



# Blow up and asymptotic behavior for a system of viscoelastic wave equations of Kirchhoff type with a delay term

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## Abstract

The focus of the current paper is to investigate the initial boundary value problem for a system of viscoelastic wave equations of Kirchhoff type with a delay term in a bounded domain. At first, the energy decay rate is proved by Nakao's technique and expressed polynomially and exponentially depending on the parameter  $m$ . However and in the unstable set, for certain initial data, the blow up of solutions is obtained.

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## 1. Introduction

In this paper, we are concerned with the following problem:

$$u_{tt} - M(\|\nabla u\|_2^2)\Delta u + \int_0^t g(t-s)\Delta u(s)ds + \mu_1|u_t(x,t)|^{m-1}u_t(x,t) + \mu_2|u_t(x,t-\tau)|^{m-1}u_t(x,t-\tau) = |u|^{p-1}u, \quad (1.1)$$

$$u_t(x,t-\tau) = f_0(x,t-\tau), \quad x \in \Omega, \quad t \in (0,\tau), \quad (1.2)$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \quad (1.3)$$

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$$u(x, t) = 0, x \in \partial\Omega, t \geq 0. \tag{1.4}$$

Here  $\Omega$  is a bounded domain in  $\mathbb{R}^n, n \in \mathbb{N}^*$ , with a smooth boundary  $\partial\Omega$ .  $\tau > 0$  is a time delay term and  $\mu_1, \mu_2$  are positive real numbers and  $p > 1$ . The initial datum  $(u_0, u_1, f_0)$  belong to a suitable spaces.

To motivate our work, let us recall some results regarding wave equations of Kirchhoff type.

This type of problem without delay (i.e.,  $\mu_2 = 0$ ) has been considered by many authors during the past decades and many results have been obtained (see [9], [18], [22], [24] ) and the references therein.

The problem (1.1)-(1.4) without the viscoelastic term and delay (i.e.,  $g = 0, \mu_2 = 0$ ) has been extensively studied and decay and blow-up have been established. For example, the following equation

$$u_{tt} - M(\|\nabla u\|_2^2)\Delta u + g(u_t) = f(u), \tag{1.5}$$

has been considered by Matsuyama and Ikehata in [18], for  $g(u_t) = \delta|u_t|^p u_t$  and  $f(u) = \xi|u|^p u$ . The authors proved existence of the global solutions by using Faedo-Galerkin’s method and the decay of energy based on the method of Nakao [20]. Later, Ono [21] investigated equation (1.5) for  $M(s) = bs, f(u) = \xi|u|^p u$  and  $g(u_t) = -\Delta u_t$ . They showed that the solutions blow up in finite time with  $E(0) \leq 0$ . For  $M(s) = a + bs$  and  $g(u_t) = u_t$ , this model was considered by the same author in [22]. By applying the potential well method, he obtained the blow-up properties with positive initial energy  $E(0)$ .

Recently, Zeng et al. [24] have studied equation (1.5) for the case  $g(u_t) = u_t$  with initial condition and zero Dirichlet boundary condition. By using the concavity argument, they proved that the solutions to equation (1.5) blows up in finite time with arbitrarily high energy.

When  $g \neq 0$  and  $M$  is not a constant function, problems related to (1.5) have been treated by many authors. Wu and Tsai [23] considered the global existence, asymptotic behavior and blow-up properties for the following equation

$$u_{tt} - M(\|\nabla u\|_2^2)\Delta u + \int_0^t g(t-s)\Delta u(s)ds - \Delta u_t = f(u), \tag{1.6}$$

where  $(x, t) \in \Omega \times (0, \infty)$  and with the same initial and boundary conditions as that of problem (1.1)-(1.4). To obtain the decay result, they assumed that the nonnegative kernel  $g'(t) \leq -rg(t) \forall t \geq 0$  for some  $r > 0$ . Later, Wu [25], extended the result of [23] under a weaker condition on  $g$  (i.e  $g'(s) \leq 0$  for  $t \geq 0$ ).

In the present paper, we analyze the influence of the viscoelastic, damping and delay terms on the solutions to (1.1)-(1.4). Under suitable assumptions on the function  $g$ , the initial data and the parameters in the equations, we establish several results concerning asymptotic behavior and finite blow-up of solutions to (1.1)-(1.4) for both negative and positive initial energy.

The paper is organized as follows. In Section 2, we present the preliminaries and some lemmas. In Section 3, the decay property is discussed. Finally, in Section 4, the blow-up results of (1.1)-(1.4) are obtained on different cases of the sign of the initial energy  $E(0)$ .

## 2. Preliminary Results

In this section, we present some material for the proof of our result. We assume

(A<sub>0</sub>)  $M(s)$  is a nonnegative  $C^1$  function for  $s \geq 0$  satisfying

$$M(s) = m_0 + \alpha s^\gamma, m_0 > 0, \alpha \geq 0 \text{ and } \gamma \geq 0.$$

(A<sub>1</sub>)  $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is a bounded  $C^1$  function satisfying

$$g(s) \geq 0, m_0 - \int_0^\infty g(s)ds = l > 0, g'(t) \leq -rg(t), \forall t \geq 0, \text{ and } g'(s) \leq 0.$$

(A<sub>2</sub>) We assume that

$$\beta \overline{M}(s) - \left[ M(s) + \eta \int_0^t g(\nu) d\nu \right] s \geq m_1 s, \quad \forall s \geq 0, \tag{2.1}$$

where  $\overline{M}(s) = \int_0^s M(\nu) d\nu$ ,  $m_1 > 0$ ,  $\beta$  and  $\eta$  are positive constants that will be specified later.

**Lemma 2.1.** (Sobolev-Poincaré inequality) Let  $2 \leq m \leq \frac{2n}{n-2}$ . The inequality

$$\|u\|_m \leq c_s \|\nabla u\|_2 \quad \text{for } u \in H_0^1(\Omega),$$

holds with some positive constant  $c_s$ .

**Lemma 2.2.** ([14]) Let  $\phi(t)$  be a nonincreasing and nonnegative function on  $[0, T]$ ,  $T > 1$ , such that

$$\phi(t)^{1+r} \leq \omega_0(\phi(t) - \phi(t+1)), \quad \text{on } [0, T],$$

where  $\omega_0 > 1$  and  $r \geq 0$ . Then we have, for all  $t \in [0, T]$

(i) if  $r = 0$ , then

$$\phi(t) \leq \phi(0)e^{-\omega_1[t-1]^+},$$

(ii) if  $r > 0$ , then

$$\phi(t) \leq (\phi(0)^{-r} + \omega_0^{-1}r[t-1]^+)^{\frac{-1}{r}},$$

where  $\omega_1 = \ln\left(\frac{\omega_0}{\omega_0-1}\right)$  and  $[t-1]^+ = \max(t-1, 0)$ .

**Lemma 2.3.** ([26]) For any  $g \in C^1$  and  $\varphi \in H^1(0, T)$ , we have

$$\int_0^t \int_{\Omega} g(t-s)\varphi\varphi_t dx ds = -\frac{1}{2} \frac{d}{dt} \left( go\varphi(t) + \int_0^t g(s) ds \|\varphi\|_2^2 \right) - g(t)\|\varphi\|_2^2 + (g'o\varphi)(t),$$

where

$$(go\varphi)(t) = \int_0^t g(t-s) \int_{\Omega} |\varphi(s) - \varphi(t)|^2 dx ds.$$

**Lemma 2.4.** ([26]). For  $u \in H_0^1(\Omega)$ , we have

$$\int_{\Omega} \left( \int_0^t g(t-s)(u(t) - u(s)) ds \right)^2 dx \leq (1-l)c_s^2 (go\nabla u)(t), \tag{2.2}$$

where  $c_s^2$  is the Poincaré constant and  $l$  is given in (A<sub>1</sub>).

### 3. Global existence and asymptotic behavior

In order to prove the global existence result, we introduce the new variable  $z$  as in [12],

$$z(x, k, t) = u_t(x, t - \tau k), \quad x \in \Omega, \quad k \in (0, 1),$$

which implies that

$$\tau z_t(x, k, t) + z_k(x, k, t) = 0, \quad \text{in } \Omega \times (0, 1) \times (0, \infty).$$

Therefore, problem (1.1)-(1.4) can be transformed as follows

$$\begin{cases} u_{tt} - M(\|\nabla u\|_2^2)\Delta u + \int_0^t g(t-s)\Delta u(s)ds \\ + \mu_1|u_t(x, t)|^{m-1}u_t(x, t) + \mu_2|z(x, 1, t)|^{m-1}z(x, 1, t) \\ = |u|^{p-1}u, \quad \text{in } \Omega \times (0, \infty), \\ \tau z_t(x, k, t) + z_k(x, k, t) = 0, \quad \text{in } \Omega \times (0, 1) \times (0, \infty), \\ z(x, k, 0) = f_0(x, -\tau k), \quad x \in \Omega, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega, \\ u(x, t) = 0, \quad x \in \partial\Omega, \quad t \geq 0. \end{cases} \quad (3.1)$$

For any regular solution of (3.1), we define the energy as

$$\begin{aligned} E(t) &= \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t) + \frac{1}{2} \left( m_0 - \int_0^t g(s)ds \right) \|\nabla u\|_2^2 \\ &- \frac{1}{p+1}\|u\|_{p+1}^{p+1} + \frac{\alpha}{2(\gamma+1)}(\|\nabla u\|_2^2)^{\gamma+1} + \frac{\zeta}{2} \int_{\Omega} \int_0^1 z^{m+1}(x, k, s)dkdx, \end{aligned} \quad (3.2)$$

such that  $\zeta$

$$\tau \frac{\mu_2 m}{m+1} < \zeta < \tau \frac{\mu_1(m+1) - \mu_2}{m+1}. \quad (3.3)$$

