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Exponential stabilization of solutions for the 1-D transmission wave equation with boundary feedback

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Abstract

The purpose of this work is to study the exponential decay of the energy for the one-dimensional transmission wave equation with a boundary velocity feedback.
Thanks to the perturbed energy method developed by some authors in several contexts, and under certain conditions, we prove that the feedback controller exponentially stabilizes the equilibrium to zero of the system below, *i.e.* the feedback leads to faster energy decay.

Keywords: Boundary feedback, decay rate of energy, exponential stabilization, perturbed energy, transmission wave equation.
2010 MSC: 35L05, 93D15, 93B52.

1. Introduction

In this paper we are concerned with the following system:

$$u_{tt} = a^2 u_{xx} \qquad \text{in} \quad (0, L/2) \times (0, \infty), \tag{1.1}$$

$$v_{tt} = b^2 v_{xx} \qquad \text{in} \quad (L/2, L) \times (0, \infty), \tag{1.2}$$

$$u(0,t) = 0; \quad b^2 v_x(L,t) = -\lambda v_t(L,t); \quad t \geq 0, \tag{1.3}$$

$$u(L/2,t) = v(L/2,t); \quad a^2 u_x(L/2,t) = b^2 v_x(L/2,t); \quad t \geq 0, \tag{1.4}$$

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$$u(x,0) = u_0(x); \quad v(x,0) = v_0(x); \quad u_t(x,0) = u_1(x); \quad v_t(x,0) = v_1(x), \tag{1.5}$$

called the transmission problem of the wave equation with a boundary velocity feedback control, where a, b, λ are positive constants.

The two constants a and b called the wave speeds in (0, L/2), (L/2, L) respectively, λ is the control gain, and the function $\phi = -\lambda v_t(L,t)$ represents the feedback control.

Let us point out that in physics, feedback means the return of a portion of the output of a circuit or device to its input, and a system in which the value of some output quantity is controlled by feeding back the value of the controlled quantity and using it to manipulate an input quantity so as to bring the value of the controlled quantity closer to a desired value. Also known as closed-loop control system (see [15]).

In recent years, questions of stabilization and decay of energy of solutions for hyperbolic equations, in particular, wave models, have been studied by many mathematicians, by using methods different.

In our article we interested to the perturbed energy method who developed in $\Big[$ [2], [3], [4], [12], [13], [14] $\Big]$.

There exists several degrees of stability that one can study. The first degree consists at analyze merely the decreasing of the energy of the solutions towards zero, *i.e.* :

$$E(t) \to 0 \quad \text{when} \quad t \to +\infty.$$

For the second, one studies intermediate situations in which the solutions decreases of the polynomial type for example:

$$E(t) \leq \frac{C}{t^\alpha}, \quad \text{for} \;\; t > 0,$$

Where C And α Are positive constants with C depends on the initial data. In this case, one must take initial data more regular in the operator's domain.

As for the third, one is been interested in the decreasing of the fastest energy, namely when this one tends to 0 in an exponential manner *i.e.* :

$$E(t) \leq Ce^{-\delta t} \quad \text{for} \;\; t > 0,$$

where C and δ are positive constants with C depends on the initial data.

We wish to stabilze the system ((1.1) - (1.5)), we seek a suitable feedback such that for any initial data (of finite energy $E(0) < \infty$), the energy of the solution of the problem ((1.1) - (1.5)) *tends to zero exponentially as* $t \to 0$ (see [8]).

In this research we show how the feedback controller exponentially stabilizes the system ((1.1) - (1.5)), under suitable conditions.

The well-posedness of problem ((1.1) - (1.5)) is by now well known in the case where $a = b$ (see [2], [10]), and can be similarly treated without any difficulty in the case where $a \neq b$.

