *N maps bounded sets into equicontinuous sets of $C([1,T],\mathbb{R})$.*

Let $t_1, t_2 \in J$, $t_1 < t_2$, B_{μ^*} be a bounded set of $C([1,T],\mathbb{R})$ as in Step 2, and let $y \in B_{\mu^*}$. Then

$$\begin{array}{rcl} |(Ny)(t_2) - (Ny)(t_1)| & \le & \frac{1}{\Gamma(\alpha)} \int_1^{t_1} \left[\left(\log \frac{t_2}{s}\right)^{\alpha-1} - \left(\log \frac{t_1}{s}\right)^{\alpha-1} \right] |f(s,y(s))| \frac{ds}{s} \\ & + & \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{\alpha-1} |f(s,y(s))| \frac{ds}{s} \\ & \le & \frac{M}{\Gamma(\alpha+1)} \left[(\log t_2)^\alpha - (\log t_1)^\alpha \right]. \end{array}$$

As $t_1 \longrightarrow t_2$, the right-hand side of the above inequality tends to zero. As consequence of Step1 to Step 3, together with the Arzela-Ascoli theorem, we can conclude that N is continuous and completely continuous.

Step 4: *Apriori bounds.*
Now it remains to show that the set

$$\epsilon := \{y \in C(J,\mathbb{R}) : y = \mu N(y) \text{ for some } 0 < \mu < 1\}$$

is bounded. For such a $y \in \epsilon$,

$$\begin{array}{rcl} y(t) & = & \mu \Big[\frac{1}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{s})^{\alpha-1} f(s,y(s)) \frac{d}{ds} - \frac{b}{\Gamma(\alpha)(a+b)} \int_1^T (\log \frac{T}{s})^{\alpha-1} f(s,y(s)) \frac{d}{ds} \\ & + & \frac{c}{(a+b)} \Big]. \end{array}$$

For $\mu \in [0,1]$, let y be such that for each $t \in [1,T]$

$$\begin{array}{rcl} |(Ny)(t)| & \leq & \frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}|f(s,y(s))|\frac{d}{ds} \\ & + & \frac{|b|}{\Gamma(\alpha)|a+b|}\int_1^T(\log\frac{T}{s})^{\alpha-1}|f(s,y(s))|\frac{d}{ds} \\ & + & \frac{|c|}{|a+b|} \\ & \leq & M\frac{(\log T)^\alpha}{\Gamma(\alpha+1)}[1+\frac{|b|}{|a+b|}]+\frac{|c|}{|a+b|} \\ & := & R. \end{array}$$

Thus

$$\|N(y)\|_\infty \leq R.$$

This implies that the set ϵ is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that N has a fixed point which is a solution of the problem(1)-(2)

□

3.3. Existence result via the Leray-Schauder nonlinear alternative

Theorem 3.5. *Assume the following hypotheses:*

(H4) There exist $\phi_f \in L^1(J,\mathbb{R}_+)$ and $\psi : [0,\infty) \longrightarrow (0,\infty)$ continuous and nondecreasing such that

$$|f(t,u)| \leq \phi_f(t)\psi(|u|) \ \ \textit{for a.e. } t\in J \textit{ and each } u \in \mathbb{R}.$$

(H5) there exists a number $M^* > 0$ such that

$$\frac{M^*}{[1+\frac{|b|}{|a+b|}]\psi(M^*)\ {}_HI^\alpha\phi_f(T)+\frac{|c|}{|a+b|}} > 1.$$

Then the boundary value problem (1.1)-(1.2) has at least one solution on $[1,T]$.

Proof. We shall use the Leray-Schauder theorem to prove that N defined by (3.5) has a fixed point. As shown in Theorem 3.4, we see that the operator N is continuous, uniformly bounded, and maps bounded sets into equicontinuous sets. So by the Arzela-Ascoli theorem N is completely continuous.