**Lemma 3.1.** *Let  $(u, z)$  be the solution of (3.1), then the energy satisfies*

$$\begin{aligned} E'(t) &= \frac{1}{2}(g' \circ \nabla u)(t) - \left( \mu_1 - \frac{\zeta}{2\tau} - \frac{\mu_2}{m+1} \right) \|u_t(t)\|_{m+1}^{m+1} \\ &- \left( \frac{\zeta}{\tau} - \frac{\mu_2 m}{m+1} \right) \int_{\Omega} z^{m+1}(x, 1, t)dx - \frac{1}{2}g(t)\|\nabla u(t)\|_2^2 \leq 0. \end{aligned} \quad (3.4)$$

*Proof.* Multiplying the first equation in (3.1) by  $u_t$ , integrating over  $\Omega$  and using integration by parts, we get

$$\begin{aligned} &\frac{d}{dt} \left[ \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}m_0\|\nabla u(t)\|_2^2 + \frac{\alpha}{2(\gamma+1)}(\|\nabla u\|_2^2)^{\gamma+1} \right. \\ &\left. - \frac{1}{p+1}\|u\|_{p+1}^{p+1} \right] + \mu_1\|u_t(t)\|_{m+1}^{m+1} \\ &+ \int_{\Omega} \mu_2|z(x, 1, t)|^{m-1}z(x, 1, t)u_t(x, t)dx \\ &- \int_0^t g(t-s) \int_{\Omega} \nabla u(s)\nabla u_t(t)dxds = 0. \end{aligned} \quad (3.5)$$

Using Lemma 2.3 on the last term of the left hand side of (3.5), we find

$$\begin{aligned} & \frac{d}{dt} \left[ \frac{1}{2} \|u_t(t)\|_2^2 + \frac{1}{2} \left( m_0 - \int_0^t g(s) ds \right) \|\nabla u(t)\|_2^2 \right. \\ & \left. + \frac{1}{2} (g \circ \nabla u)(t) + \frac{\alpha}{2(\gamma + 1)} (\|\nabla u\|_2^2)^{\gamma+1} - \frac{1}{p+1} \|u\|_{p+1}^{p+1} \right] \\ & + \mu_1 \|u_t(t)\|_{m+1}^{m+1} + \mu_2 \int_{\Omega} |z(x, 1, t)|^{m-1} z(x, 1, t) u_t(x, t) dx \\ & + \frac{1}{2} g(t) \|\nabla u(t)\|_2^2 - \frac{1}{2} (g' \circ \nabla u)(t) = 0. \end{aligned} \tag{3.6}$$

Integrating (3.6) over  $(0, t)$  we arrive at

$$\begin{aligned} & \left[ \frac{1}{2} \|u_t(t)\|_2^2 + \frac{1}{2} \left( m_0 - \int_0^t g(s) ds \right) \|\nabla u(t)\|_2^2 + \frac{1}{2} (g \circ \nabla u)(t) \right. \\ & \left. + \frac{\alpha}{2(\gamma + 1)} (\|\nabla u\|_2^2)^{\gamma+1} - \frac{1}{p+1} \|u\|_{p+1}^{p+1} \right] + \int_0^t \mu_1 \|u_s(s)\|_{m+1}^{m+1} ds \\ & + \mu_2 \int_0^t \int_{\Omega} |z(x, 1, s)|^{m-1} z(x, 1, s) u_s(x, s) ds dx \\ & - \frac{1}{2} \int_0^t g(s) (g' \circ \nabla u)(s) ds + \frac{1}{2} \int_0^t g(s) \|\nabla u(s)\|_2^2 ds \\ & = \frac{1}{2} \|u^1\|_2^2 + \frac{1}{2} \|\nabla u^0\|_2^2. \end{aligned} \tag{3.7}$$

Multiplying the second equation in (3.1) by  $\zeta |z|^{m-1} z$  and integrating the result over  $\Omega \times (0, 1)$  we obtain

$$\begin{aligned} & \zeta \frac{d}{dt} \int_{\Omega} \int_0^1 |z(x, k, t)|^{m-1} z(x, k, t) dk dx = \\ & - \frac{\zeta}{\tau(m+1)} \int_{\Omega} \int_0^1 \frac{\partial}{\partial k} |z(x, k, t)|^{m+1} dk dx \\ & = - \frac{\zeta}{\tau(m+1)} \int_{\Omega} [|z(x, 1, t)|^{m+1} - |z(x, 0, t)|^{m+1}] dx \\ & = - \frac{\zeta}{\tau} \int_{\Omega} |z(x, 1, t)|^{m+1} dx + \frac{\zeta}{\tau} \|u_t(t)\|_{m+1}^{m+1}. \end{aligned} \tag{3.8}$$

Combining (3.7) and (3.8) together, we get

$$\begin{aligned} & E(t) + \left( \mu_1 - \frac{\zeta}{\tau} \right) \int_0^t \|u_s(s)\|_{m+1}^{m+1} ds + \frac{\zeta}{\tau} \int_0^t \int_{\Omega} |z(x, 1, s)|^{m+1} ds dx \\ & - \mu_2 \int_0^t \int_{\Omega} |z(x, 1, s)|^{m-1} z(x, 1, s) u_s(x, s) ds dx \\ & + \frac{1}{2} \int_0^t g(s) \|\nabla u(s)\|_2^2 ds - \frac{1}{2} \int_0^t (g' \circ \nabla u)(s) ds = E(0). \end{aligned} \tag{3.9}$$

Making use of the Young’s inequality on the fourth term of the left hand side of (3.9), we deduce that

$$\begin{aligned} & E(t) + \left( \mu_1 - \frac{\zeta}{\tau} - \frac{\mu_2}{m+1} \right) \int_0^t \|u_s(s)\|_{m+1}^{m+1} ds \\ & - \left( \frac{\zeta}{\tau} - \frac{\mu_2 m}{m+1} \right) \int_{\Omega} \int_0^t z^{m+1}(x, 1, s) ds dx \\ & + \frac{1}{2} \int_0^t g(s) \|\nabla u(s)\|_2^2 ds - \frac{1}{2} \int_0^t (g' \circ \nabla u)(s) ds = E(0). \end{aligned} \tag{3.10}$$

After deriving (3.10), we get the desired result. This completes the proof.  $\square$

*Remark 3.2.* Due to the condition (3.3) we have

$$c_1 = \left( \mu_1 - \frac{\zeta}{\tau} - \frac{\mu_2}{m+1} \right) > 0, \quad c_2 = \left( \frac{\zeta}{\tau} - \frac{\mu_2 m}{m+1} \right) > 0.$$

*Remark 3.3.* The conditions  $(A_2)$  is used to derive the blow up of solution to problem (3.1).

We show that when  $m \geq p$  the solution of the problem (3.1) is global if

$$\frac{n}{n-2} > m \geq p > 1, \quad n \geq 3; \quad (1 < p < m, \quad n \leq 2). \tag{3.11}$$

Now, we center our attention on the global existence of the solutions to the problem (3.1). In order to do so, as in [6], we define

$$I_1(t) = \left( m_0 - \int_0^t g(s) ds \right) \|\nabla u\|_2^2 + (g \circ \nabla u)(t) - \|u\|_{p+1}^{p+1} + \zeta \int_{\Omega} \int_0^1 z^{m+1}(x, k, s) dk dx, \tag{3.12}$$

$$I_2(t) = \left( m_0 - \int_0^t g(s) ds \right) \|\nabla u\|_2^2 + (g \circ \nabla u)(t) - \|u\|_{p+1}^{p+1} + \alpha (\|\nabla u\|_2^2)^{\gamma+1} + \zeta \int_{\Omega} \int_0^1 z^{m+1}(x, k, s) dk dx, \tag{3.13}$$

and

$$J(t) = \frac{1}{2}(g \circ \nabla u)(t) + \frac{1}{2} \left( m_0 - \int_0^t g(s) ds \right) \|\nabla u\|_2^2 - \frac{1}{p} \|u\|_p^p + \frac{\alpha}{2(\gamma+1)} (\|\nabla u\|_2^2)^{\gamma+1} + \frac{\zeta}{m+1} \int_{\Omega} \int_0^1 z^{m+1}(x, k, s) dk dx. \tag{3.14}$$

We observe that

$$E(t) = J(t) + \frac{1}{2} \|u_t\|_2^2, \tag{3.15}$$

and

$$I_2(t) = I_1(t) + \alpha (\|\nabla u\|_2^2)^{\gamma+1}. \tag{3.16}$$

**Lemma 3.4.** *Suppose that  $(A_0) - (A_1)$  and (3.11) hold. Let  $(u, z)$  be the solution of the problem (3.1). Assume further that  $I_1(0) > 0$  and*

$$\alpha = \left( \frac{2(p+1)}{p-1} E(0) \right)^{\frac{p-1}{2}} < 1. \tag{3.17}$$

Then  $I_1(t) > 0$  for all  $t \geq 0$ .