We define the energy functional $E(t)$ of the system ((1.1) - (1.5)):(see [16])

$$E(t) = \frac{1}{2}\int_0^{L/2} \Big(|u_t(x,t)|^2 + a^2 |u_x(x,t)|^2 \Big) dx + \frac{1}{2}\int_{L/2}^{L} \Big(|v_t(x,t)|^2 + b^2 |v_x(x,t)|^2 \Big) dx,$$

and construct the following perturbed energy functional E_ϵ (see [7])

$$F(t) = 2\int_0^{L/2} x u_t(x,t) u_x(x,t) dx + 2\int_{L/2}^{L} x v_t(x,t) v_x(x,t) dx, \tag{1.6}$$

$$E_\epsilon(t) = E(t) + \epsilon F(t), \tag{1.7}$$

where ϵ is a positive constant, choosing sufficiently small.

2. Preliminaries

Before proving the below main result theorem, we first establish the following lemmas.

Lemma 2.1. $\Big($***Young's inequality***$\Big)$
Let $0 < p, q < \infty, \frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \quad (a, b > 0).$$

The proof of the lemma above is referred to ([6] p 622-625)

Lemma 2.2. *The energy* $E(t)$ *of the system* (1.1)-(1.5) *is decreasing function for all* $t \geq 0$.

Proof. We examine the derivative of the energy

$$\frac{dE}{dt} = \frac{1}{2}\int_0^{L/2} \frac{\partial}{\partial t}\Big(u_t^2(x,t)\Big) + a^2\frac{\partial}{\partial t}\Big(u_x^2(x,t)\Big)dx + \frac{1}{2}\int_{L/2}^{L} \frac{\partial}{\partial t}\Big(v_t^2(x,t)\Big) + b^2\frac{\partial}{\partial t}\Big(v_x^2(x,t)\Big)dx,$$

using the identities

$$u_t u_{tt} = \frac{1}{2}\frac{\partial}{\partial t}(u_t^2), \quad \text{and} \quad u_t u_{xx} = \frac{\partial}{\partial x}(u_x u_t) - \frac{1}{2}\frac{\partial}{\partial t}(u_x^2),$$

we get

$$\begin{aligned}
\frac{dE}{dt} &= \int_0^{L/2} u_t u_{tt} + a^2(\frac{\partial}{\partial x}(u_x u_t) - u_t u_{xx})dx + \int_{L/2}^{L} v_t v_{tt} + b^2(\frac{\partial}{\partial x}(v_x v_t) - v_t v_{xx})dx \\
&= \int_0^{L/2} u_t(u_{tt} - a^2 u_{xx})dx + \int_{L/2}^{L} v_t(v_{tt} - b^2 v_{xx})dx \\
&+ a^2\Big(u_x(L/2,t)u_t(L/2,t) - u_x(0,t)u_t(0,t)\Big) + b^2\Big(v_x(L,t)v_t(L,t) - v_x(L/2,t)v_t(L/2,t)\Big)
\end{aligned}$$

using (1.1)-(1.2) we get

$$\frac{dE}{dt} = a^2\Big(u_x(L/2,t)u_t(L/2,t) - u_x(0,t)u_t(0,t)\Big) + b^2\Big(v_x(L,t)v_t(L,t) - v_x(L/2,t)v_t(L/2,t)\Big)$$

finally, using (1.3)-(1.4) yields

$$\frac{dE}{dt} = -\lambda\,|v_t(L,t)|^2 \leq 0, \tag{2.1}$$

and then the energy is decreasing with time, *i.e.*,

$$E(t) \leq E(0) \quad \text{for all} \quad t \geq 0.$$

□

Lemma 2.3. *The perturbed energy satisfies*

$$\Big(1 - \frac{2L\epsilon}{\min(a,b)}\Big)E(t) \leq E_\epsilon(t) \leq \Big(1 + \frac{2L\epsilon}{\min(a,b)}\Big)E(t), \tag{2.2}$$

where ϵ *is small enough, such that* $0 < \epsilon < \dfrac{\min(a,b)}{2L}$.