Let y be such that for each $t \in [1,T]$

$$\begin{array}{rcl} |y(t)| & \leq & \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\phi_f(s)\psi(|y|)\frac{ds}{s} \\ & + & \frac{|b|}{\Gamma(\alpha)(|a+b|)}\int_1^T(\log\frac{T}{s})^{\alpha-1}\phi_f(s)\psi(|y|)\frac{d}{ds}+\frac{|c|}{|a+b|} \\ & \leq & [1+\frac{|b|}{|a+b|}]\psi(\|y\|_\infty)I^\alpha\phi_f(T)+\frac{|c|}{|a+b|}. \end{array}$$

Thus

$$\frac{\|y\|_\infty}{[1+\frac{|b|}{|a+b|}]\psi(\|y\|_\infty)I^\alpha\phi_f(T)+\frac{|c|}{|a+b|}} \leq 1.$$

Then by condition (H5), there exists M^* such that $\|y\|_\infty \neq M^*$.

Let $B_{M^*} := \{y \in C([1,T],\mathbb{R}) : \|y\|_\infty < M^*\}$. The operator N is completely continuous. From the choice of B_{M^*} there is no $y \in \partial B_{M^*}$ such that $y = \mu N(y)$, for some $\mu \in (0,1)$. As consequence of the nonlinear alternative of Leray-Schauder type, we deduce that N has a fixed point $y \in B_{M^*}$, which is a solution of the (1.1)-(1.2).

This completes the proof.

□

4. Examples

In this section, we present some examples to illustrate our results of the previous section.

4.1. Example 1

We consider the fractional boundary value problem,

$$ {}^c_H D^{\frac{1}{2}} y(t) = \frac{\cos^2 t}{(e^{-t}+3)^2 |y(t)|}, \text{ for a.e. } (t,y) \in ([1,e], \mathbb{R}_+), \tag{4.1} $$

$$ y(1) + y(e) = 0, \tag{4.2} $$

where $\alpha = \frac{1}{2}$, $T = e,\ a = b = 1$, $c = 0$, and $f(t,y) := \frac{\cos^2 t}{(e^{-t}+3)^2|y|}$.

We have

$$ |f(t,y)| = |\frac{\cos^2 t}{(e^{-t}+3)^2|y|}| \leq \frac{1}{9}. $$

Choosing $M = \frac{1}{9}$, then by Theorem 3.4, the problem (4.1)-(4.2) has a solution on $[1,e]$.

4.2. Example 2

We consider the fractional boundary value problem,

$$ {}^c_H D^{\frac{3}{2}} y(t) = (\log t)^4 \frac{y^2 e^{-y}}{2|y|(e^{-y}+2)^2}, \text{ for a.e. } (t,y) \in ([1,e], \mathbb{R}_+), \tag{4.3} $$

$$ y(1) + 2y(e) = -1, \tag{4.4} $$

where $\alpha = \frac{1}{2}$, $T = e,\ a = 1,\ b = 2,\ c = -1$, and

$$ f(t,y) = (\log t)^4 \frac{y^2 e^{-y}}{2|y|(e^{-y}+2)^2}. $$

Then,

$$ |f(t,y)| = |(\log t)^4 \frac{y^2 e^{-y}}{2|y|(e^{-y}+2)^2}| \leq (\log t)^4 \frac{|y|}{8}. $$

Choosing $\psi(|y|) = \frac{|y|}{8}$, $\phi_f(t) = (\log t)^4$, and ${}_H I^{\frac{1}{2}} \phi_f(e) = \frac{256}{63\sqrt{\pi}}$, if

$$ \frac{M^*}{[1+\frac{|b|}{|a+b|}]\psi(M^*)\ {}_H I^\alpha \phi_f(T) + \frac{|c|}{|a+b|}} $$

$$ = \frac{M^*}{\frac{5}{24} M^* \frac{256}{63\sqrt{\pi}} + \frac{1}{3}} $$

$$ = \frac{M^*}{M^* \frac{1280}{1512\sqrt{\pi}} + \frac{1}{3}} > 1, $$

then we can find $M^* \approx 0.638$ so that all the conditions of the Theorem 3.5 are satisfied. And so there exists at least one solution of problem (4.3)-(4.4) on $[1,e]$.

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