*Proof.* Since  $I(0) > 0$ , then there exists (by continuity of  $u(t)$ )  $T^* < T$  such that

$$I(t) \geq 0, \quad \text{for all } t \in [0, T^*]. \tag{3.18}$$

From (3.14) and (3.15) we get easily

$$\begin{aligned}
 J(t) &\geq \frac{p-1}{2(p+1)} \left[ \left( m_0 - \int_0^t g(s) ds \right) \|\nabla u\|_2^2 + (go\nabla u)(t) \right] \\
 &\quad + \frac{p-1}{2(p+1)} \left[ \zeta \int_0^1 \int_{\Omega} z^{m+1}(x, k, t) dk dx \right] + \frac{1}{p+1} I_1(t) \\
 &\geq \frac{p-1}{2(p+1)} \left[ \left( m_0 - \int_0^t g(s) ds \right) \|\nabla u\|_2^2 + (go\nabla u)(t) \right. \\
 &\quad \left. + \zeta \int_0^1 \int_{\Omega} z^{m+1}(x, k, t) dk dx \right] \geq \frac{p-1}{2(p+1)} l \|\nabla u\|_2^2.
 \end{aligned}
 \tag{3.19}$$

Thus by (3.10) and (3.19) we deduce  $\forall t \in [0, T^*]$

$$l \|\nabla u\|_2^2 \leq \left( 1 - \int_0^t g(s) ds \right) \|\nabla u\|_2^2 \leq \frac{2(p+1)}{(p-1)} E(t) \leq \frac{2(p+1)}{(p-1)} E(0).
 \tag{3.20}$$

Exploiting Lemma 2.1 and formula (3.17), we obtain

$$\begin{aligned}
 \|u\|_{p+1}^{p+1} &\leq c_s^{p+1} \|\nabla u\|_2^{p+1} \leq \frac{c_s^{p+1}}{l} \|\nabla u\|_2^{p-1} \|\nabla u\|_2^2 \\
 &\leq \frac{c_s^{p+1}}{l} \left( \frac{2(p+1)}{(p-1)l} E(0) \right)^{\frac{p-1}{2}} l \|\nabla u\|_2^2 \\
 &= \alpha l \|\nabla u\|_2^2 < \left( m_0 - \int_0^t g(s) ds \right) \|\nabla u\|_2^2, \quad \forall t \in [0, T^*].
 \end{aligned}
 \tag{3.21}$$

Whereupon

$$I_1(t) = \left( m_0 - \int_0^t g(s) ds \right) \|\nabla u\|_2^2 + (go\nabla u)(t) - \|u\|_{p+1}^{p+1} + \zeta \int_0^1 \int_{\Omega} z^{m+1}(x, k, t) dk dx > 0, \forall t \in [0, T^*].$$

Repeating this procedure and using the fact that

$$\lim_{t \rightarrow T^*} \frac{c_s^{p+1}}{l} \left( \frac{2(p+1)}{2l(p-1)} E(u(t)) \right)^{\frac{p-1}{2}} \leq \alpha < 1.$$

We can take  $T^* = T$ . This completes the proof. □

**Theorem 3.5.** Assume that the assumptions  $(A_0) - (A_1)$  are fulfilled. Let  $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ ,  $u_1 \in H_0^1(\Omega)$  and  $f_0 \in L^2(\Omega \times (0, 1))$  be given. Suppose that (3.11) and  $\mu_2 < \mu_1$  hold. Then the solution of the problem (3.1) is global and bounded in time. Furthermore, if

$$m_0 < \frac{2\theta l + l^2 + 4l}{4 + l},
 \tag{3.22}$$

then we have the following decay estimates:

(i) if  $m = 1$

$$\phi(t) \leq \phi(0) e^{-\omega_1 [t-1]^+},$$

where  $\omega_1 = \ln \left( \frac{\omega_0}{\omega_0 - 1} \right)$ , and  $[t - 1]^+ = \max(t - 1, 0)$ ,

(ii) if  $m > 1$

$$\phi(t) \leq (\phi(0)^{-m} + \tau_2 m [t - 1]^+)^{-\frac{1}{m}} \quad \text{for } t \geq 0,$$

with  $\tau_2 = c_{14}^{-1}(E(0))$ .

*Proof.* First, we prove that  $T = \infty$ . It is sufficient to show that  $l\|\nabla u\|_2^2$  is bounded independently of  $t$ . We have from (3.14) and (3.15)

$$E(0) \geq E(t) = \frac{1}{2}\|u_t\|_2^2 + J(t) \geq \frac{1}{2}\|u_t\|_2^2 + \left(\frac{p-1}{2(p+1)}\right)(l\|\nabla u\|_2^2 \geq (l\|\nabla u\|_2^2).$$

Therefore

$$(l\|\nabla u\|_2^2) \leq \xi E(0),$$

where  $\xi$  is a positive constant which depends only on  $p$ , thus we obtain the global existence result. This ends the proof. □

From now and on, we focus our attention to the decay rate of the solutions to problem (3.1). In order to do so, we will derive the decay rate of the energy function for problem (3.1) by Nakao’s method [14]. For this aim, we have to show that the energy function defined by (3.15) satisfies the hypotheses of Lemma 2.2. By integrating (3.4) over  $[t, t + 1]$ , we have

$$E(t) - E(t + 1) = [D(t)]^{m+1}, \tag{3.23}$$

where

$$\begin{aligned} [D(t)]^{m+1} &= c_1 \int_t^{t+1} \|u_t\|_{m+1}^{m+1} ds + c_2 \int_t^{t+1} \int_{\Omega} z^{m+1}(x, 1, s) dx ds \\ &+ \int_t^{t+1} \frac{1}{2}(g' \circ \nabla u)(s) ds - \int_t^{t+1} \frac{1}{2}g(s)\|\nabla u(s)\| ds. \end{aligned} \tag{3.24}$$

By virtue of (3.24) and Holder’s inequality, we observe that

$$\int_t^{t+1} |u_t|^2 dx dt + \int_t^{t+1} \int_{\Omega} |z(x, 1, s)|^2 dx ds \leq c(\Omega)[D(t)]^2, \tag{3.25}$$

where  $c(\Omega) = vol(\Omega)$ . Applying the mean value, there exist  $t_1 \in [t, t + \frac{1}{4}]$  and  $t_2 \in [t + \frac{3}{4}, t + 1]$  such that

$$\|u_t(t_i)\|_2^2 + \|z(x, 1, t_i)\|_2^2 \leq c(\Omega)D(t)^2, i = 1, 2. \tag{3.26}$$

Multiplying the first equation in (3.1) by  $u$  and integrating over  $\Omega \times [t_1, t_2]$ . Using integration by parts, Holder’s inequality, adding and subtracting the following term  $\int_{\Omega} \zeta z^{m+1}(x, k, t) dk dx$ , we obtain

$$\begin{aligned} \int_{t_1}^{t_2} I_2(t) dt &\leq \sum_{i=1}^2 \|u_t(t_i)\|_2 \|u(t_i)\|_2 + \int_{t_1}^{t_2} \|u_t\|_2^2 dt \\ &- \mu_1 \int_{t_1}^{t_2} \int_{\Omega} |u_t|^{m-1} |u_t| u dx dt + \int_{t_1}^{t_2} (g \circ \nabla u)(t) dt \\ &+ \int_{t_1}^{t_2} \int_{\Omega} \int_0^t g(t-s) \nabla u(t) (\nabla u(s) - \nabla u(t)) ds dx dt \\ &+ \int_{t_1}^{t_2} \int_0^1 \zeta \int_{\Omega} z^{m+1}(x, k, t) dk dx dt \\ &- \mu_2 \int_{\Omega} \int_{t_1}^{t_2} |z(x, 1, t)|^{m-1} z(x, 1, t) u dt dx. \end{aligned} \tag{3.27}$$



Since

$$\begin{aligned}
 & \int_{\Omega} \int_0^t g(t-s) \nabla u(t) (\nabla u(s) - \nabla u(t)) ds dx \\
 &= \frac{1}{2} \left[ \int_0^t g(t-s) (\|\nabla u(s)\|_2^2 + \|\nabla u(t)\|_2^2) ds \right. \\
 & \left. - \int_0^t g(t-s) (\|\nabla u(s)\|_2^2 + \|\nabla u(t)\|_2^2) ds \right] \\
 & - \int_{\Omega} \int_0^t g(s) |\nabla u(t)|^2 ds dx = -\frac{1}{2} \int_{\Omega} \int_0^t g(s) |\nabla u(s)|^2 ds dx \\
 & + \frac{1}{2} \int_0^t g(t-s) ds \|\nabla u(s)\|_2^2 ds - \frac{1}{2} (g \circ \nabla u)(t).
 \end{aligned} \tag{3.28}$$

Then (3.27) takes the form

$$\begin{aligned}
 \int_{t_1}^{t_2} I_2(t) dt &\leq \sum_{i=1}^2 \|u_t(t_i)\|_2 \|u(t_i)\|_2 + \int_{t_1}^{t_2} \|u_t\|_2^2 dt \\
 & + \mu_1 \int_{t_1}^{t_2} \int_{\Omega} |u_t|^{m-1} |u_t| u dx dt + \frac{1}{2} \int_{t_1}^{t_2} (g \circ \nabla u)(t) dt \\
 & - \frac{1}{2} \int_{\Omega} \int_0^t g(s) |\nabla u(s)|^2 ds dx + \frac{1}{2} \int_0^t g(t-s) ds \|\nabla u(s)\|_2^2 ds \\
 & + \int_{t_1}^{t_2} \int_0^1 \zeta \int_{\Omega} z^{m+1}(x, k, t) dk dx dt \\
 & - \mu_2 \int_{\Omega} \int_{t_1}^{t_2} |z(x, 1, t)|^{m-1} z(x, 1, t) u dt dx.
 \end{aligned} \tag{3.29}$$

Now we will estimate the right hand side of (3.29). First by (3.20), (3.26) and Lemma 2.1, we have

$$\begin{aligned}
 \|u_t(t_i)\|_2 \|u(t_i)\|_2 &\leq c_s c(\Omega)^{\frac{1}{2}} D(t) \sup_{t_1 \leq s \leq t_2} \|\nabla u(s)\|_2 \\
 &\leq D(t) c_s c(\Omega)^{\frac{1}{2}} \left( \frac{2(p+1)}{p-1} \right)^{\frac{1}{2}} \sup_{t_1 \leq s \leq t_2} (E(s))^{\frac{1}{2}} \\
 &\leq D(t) c_s c(\Omega)^{\frac{1}{2}} \left( \frac{2(p+1)}{p-1} \right)^{\frac{1}{2}} (E(t))^{\frac{1}{2}}.
 \end{aligned} \tag{3.30}$$

Employing Young’s inequality for convolution  $\|\varphi * \phi\| \leq \|\varphi\| \|\phi\|$  and noting that

$$l \|\nabla u(t)\|_2^2 \leq \frac{1}{\theta} I_2(t), \tag{3.31}$$

then we have

$$\begin{aligned}
 \int_0^t \int_{t_1}^{t_2} g(t-s) \|\nabla u(s)\|_2^2 ds dt &\leq \int_{t_1}^{t_2} g(t) dt \int_{t_1}^{t_2} \|\nabla u(t)\|_2^2 dt \\
 &\leq (m_0 - l) \int_{t_1}^{t_2} \|\nabla u(t)\|_2^2 dt \leq \frac{m_0 - l}{l\theta} \int_{t_1}^{t_2} I_2(t) dt.
 \end{aligned} \tag{3.32}$$

In the same manner

$$\begin{aligned}
 \int_0^t \int_{t_1}^{t_2} g(s) \|\nabla u(t)\|_2^2 ds dt &\leq \int_0^{\infty} g(s) ds \int_{t_1}^{t_2} \|\nabla u(t)\|_2^2 dt \\
 &\leq \frac{m_0 - l}{l\theta} \int_{t_1}^{t_2} I_2(t) dt.
 \end{aligned} \tag{3.33}$$

Exploiting (3.31) and (3.33) to obtain

$$\begin{aligned} \frac{1}{2} \int_{t_1}^{t_2} (g \circ \nabla u)(t) dt &= \frac{1}{2} \int_0^t \int_{t_1}^{t_2} g(t-s) \|\nabla u(s) - \nabla u(t)\| ds dt \\ &\leq \int_0^t g(t) dt \int_{t_1}^{t_2} \|\nabla u(t)\| dt \leq \int_0^t g(t) dt \int_{t_1}^{t_2} l \|\nabla u(t)\| dt \\ &\leq \frac{(m_0 - l)}{2\theta} \int_{t_1}^{t_2} I_2(t) dt. \end{aligned} \tag{3.34}$$

It follows by using (3.8) that

$$\begin{aligned} \int_{t_1}^{t_2} \int_0^1 z^{m+1}(x, k, t) dk dx dt &\leq \frac{1}{2\tau} \int_{t_1}^{t_2} \int_{t_1}^{t_2} \|u_t(s)\|_{m+1}^{m+1} ds d\nu \\ &\leq \left( \int_{t_1}^{t_2} d\nu \right) \left( \int_{t_1}^{t_2} \|u_t(s)\|_{m+1}^{m+1} ds \right) \leq (t_2 - t_1) D(t)^{m+1}. \end{aligned} \tag{3.35}$$

Using Sobolev’s inequality, we get

$$\begin{aligned} \left| \int_{t_1}^{t_2} \int_{\Omega} |u_t|^{m-1} |u_t| u dx dt \right| &\leq \int_{t_1}^{t_2} \|u_t\|_{m+1}^m \|u\|_{m+1} dt \\ &\leq c_s \|u_t\|_{m+1}^m \|\nabla u\|_2 dt \leq c_s \left( \frac{2(p+1)}{l(p-1)} \right)^{\frac{1}{2}} \sup_{t_1 \leq s \leq t_2} \left( (E(s))^{\frac{1}{2}} \right) \\ &\times \int_{t_1}^{t_2} \|u_t\|_{m+1}^m dt \leq c_s \left( \frac{2(p+1)}{l(p-1)} \right)^{\frac{1}{2}} E(t)^{\frac{1}{2}} [D(t)]^m. \end{aligned} \tag{3.36}$$

Also we have

$$\begin{aligned} \left| \int_{t_1}^{t_2} |z(x, 1, t)|^{m-1} z(x, 1, t) u dx dt \right| &\leq \int_{t_1}^{t_2} \|z(t, 1, t)\|_{m+1}^m \|u_t\|_{m+1} dt \\ &\leq c_s \left( \frac{2(p+1)}{l(p-1)} \right)^{\frac{1}{2}} \sup_{t_1 \leq s \leq t_2} \left( (E(s))^{\frac{1}{2}} \right) \int_{t_1}^{t_2} \|z(x, 1, t)\|_{m+1}^m dt \\ &\leq c_s \left( \frac{2(p+1)}{l(p-1)} \right)^{\frac{1}{2}} (E(t))^{\frac{1}{2}} [D(t)]^m. \end{aligned} \tag{3.37}$$

Therefore, from (3.30)-(3.37) we deduce

$$\begin{aligned} \int_{t_1}^{t_2} I_2(t) dt &\leq 2c_s \left( \frac{2(p+1)}{l(p-1)} \right)^{\frac{1}{2}} (E(t))^{\frac{1}{2}} [D(t)]^m + (t_2 - t_1) [D(t)]^{m+1} \\ &+ c(\Omega) [D(t)]^2 + c_s c(\Omega)^{\frac{1}{2}} \left( \frac{2(p+1)}{l(p-1)} \right)^{\frac{1}{2}} (E(t))^{\frac{1}{2}} D(t) \\ &+ \left[ \frac{(2m_0 - 2l)}{l\theta} + \frac{m_0 - l}{2\theta} \right] \int_{t_1}^{t_2} I_2(t) dt. \end{aligned} \tag{3.38}$$

Then, rewriting (3.29), we obtain

$$\begin{aligned} c_5 \int_{t_1}^{t_2} I_2(t) dt &\leq c(\Omega) [D(t)]^2 + c_3 (E(t))^{\frac{1}{2}} D(t) \\ &+ c_6 E(t)^{\frac{1}{2}} [D(t)]^m + c_4 [D(t)]^{m+1}, \end{aligned} \tag{3.39}$$

with

$$c_5 = \left[ 1 - \left( \frac{2m_0 - 2l}{l\theta} + \frac{m_0 - l}{2\theta} \right) \right], c_3 = c_s c(\Omega) \left( \frac{2(p+1)}{l(p-1)} \right)^{\frac{1}{2}}, c_4 = (t_2 - t_1), c_6 = 2c_5.$$

From the condition (3.22) and observing that is equivalent to  $c_5 > 0$ , then we get

$$\int_{t_1}^{t_2} I_2(t) dt \leq c_7 \left( [D(t)]^2 + (E(t))^{\frac{1}{2}} D(t) + (E(t))^{\frac{1}{2}} [D(t)]^m + [D(t)]^{m+1} \right), \tag{3.40}$$

where  $c_7 = \frac{\max(c_3, c_4, c_6, c(\Omega))}{c_5}$ . On the other hand, from the definition of  $E(t)$  and by (3.15) and () we have

$$\begin{aligned} E(t) &= \frac{p-1}{2(p+1)} \left[ (go\nabla u)(t) + \left( m_0 - \int_0^t g(s) ds \right) \|\nabla u\|_2^2 \right. \\ &\quad \left. + \int_{\Omega} \int_0^1 z^{m+1}(x, k, s) dk dx \right] + \frac{1}{2} \|u_t(t)\|_2^2 + \frac{\alpha}{2(\gamma+1)} (\|\nabla u\|_2^2)^{\gamma+1} \\ &+ \frac{1}{p+1} I_1(t) \leq \frac{p-1}{2(p+1)} \left[ (go\nabla u)(t) + \left( m_0 - \int_0^t g(s) ds \right) \|\nabla u\|_2^2 \right. \\ &\quad \left. + \int_{\Omega} \int_0^1 z^{m+1}(x, k, s) dk dx \right] + \left( \frac{1}{p+1} + \frac{1}{2(\gamma+1)} \right) I_2(t) \\ &\quad + \frac{1}{2} \|u_t(t)\|_2^2, \end{aligned} \tag{3.41}$$

where  $c_8 = \frac{1}{p+1} + \frac{1}{2(\gamma+1)}$ . Hence, integrating (3.41) over  $(t_1, t_2)$  and using (3.25), (3.31), (3.34) and (3.39), we deduce that

$$\begin{aligned} \int_{t_1}^{t_2} E(t) dt &\leq \frac{p-1}{2(p+1)} \int_{t_1}^{t_2} \left[ (go\nabla u)(t) \right. \\ &\quad \left. + \left( m_0 - \int_0^t g(s) ds \right) \|\nabla u\|_2^2 \right] dt \\ &\quad + \frac{p-1}{2(p+1)} \int_{t_1}^{t_2} \left[ \int_{\Omega} \int_0^1 z^{m+1}(x, k, s) dk dx + \right] dt \\ &\quad + \left( \frac{1}{p+1} + \frac{1}{2(\gamma+1)} \right) \int_{t_1}^{t_2} I_2(t) dt + \int_{t_1}^{t_2} \frac{1}{2} \|u_t(t)\|_2^2 dt \\ &\leq c_9 \int_{t_1}^{t_2} I_2(t) dt + c_4 [D(t)]^{m+1} + c(\Omega) [D(t)]^2 \\ &\leq c_{10} \left[ c(\Omega) [D(t)]^2 + (E(t))^{\frac{1}{2}} D(t) + (E(t))^{\frac{1}{2}} [D(t)]^m + [D(t)]^{m+1} \right], \end{aligned} \tag{3.42}$$

where

$$c_9 = \frac{p-1}{2(p+1)} \left[ \left( \frac{m_0 - l}{2\theta} \right) + \frac{1}{\theta} \right] + \frac{1}{p} + \frac{1}{2(\gamma+1)}, c_{10} = \max(c_8, c(\Omega), c_4).$$

Moreover, integrating (3.42) over  $(t, t_2)$ , using (3.24) and taking into account the fact that  $E(t_2) \leq 2 \int_{t_1}^{t_2} E(t) dt$ ,

due to  $t_2 - t_1 \geq \frac{1}{2}$ , we obtain

$$\begin{aligned}
 E(t) &= E(t_2) + \int_t^{t_2} \frac{1}{2}(g' \circ \nabla u)(t) dt + \int_t^{t_2} \frac{1}{2}g(t)\|\nabla u(t)\|_2^2 dt \\
 &+ \int_t^{t_2} \left( \mu_1 - \frac{\zeta}{\tau} - \frac{\mu_1}{m+1} \right) \|u'(t)\|_{m+1}^{m+1} dt \\
 &+ \int_t^{t_2} \left( \frac{\zeta}{\tau} - \frac{\mu_2 m}{m+1} \right) \int_{\Omega} z^{m+1}(x, 1, t) dx dt \\
 &\leq 2 \int_{t_1}^{t_2} E(t) dt + [D(t)]^{m+1},
 \end{aligned}
 \tag{3.43}$$

which implies that

$$E(t) \leq c_{11} \left[ [D(t)]^2 + [E(t)]^{\frac{1}{2}} D(t) + E(t)^{\frac{1}{2}} [D(t)]^m + [D(t)]^{m+1} \right].
 \tag{3.44}$$

Then a simple application of Young’s inequality gives, for all  $t \geq 0$

$$E(t) \leq c_{12} \left[ [D(t)]^2 + [D(t)]^{2m} + [D(t)]^{m+1} \right],
 \tag{3.45}$$

where  $c_{11}, c_{12}$  are some positive constants. Therefore, we have the following decay estimate:

(i) If  $m = 1$ . From (3.45), we get

$$E(t) \leq c_{12}[E(t) - E(t + 1)],$$

here we choose  $c_{12} > 1$ . Thus by Lemma 2.2, we obtain

$$E(t) \leq E(0)e^{-\tau_1 t} \quad \text{for } t \geq 0,$$

with  $\tau_1 = \ln \left( \frac{c_{12}}{c_{12}-1} \right)$ .

(ii) If  $m > 1$ . It follows from (3.45) that, for all  $t \geq 0$ ,

$$E(t) \leq c_{12} \left( (1 + [D(t)]^{m-1} + [D(t)]^{2m-2}) [D(t)]^2 \right).
 \tag{3.46}$$

As

$$D(t) \leq E(t)^{\frac{1}{m+1}} \leq E(0)^{\frac{1}{m+1}},
 \tag{3.47}$$

then by (3.45) and (3.46), we have, for all  $t \geq 0$

$$E(t) \leq c_{11} \left[ (1 + E(0)^{\frac{m-1}{m+1}} + E(0)^{\frac{2m-2}{m+1}}) [D(t)]^2 \leq c_{12}(E(0)) [D(t)]^2 \right],
 \tag{3.48}$$

where  $c_{12} = \left( \frac{1}{E(0)} + E(0)^{\frac{-2}{m+1}} + E(0)^{\frac{m-3}{m+1}} \right)$ . We get easily that

$$\begin{aligned}
 (E(t))^{1+m} &\leq [c_{13}(E(0))D(t)^2]^{1+m} \leq c_{14}(E(0)) [D(t)]^{2m+2} \\
 &\leq c_{14}E(0) [(D(t))^{m+1}] = c_{15}(E(0))(E(t) - E(t + 1)),
 \end{aligned}
 \tag{3.49}$$

where

$$c_{14}(E(0)) = 2^m (c_{13}(E(0)))^{m+1}, \quad c_{15}(E(0)) = c_{14}(E(0))E(0).$$

Applying the Lemma 2.2 to (3.49) yields

$$\phi(t) \leq (\phi(0))^{-m} + \tau_2 m [t - 1]^+ \frac{-1}{m}, \quad \text{for } t \geq 0,$$

with  $\tau_2 = c_{14}^{-1}(E(0))$ . The proof of Theorem 3.1 is completed.

#### 4. Blow up result

In this section we have a condition on parameters  $m$  and  $p$  as follows

$$\frac{n}{(n-2)} > p > m \geq 1, \quad n \geq 3(1 < p, n \leq 2). \tag{4.1}$$

2– **Nonlinear case:**  $m > 1$ , with  $E(0) < 0$ .

For this case our result reads as follows:

**Theorem 4.1.** *Let  $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ ,  $u_1 \in H_0^1(\Omega)$  and  $f_0 \in L^2(\Omega \times (0, 1))$  be given. Assume that the assumptions  $(A_0) - (A_2)$  are fulfilled. Let  $p > 2$  and suppose that (4.1) holds. Then for any initial data satisfying  $E(0) < 0$ , the solution of (3.1) blows up in finite time.*

*Proof.* We set

$$H(t) = -E(t). \tag{4.2}$$

Using the Lemma 3.1, to get

$$\begin{aligned} H'(t) \geq & \left( \mu_1 - \frac{\zeta}{\tau} - \frac{\mu_2}{m+1} \right) \|u_t\|_m^m + \frac{1}{2}g(t)\|\nabla u\|_2^2 \\ & + \left( \frac{\zeta}{\tau} - \frac{\mu_2 m}{m+1} \right) \int_{\Omega} z^{m+1}(x, 1, t)dx - \frac{1}{2}(g' \circ \nabla u)(s)ds, \end{aligned} \tag{4.3}$$

hence,

$$0 < H(0) \leq H(t) \leq \frac{1}{p+1} \|u\|_{p+1}^{p+1}, \quad t > 0. \tag{4.4}$$

Let

$$M(t) = \|u\|_2^2. \tag{4.5}$$

Differentiating (4.5) twice, we obtain

$$M'(t) = 2 \int_{\Omega} u_t u dx,$$

and

$$M''(t) = 2\|u_t\|_2^2 + 2 \int_{\Omega} u_{tt} u dx. \tag{4.6}$$

Using the first equation in (3.1), to get

$$\begin{aligned} M''(t) = & 2\|u_t\|_2^2 + 2 \int_{\Omega} \int_0^t g(t-s) \nabla u(s) \nabla u(t) dx ds \\ & - 2\mu_1 \int_{\Omega} |u_t(x, t)|^{m-1} u u_t(x, t) dx - 2M(\|\nabla u\|_2^2) \|\nabla u\|_2^2 \\ & - 2\mu_2 \int_{\Omega} |z(x, 1, t)|^{m-1} u z(x, 1, t) dx + 2\|u\|_{p+1}^{p+1}. \end{aligned} \tag{4.7}$$

Using Holder’s and Young’s inequalities to get

$$\begin{aligned} & \int_{\Omega} \int_0^t g(t-s) \nabla u(s) \nabla u(t) dx ds = \int_0^t g(t-s) \|\nabla u(t)\|_2^2 ds \\ & + \int_{\Omega} \int_0^t g(t-s) \nabla u(t) (\nabla u(s) - \nabla u(t)) ds dx \\ & \geq -(g \circ \nabla u)(t) + \frac{3}{4} \int_0^t g(s) ds \|\nabla u(t)\|_2^2. \end{aligned} \tag{4.8}$$

Noticing the estimate (4.8), then (4.7) takes the form

$$\begin{aligned}
 M''(t) &\geq (p + 1)\|u_t\|_2^2 + 2(p + 1)H(t) + (p - 1)(go\nabla u)(t) \\
 &+ \frac{(1 - 2p)}{2} \int_0^t g(s)ds\|\nabla u(t)\|_2^2 - 2m_0\|\nabla u(t)\|_2^2 - 2\alpha(\|\nabla u(t)\|_2^2)^{\gamma+1} \\
 &+ (p + 1)m_0\|\nabla u(t)\|_2^2 + \frac{(p + 1)\alpha}{\gamma + 1}(\|\nabla u(t)\|_2^2)^{\gamma+1} \\
 &+ m_1\|\nabla u\|_2^2 - 2\mu_1 \int_{\Omega} |u_t(x, t)|^{m-1}uu'(x, t)dx \\
 &- 2\mu_2 \int_{\Omega} |z(x, 1, t)|^{m-1}uz(x, 1, t)dx \\
 &+ (p + 1)\zeta \int_{\Omega} \int_0^1 z^{m+1}(x, k, t)dkdx.
 \end{aligned} \tag{4.9}$$

Using (2.1) to get

$$\begin{aligned}
 M''(t) &\geq (p + 1)\|u_t\|_2^2 + 2(p + 1)H(t) + (p - 1)(go\nabla u)(t) \\
 &- 2\mu_1 \int_{\Omega} |u_t(x, t)|^{m-1}uu'(x, t)dx \\
 &- 2\mu_2 \int_{\Omega} |z(x, 1, t)|^{m-1}uz(x, 1, t)dx \\
 &+ (p + 1)\zeta \int_{\Omega} \int_0^1 z^{m+1}(x, k, t)dkdx + m_1\|\nabla u\|_2^2.
 \end{aligned} \tag{4.10}$$

Now we define the functional

$$L(t) = H(t)^{1-\kappa} + 2\epsilon M'(t). \tag{4.11}$$

Differentiating (4.11), we obtain

$$L'(t) = (1 - \kappa)H(t)^{-\kappa}H'(t) + 2\epsilon M''(t). \tag{4.12}$$

Replacing (4.10) in (4.12), we get

$$\begin{aligned}
 L'(t) &\geq (1 - \kappa)H(t)^{-\kappa}H'(t) + 2\epsilon(p + 1)\|u'\|_2^2 + 2\epsilon(p + 1)H(t) \\
 &- 2\epsilon\mu_1 \int_{\Omega} |u'(x, t)|^{m-1}uu'(x, t)dx + m_1\|\nabla u\|_2^2 \\
 &- 2\epsilon\mu_2 \int_{\Omega} |z(x, 1, t)|^{m-1}uz(x, 1, t) + \epsilon(p - 1)(go\nabla u)(t)dx \\
 &+ \epsilon(p + 1)\zeta \int_{\Omega} \int_0^1 z^{m+1}(x, k, t)dkdx.
 \end{aligned} \tag{4.13}$$

Note that by (4.4) and using Holder inequality, we obtain

$$\begin{aligned}
 \left| \int_{\Omega} u|u_t|^{m-1}u_t dx \right| &\leq \|u_t\|_{m+1}^m \|u\|_m \leq c_1 \|u\|_{m+1}^{\frac{m+1}{p+1}} \|u\|_{m+1}^{1-\frac{m+1}{p+1}} \|u_t\|_{m+1}^m \\
 &\leq c_2 \|u\|_{m+1}^{\frac{m+1}{p+1}} H(t)^{\frac{1}{p+1} - \frac{m+1}{(p+1)^2}} \|u_t\|_{m+1}^m,
 \end{aligned} \tag{4.14}$$

By Young’s inequality and (4.3), produces

$$\begin{aligned}
 \left| \int_{\Omega} u|u_t|^{m-1}u_t dx \right| &\leq c_3 \left( \rho^{\frac{1}{1+p}} \|u\|_{m+1}^{m+1} H(0)^{-\bar{k}} \right. \\
 &\quad \left. + \rho^{-m'} H(0)^{k-\bar{k}} H'(t)H(t)^{-k} \right),
 \end{aligned} \tag{4.15}$$

where  $\bar{k} = \frac{1}{p+1} - \frac{m+1}{(p+1)^2} > 0$ ,  $\rho > 0$ ,  $m' = \frac{m+1}{m}$ . Letting  $0 < k < \bar{k}$ . Similarly we have

$$\left| \int_{\Omega} u|z(x, 1, t)|^{m-1}z(x, 1, t)dx \right| \leq c_3 \left( \rho^{\frac{1}{1+p}} \|u\|_{m+1}^{m+1} H(0)^{-\bar{k}} + \rho^{-m'} H(0)^{k-\bar{k}} H'(t)H(t)^{-k} \right). \tag{4.16}$$

Exploiting (4.14) and (4.15), to get

$$\begin{aligned} L'(t) &\geq \left[ (1-k) - 2\epsilon(\mu_1 + \mu_2)H^{k-\bar{k}}(0)\rho^{-m'} \right] H(t)^{-k} H'(t) \\ &\quad - 2\epsilon(\mu_1 + \mu_2)H(0)^{-\bar{k}}\rho^{\frac{1}{p+1}} \|u\|_{m+1}^{m+1} + \epsilon m_1 \|\nabla u\|_2^2 + 2\epsilon(p+1)H(t) \\ &\quad + \epsilon(p+2)\|u_t\|_2^2 + \epsilon(p-1)(go\nabla u)(t) \\ &\quad + \epsilon(p+1)\zeta \int_{\Omega} \int_0^1 z^{m+1}(x, k, t)dkdx, \end{aligned} \tag{4.17}$$

for  $\epsilon$  sufficiently small, we have

$$\left[ (1-k) - 2\epsilon(\mu_1 + \mu_2)H^{k-\bar{k}}(0)\rho^{-m'} \right] \geq 0.$$

We set  $s = m + 1 \leq p + 1$  such that  $\|u\|_{m+1}^s \leq c_1(\|\nabla u\|_2^2 + \|u\|_{p+1}^{p+1})$ , where  $c = 2(\mu_1 + \mu_2)H(0)^{-\bar{k}}\rho^{\frac{1}{p+1}}c_1$  and taking  $m_1 > c$ . Then

$$\begin{aligned} L'(t) &\geq \epsilon(m_1 - c)\|\nabla u\|_2^2 - \epsilon c \|u\|_{p+1}^{p+1} + 2\epsilon(p+1)H(t) + \epsilon(p+2)\|u_t\|_2^2 \\ &\quad + \epsilon(p-1)(go\nabla u)(t) + \epsilon(p+1)\zeta \int_{\Omega} \int_0^1 z^{m+1}(x, k, t)dkdx. \end{aligned} \tag{4.18}$$

Using this notation  $a_1 = m_1 - c$ ,  $a_2 = c$ ,  $a_3 = 2(p+1)$ ,  $a_4 = p+2$ ,  $a_5 = p-1$ , then (4.18) becomes

$$\begin{aligned} L'(t) &\geq a_1\epsilon\|\nabla u\|_2^2 - \epsilon a_2 \|u\|_{p+1}^{p+1} + \epsilon a_3 H(t) + \epsilon a_4 \|u_t\|_2^2 \\ &\quad + \epsilon a_5 (go\nabla u)(t). \end{aligned} \tag{4.19}$$

Following the approach of Messaoudi [13] we can suppose that  $p = 2a_6 + (p - 2a_6)$ , where  $a_6 < \min(a_1, a_2, a_3, a_4, a_5)$ , then (4.19) takes the form

$$\begin{aligned} L'(t) &\geq (a_1 - a_6)\epsilon\|\nabla u\|_2^2 + \epsilon(a_6 - a_2)\|u\|_{p+1}^{p+1} + \epsilon(a_3 - a_6)H(t) \\ &\quad + \epsilon(a_4 - a_6)\|u_t\|_2^2 + \epsilon(a_5 - a_6)(go\nabla u)(t), \end{aligned} \tag{4.20}$$

it produces

$$L'(t) \geq \delta\epsilon \left[ \|\nabla u\|_2^2 + \|u\|_{p+1}^{p+1} + H(t) + \|u_t\|_2^2 + (go\nabla u)(t) \right], \tag{4.21}$$

finally

$$L'(t) \geq \delta\epsilon \left[ \|u\|_{p+1}^{p+1} + H(t) + \|u_t\|_2^2 \right], \tag{4.22}$$

where  $\delta > 0$  is the minimum of the coefficients of  $\|u\|_{p+1}^{p+1}$ ,  $H(t)$ ,  $\|u_t\|_2^2$ . We pick out  $\epsilon$  so that

$$L(0) = H(t)^{1-\kappa}(0) + 2\epsilon \int_{\Omega} u^1 u^0 dx > 0. \tag{4.23}$$

Consequently we have

$$L(t) \geq L(0) > 0, \quad \forall t \geq 0. \tag{4.24}$$

Now, we set  $\varpi = \frac{1}{1-k}$ , and since  $k < \bar{k} < 1$  it is not difficult to see that  $1 < \varpi < \frac{1}{1-k}$ . We set

$$L(t) = H(t)^{1-k} + 2\epsilon \int_{\Omega} u_t u dx, \tag{4.25}$$

then

$$L(t) = H(t)^{\frac{1}{\varpi}} + 2\epsilon \int_{\Omega} u_t u dx \leq H(t)^{\frac{1}{\varpi}} + 2\epsilon \int_{\Omega} u_t u dx. \tag{4.26}$$

By using Young, Holder’s inequalities and (4.26) it follows that

$$\begin{aligned} L(t)^{\varpi} &\leq H(t) + \left[ 2\epsilon \int_{\Omega} u_t u dx + 2E_1(\|u\|_{p+1})^{\frac{p+1}{\varpi}} \right]^{\varpi} \\ &\leq 2^{\varpi-1} [H(t) + 2^{\varpi-1} (\beta^{\varpi} \|u_t\|_2^{\varpi} \|u\|_2^{\varpi})] \leq c_2 [H(t) + \|u_t\|_2^{\varpi} \|u\|_2^{\varpi}], \end{aligned} \tag{4.27}$$

where  $c_2 = \max \{2^{\varpi-1}, \beta^{\varpi}\}$ . On the other hand for  $p > 1$ , using Holder and Young inequalities we have

$$\|u_t\|_2^{\varpi} \|u\|_2^{\varpi} \leq c_3 \|u_t\|_2^{\varpi} \|u\|_{p+1}^{\varpi} \leq c_4 \left( \|u_t\|_2^2 + \|u\|_{p+1}^{\frac{2}{1-2k}} \right), \tag{4.28}$$

and

$$\|u\|_{p+1}^{\frac{2}{1-2k}} = \|u\|_{p+1}^{p+1} \|u\|_2^{\frac{2}{1-2k} - \frac{1}{p+1}} \leq c_5 H(0)^{\frac{2}{(1-2k)(p+1)} - 1} \|u\|_{p+1}^{p+1}. \tag{4.29}$$

Then (4.27) takes the form

$$L(t)^{\varpi} \leq c_6 \left( H(t) + \|u_t\|_2^2 + \|u\|_{p+1}^{p+1} \right). \tag{4.30}$$

Combining (4.22) and (4.30) we deduce that

$$L'(t) \geq c_7 L(t)^{\varpi}, \quad c_7 > 0, \quad \varpi > 1. \tag{4.31}$$

A simple integration over  $(0, t)$  yields the desired result. □

### 3– Nonlinear case: $m > 1$ , with $E(0) > 0$ .

In this subsection, we investigate the blow up of solutions with nonlinear terms and for positive initial energy  $E(0) > 0$ . For this purpose, we define the functional

$$G(x) = \frac{1}{2}x^2 - \frac{B}{p+1}x^{p+1}. \tag{4.32}$$

Moreover, we introduce the following notations

$$B = c_s^{p+1}l, \quad \lambda_1 = c_s^{-\frac{p+1}{p-2}}, \quad E_1 = \left( \frac{1}{2} - \frac{1}{p+1} \right) \lambda_1, \tag{4.33}$$

where

$$\begin{aligned} \varphi(t) &= \left[ l \|\nabla u\|_2^2 + (g \circ \nabla u)(t) + \frac{\alpha}{\gamma+1} (\|\nabla u\|_2^2)^{\gamma+1} \right. \\ &\quad \left. + \zeta \int_0^1 \int_{\Omega} z^{m+1}(x, k, t) dk dx \right]^{\frac{1}{2}}. \end{aligned} \tag{4.34}$$



**Lemma 4.2.** Assume  $(A_1)$  holds. Let  $(u, z)$  be a solution of (3.1) with initial data satisfying  $E(0) < E_1$  and  $\varphi(0) > \lambda_1$  i.e.

$$\left[ l\|\nabla u(0)\|_2^2 + \frac{\alpha}{\gamma + 1}(\|\nabla u(0)\|_2^2)^{\gamma+1} + \zeta \int_0^1 \int_{\Omega} z^{m+1}(x, k, 0)dkdx \right]^{\frac{1}{2}} > \lambda_1. \tag{4.35}$$

Then there exists  $\lambda_2 > \lambda_1$  such that, for all  $t \geq 0$

$$\varphi(t) \geq \lambda_2, \quad \text{and} \quad \|u\|_{p+1}^{p+1} \geq \lambda_3 = c_s^{p+1}\lambda_2^{p+1}, \tag{4.36}$$

*Proof.* As in [6] but here we give slight modification. We observe that  $G$  is increasing for  $0 < \lambda < \lambda_1$ , decreasing for  $\lambda > \lambda_1$  and  $G(\lambda) \rightarrow -\infty$  as  $\lambda \rightarrow \infty$ . Thus, as  $E(0) < E_1$ , there exist  $\lambda'_2 < \lambda_1 < \lambda_2$  such that  $G(\lambda'_2) = G(\lambda_2) = E(0)$ , which together with  $\varphi(0) \geq \lambda_1$  and (4.32) infer that

$$G(\varphi(0)) \leq E(0) = G(\lambda_2).$$

This implies that  $\varphi(0) \geq \lambda_2$ . To establish the first estimate in (4.36), we argue by contradiction. Suppose that the first estimate in (4.36) does not hold, then there exists  $t^* > 0$  such that

$$\varphi(t^*) < \lambda_2.$$

We easily observe from (3.14) and (3.15) and Lemma 3.2 that

$$\begin{aligned} E(t) \geq J(t) &= \frac{1}{2}\varphi(t)^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1} \geq \frac{1}{2}\varphi(t)^2 \\ &- \frac{1}{p+1}c_s^{p+1}l\|\nabla u\|_2^{p+1} \geq \frac{1}{2}\varphi(t)^2 - \frac{c_s^{p+1}l}{p+1}\varphi(t)^{p+1} = G(\varphi(t)). \end{aligned} \tag{4.37}$$

**Case 1:** if  $\lambda_1 < \varphi(t^*) < \lambda_2$ , then

$$G(\varphi(t^*)) > G(\lambda_2) = E(0) \geq E(t^*).$$

This contradicts the first estimate in (4.36).

**Case 2:** if  $\varphi(t^*) \leq \lambda_1$ , then by continuity of the function  $\varphi(t)$ , there exists  $0 < t_1 < t^*$  such that

$$\lambda_1 < \varphi(t_1) < \lambda_2,$$

then

$$G(\varphi(t_1)) > G(\lambda_2) = E(0) \geq E(t_1).$$

This is also a contradiction of the first estimate of (4.36). Thus, we have proved the first estimate in (4.36). To prove the second estimate in (4.36) we have

$$\begin{aligned} E(t) &= \frac{1}{2}\|u'(t)\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t) + \frac{1}{2} \left( m_0 - \int_0^t g(s)ds \right) \|\nabla u\|_2^2 \\ &- \frac{1}{p+1}\|u\|_{p+1}^{p+1} + \frac{\alpha}{2(\gamma + 1)}(\|\nabla u\|_2^2)^{\gamma+1} + \frac{\zeta}{2} \int_{\Omega} \int_0^1 z^{m+1}(x, k, s)dkdx \\ &\leq E(0). \end{aligned} \tag{4.38}$$

Hence

$$-\frac{1}{p+1}\|u\|_{p+1}^{p+1} + \frac{1}{2}\|u_t\|_2^2 + \varphi(t)^2 \leq E(0). \tag{4.39}$$

Then from (4.38) we have

$$\begin{aligned} \frac{1}{p+1} \|u\|_{p+1}^{p+1} &\geq \varphi(t)^2 - E(0) \geq \frac{1}{2} \lambda_2^2 - E(0) = \frac{1}{2} \lambda_2^2 - G(\lambda_2) \\ &= \frac{c_s^{p+1}}{p+1} \lambda_2^{p+1}. \end{aligned} \tag{4.40}$$

This completes the proof. □

**Theorem 4.3.** *Let  $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ ,  $u_1 \in H_0^1(\Omega)$  and  $f_0 \in L^2(\Omega \times (0, 1))$  be given. Assume that the hypotheses  $(A_0) - (A_2)$  hold, for any initial data satisfying  $E(0) < E_1$ ,  $\varphi(0) > \lambda_1$ . Then any solution of problem (3.1) blows up at finite time  $T$ .*

*Proof.* We set

$$H(t) = E_1 - E(t), \quad t \geq 0. \tag{4.41}$$

By Lemma 3.1, we have

$$\begin{aligned} H'(t) = -E'(t) &= \left( \mu_1 - \frac{\zeta}{2\tau} - \frac{\mu_1}{m+1} \right) \|u_t\|_{m+1}^{m+1} \\ &+ \left( \frac{\zeta}{\tau} - \frac{\mu_2 m}{m+1} \right) \int_{\Omega} z^{m+1}(x, 1, t) dx - \frac{1}{2} (g' \circ \nabla u)(t) + \frac{1}{2} g(t) \|\nabla u(t)\|_2^2, \end{aligned} \tag{4.42}$$

consequently

$$\begin{aligned} 0 < E_1 - E(0) = H(0) &\leq H(t) = E_1 - \frac{1}{2} \varphi(t)^2 + \frac{1}{p+1} \|u\|_{p+1}^{p+1} \\ &\leq -\frac{1}{p+1} \lambda_1^2 + \frac{1}{p} \|u\|_{p+1}^{p+1} \leq \frac{1}{p+1} \|u\|_{p+1}^{p+1}, \quad \forall t \geq 0. \end{aligned} \tag{4.43}$$

Now we define the functional

$$L(t) = H(t)^{1-\kappa} + 2\epsilon \int_{\Omega} u_t u dx + 2tE_1. \tag{4.44}$$

Differentiating (4.44), to obtain

$$L'(t) = (1 - \kappa)H(t)^{-\kappa} H'(t) + 2\epsilon \int_{\Omega} u_{tt} u dx + 2\epsilon \|u_t\|_2^2 + 2E_1. \tag{4.45}$$

Then from (4.45) we have

$$\begin{aligned} L'(t) &= (1 - \kappa)H(t)^{-\kappa} H'(t) + 2\epsilon \|u_t\|_2^2 - 2\epsilon \|\nabla u\|_2^2 \\ &- 2\epsilon \alpha (\|\nabla u\|_2^2)^{\gamma+1} + 2\epsilon \int_{\Omega} \int_0^t g(t-s) \nabla u(s) \nabla u(t) dx ds \\ &- 2\epsilon \mu_1 \int_{\Omega} |u_t(x, t)|^{m-1} u u_t(x, t) dx \\ &- 2\epsilon \mu_2 \int_{\Omega} |z(x, 1, t)|^{m-1} u z(x, 1, t) dx + 2\epsilon \|u\|_{p+1}^{p+1} + 2E_1. \end{aligned} \tag{4.46}$$

Using Holder and Young's inequalities, to get

$$\begin{aligned} \int_{\Omega} \int_0^t g(t-s) \nabla u(s) \nabla u(t) dx ds &= \int_{\Omega} \int_0^t g(t-s) \nabla u(t) (\nabla u(s) \\ &- \nabla u(t)) ds dx + \int_0^t g(t-s) ds \|\nabla u(t)\|_2^2 \\ &\geq -(g \circ \nabla u)(t) + \frac{3}{4} \int_0^t g(s) ds \|\nabla u(t)\|_2^2. \end{aligned} \tag{4.47}$$

Using  $(A_2)$  then (4.46) takes the form

$$\begin{aligned}
 L'(t) &= (1 - \kappa)H(t)^{-\kappa}H'(t) + 2\epsilon(1 + p)\|u_t\|_2^2 + \epsilon a_1(g\circ\nabla u)(t) \\
 &+ 2E_1a_2 + 2\epsilon(p + 1)H(t) - 2\epsilon\mu_1 \int_{\Omega} |u_t(x, t)|^{m-1}uu_t(x, t)dx \\
 &- 2\epsilon\mu_2 \int_{\Omega} |z(x, 1, t)|^{m-1}uz(x, 1, t)dx \\
 &+ (p + 1)\epsilon\zeta \int_{\Omega} \int_0^1 z^{m+1}(x, k, t)dkdx + m_1\epsilon\|\nabla u\|_2^2,
 \end{aligned}
 \tag{4.48}$$

for  $\epsilon$  sufficiently small and,  $p > 1$ ,  $a_1 = (p - 1)$ ,  $a_2 = (1 - \epsilon p) > 0$ , this yields

$$\begin{aligned}
 L'(t) &\geq (1 - \kappa)H(t)^{-\kappa}H'(t) + 2\epsilon(1 + p)\|u_t\|_2^2 + \epsilon a_1(g\circ\nabla u)(t) \\
 &+ \epsilon m_1\|\nabla u\|_2^2 - 2\epsilon\mu_1 \int_{\Omega} |u_t(x, t)|^{m-1}uu_t(x, t)dx \\
 &- 2\epsilon\mu_2 \int_{\Omega} |z(x, 1, t)|^{m-1}uz(x, 1, t)dx + 2\epsilon(p + 1)H(t) \\
 &+ (p + 1)\epsilon\zeta \int_{\Omega} \int_0^1 z^{m+1}(x, k, t)dkdx.
 \end{aligned}
 \tag{4.49}$$

Using Holder inequality and (4.42), we obtain

$$\begin{aligned}
 \left| \int_{\Omega} u|u_t|^{m-1}u_t dx \right| &\leq \|u_t\|_{m+1}^m \|u\|_m \leq c_1 \|u\|_{m+1}^{\frac{m+1}{p+1}} \|u\|_{m+1}^{1-\frac{m+1}{p+1}} \|u_t\|_{m+1}^m \\
 &\leq c_2 \|u\|_{m+1}^{\frac{m+1}{p+1}} H(t)^{\frac{1}{p+1} - \frac{m+1}{(p+1)^2}} \|u_t\|_{m+1}^m.
 \end{aligned}
 \tag{4.50}$$

By Young’s inequality and (4.42), we obtain

$$\begin{aligned}
 \left| \int_{\Omega} u|u_t|^{m-1}u_t dx \right| &\leq c_3 \left( \gamma^{\frac{1}{1+p}} \|u\|_{m+1}^{m+1} H(0)^{-\bar{k}} \right. \\
 &\quad \left. + \gamma^{-m'} H(0)^{k-\bar{k}} H'(t) H(t)^{-k} \right),
 \end{aligned}
 \tag{4.51}$$

where  $\bar{k} = \frac{1}{p+1} - \frac{m+1}{(p+1)^2} > 0$ ,  $\gamma > 0$ ,  $m' = \frac{m+1}{m}$ . Letting  $0 < k < \bar{k}$ . Similarly we have

$$\begin{aligned}
 \left| \int_{\Omega} u|z(x, 1, t)|^{m-1}z(x, 1, t) dx \right| &\leq c_3 \left( \gamma^{\frac{1}{1+p}} \|u\|_{m+1}^{m+1} H(0)^{-\bar{k}} \right. \\
 &\quad \left. + \gamma^{-m'} H(0)^{k-\bar{k}} H'(t) H(t)^{-k} \right).
 \end{aligned}
 \tag{4.52}$$

Exploiting (4.51) and (4.52), to get

$$\begin{aligned}
 L'(t) &\geq \left[ (1 - k) - 2\epsilon(\mu_1 + \mu_2)H^{k-\bar{k}}(0)\gamma^{-m'} \right] H(t)^{-k}H'(t) \\
 &- 2\epsilon(\mu_1 + \mu_2)H(0)^{-\bar{k}}\gamma^{\frac{1}{p+1}} \|u\|_{m+1}^{m+1} + \epsilon m_1\|\nabla u\|_2^2 + 2\epsilon(p + 1)H(t) \\
 &+ 2\epsilon(p + 1)\|u_t\|_2^2 + \epsilon(p - 1)(g\circ\nabla u)(t) \\
 &+ \epsilon(p + 1)\zeta \int_{\Omega} \int_0^1 z^{m+1}(x, k, t)dkdx,
 \end{aligned}
 \tag{4.53}$$

for  $\epsilon$  sufficiently small such that

$$\left[ (1 - k) - 2\epsilon(\mu_1 + \mu_2)H^{k-\bar{k}}(0)\gamma^{-m'} \right] \geq 0.$$

We set  $s = m + 1 \leq p + 1$  such that  $\|u\|_{m+1}^s \leq c_1(\|\nabla u\|_2^2 + \|u\|_{p+1}^{p+1})$ , where  $c = 2(\mu_1 + \mu_2)H(0)^{-\bar{k}}\gamma^{\frac{1}{p+1}}c_1$  and taking  $m_1 > c$ , we arrive at

$$\begin{aligned} L'(t) &\geq \epsilon(m_1 - c)\|\nabla u\|_2^2 - \epsilon c\|u\|_{p+1}^{p+1} + 2\epsilon(p + 1)H(t) \\ &\quad + 2\epsilon(p + 1)\|u_t\|_2^2 + \epsilon(p - 1)(go\nabla u)(t) \\ &\quad + \epsilon(p + 1)\zeta \int_{\Omega} \int_0^1 z^{m+1}(x, k, t)dkdx. \end{aligned} \tag{4.54}$$

Using this notation  $a_1 = m_1 - c$ ,  $a_2 = c$ ,  $a_3 = 2(p + 1)$ ,  $a_4 = p + 2$ ,  $a_5 = p - 1$ , then (4.54) becomes

$$\begin{aligned} L'(t) &\geq a_1\epsilon\|\nabla u\|_2^2 - \epsilon a_2\|u\|_{p+1}^{p+1} + \epsilon a_3H(t) + \epsilon a_4\|u_t\|_2^2 \\ &\quad + \epsilon a_5(go\nabla u)(t). \end{aligned} \tag{4.55}$$

Following the approach of Messaoudi [13] we can suppose that  $p = 2a_6 + (p - 2a_6)$ , where  $a_6 < \min(a_1, a_2, a_3, a_4, a_5)$  then (4.55) takes the form

$$\begin{aligned} L'(t) &\geq (a_1 - a_6)\epsilon\|\nabla u\|_2^2 + \epsilon(a_6 - a_2)\|u\|_{p+1}^{p+1} + \epsilon(a_3 - a_6)H(t) \\ &\quad + \epsilon(a_4 - a_6)\|u_t\|_2^2 + \epsilon(a_5 - a_6)(go\nabla u)(t), \end{aligned} \tag{4.56}$$

hence

$$L'(t) \geq \delta\epsilon \left[ \|\nabla u\|_2^2 + \|u\|_{p+1}^{p+1} + H(t) + \|u_t\|_2^2 + (go\nabla u)(t) \right], \tag{4.57}$$

finally

$$L'(t) \geq \delta\epsilon \left[ \|u\|_{p+1}^{p+1} + H(t) + \|u_t\|_2^2 \right], \tag{4.58}$$

where  $\delta > 0$  is the minimum of the coefficients of  $\|u\|_{p+1}^{p+1}$ ,  $H(t)$ ,  $\|u_t\|_2^2$ . We pick out  $\epsilon$  so that

$$L(0) = H(t)^{1-\kappa}(0) + 2\epsilon \int_{\Omega} u^1 u^0 dx > 0. \tag{4.59}$$

Consequently we have

$$L(t) \geq L(0) > 0, \quad \forall t \geq 0. \tag{4.60}$$

Now, we set  $\varpi = \frac{1}{1-k}$ , and since  $k < \bar{k} < 1$  it is not hard to see that  $1 < \varpi < \frac{1}{1-\bar{k}}$ . We set

$$L(t) = H(t)^{1-k} + 2\epsilon \int_{\Omega} u_t u dx + 2t\epsilon E_1. \tag{4.61}$$

Taking into account (4.36) of Lemma 4.1 we also can choose  $\epsilon$  small enough such that  $\epsilon t < (\lambda_3)^{\frac{p+1}{\varpi}}$ , we obtain

$$\begin{aligned} L(t) &= H(t)^{\frac{1}{\varpi}} + 2\epsilon \int_{\Omega} u_t u dx + 2E_1(\lambda_3)^{\frac{p+1}{\varpi}} \\ &\leq H(t)^{\frac{1}{\varpi}} + 2\epsilon \int_{\Omega} u_t u dx + 2E_1(\|u\|_{p+1})^{\frac{p+1}{\varpi}}. \end{aligned} \tag{4.62}$$

By using Young, Holder's inequalities and (4.61) it follows that

$$\begin{aligned} L(t)^{\varpi} &\leq H(t) + [2\epsilon \int_{\Omega} u_t u dx + 2E_1(\|u\|_{p+1})^{\frac{p+1}{\varpi}}]^{\varpi} \\ &\leq 2^{\varpi-1}[H(t) + 2^{\varpi-1}(\beta^{\varpi}\|u_t\|_2^{\varpi}\|u\|_2^{\varpi} + (2E_1)^{\varpi}\|u\|_{p+1}^{p+1})] \\ &\leq c_2 \left( H(t) + \|u_t\|_2^{\varpi}\|u\|_2^{\varpi} + \|u\|_{p+1}^{p+1} \right), \end{aligned} \tag{4.63}$$

where  $c_2 = \max \{2^{\varpi-1}, \beta^{\varpi}, (2E_1)^{\varpi}\}$ . On the other hand, for  $p > 1$ , using Holder and Young inequalities we have

$$\|u_t\|_2^{\varpi} \|u\|_2^{\varpi} \leq c_3 \|u_t\|_2^{\varpi} \|u\|_{p+1}^{\varpi} \leq c_4 \left( \|u_t\|_2^2 + \|u\|_{p+1}^{\frac{2}{1-2k}} \right), \quad (4.64)$$

$$\|u\|_{p+1}^{\frac{2}{1-2k}} = \|u\|_{p+1}^{p+1} \|u\|_2^{\frac{2}{1-2k} - \frac{1}{p+1}} \leq c_5 H(0)^{\frac{2}{(1-2k)(p+1)} - 1} \|u\|_{p+1}^{p+1}. \quad (4.65)$$

Then (4.63) takes the form

$$L(t)^{\varpi} \leq c_6 \left( H(t) + \|u_t\|_2^2 + \|u\|_{p+1}^{p+1} \right). \quad (4.66)$$

Combining (4.63) and (4.66) we deduce that

$$L'(t) \geq c_7 L(t)^{\varpi}, \quad c_7 > 0, \quad \varpi > 1. \quad (4.67)$$

A simple integration over  $(0, t)$  yields the desired result.  $\square$

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