Proof. We have

$$\begin{aligned}
|F(t)| &= \left|\int_0^{L/2} 2xu_t(x,t)u_x(x,t)dx + \int_{L/2}^{L} 2xv_t(x,t)v_x(x,t)dx\right| \\
&\leq \left|\int_0^{L/2} 2xu_t(x,t)u_x(x,t)dx\right| + \left|\int_{L/2}^{L} 2xv_t(x,t)v_x(x,t)dx\right| \\
&\leq \frac{1}{a}\int_0^{L/2} 2\,|x|\,|u_t(x,t)|\,|au_x(x,t)|\,dx + \frac{1}{b}\int_{L/2}^{L} 2\,|x|\,|v_t(x,t)|\,|bv_x(x,t)|\,dx \\
&\leq \frac{L}{2a}\int_0^{L/2} 2\,|u_t(x,t)|\,|au_x(x,t)|\,dx + \frac{L}{b}\int_{L/2}^{L} 2\,|v_t(x,t)|\,|bv_x(x,t)|\,dx \\
&\leq \frac{L}{a}\int_0^{L/2} 2\,|u_t(x,t)|\,|au_x(x,t)|\,dx + \frac{L}{b}\int_{L/2}^{L} 2\,|v_t(x,t)|\,|bv_x(x,t)|\,dx
\end{aligned}$$

by applying Young's inequality 2.1, we derive that

$$\begin{aligned}
|F(t)| &\leq \frac{L}{a}\int_0^{L/2} |u_t(x,t)|^2 + a^2\,|u_x(x,t)|^2\,dx + \frac{L}{b}\int_{L/2}^{L} |v_t(x,t)|^2 + b^2\,|v_x(x,t)|^2\,dx \\
&\leq \frac{2L}{\min(a,b)}\Big(\frac{1}{2}\int_0^{L/2} |u_t(x,t)|^2 + a^2\,|u_x(x,t)|^2\,dx + \frac{1}{2}\int_{L/2}^{L} |v_t(x,t)|^2 + b^2\,|v_x(x,t)|^2\,dx\Big) \\
&= \frac{2L}{\min(a,b)}E(t),
\end{aligned}$$

it therefore follows that

$$E_\epsilon(t) \leq E(t) + \epsilon\,|F(t)| \leq \Big(1 + \frac{2L\epsilon}{\min(a,b)}\Big)E(t),$$

and

$$E_\epsilon(t) \geq E(t) - \epsilon\,|F(t)| \geq \Big(1 - \frac{2L\epsilon}{\min(a,b)}\Big)E(t),$$

finally, we get

$$\Big(1 - \frac{2L\epsilon}{\min(a,b)}\Big)E(t) \leq E_\epsilon(t) \leq \Big(1 + \frac{2L\epsilon}{\min(a,b)}\Big)E(t).$$

□

3. Main results

We now in position to announce our result.

Theorem 3.1. *Assume that $b \leq a$, then there exist constants $M, \omega > 0$ such that the solution of* (1.1)-(1.5)*satisfies*

$$E(t) \leq ME(0)e^{-\omega t} \quad for \quad t \geq 0.$$

Proof. Differentiating (1.6) with respect to t, we obtain

$$\frac{dF}{dt} = \int_0^{L/2} 2xu_{tt}u_x dx + \int_0^{L/2} 2xu_t u_{xt} dx + \int_{L/2}^{L} 2xv_{tt}v_x dx + \int_{L/2}^{L} 2xv_t v_{xt} dx.$$

Moreover, by (1.1) yields

$$\begin{aligned}
\int_0^{L/2} 2xu_{tt}u_x dx &= \int_0^{L/2} 2xa^2 u_{xx}u_x dx \\
&= \int_0^{L/2} a^2 x\frac{\partial}{\partial x}(u_x^2)dx,
\end{aligned}$$

by integrating by parts, we obtain

$$\int_0^{L/2} 2xu_{tt}u_x dx = a^2\frac{L}{2}u_x^2(L/2,t) - a^2\int_0^{L/2} u_x^2 dx,$$

and

$$\begin{aligned}\int_0^{L/2} 2xu_tu_{xt}dx &= \int_0^{L/2} x\frac{\partial}{\partial x}(u_t^2)dx\\ &= \frac{L}{2}u_t^2(L/2,t) - \int_0^{L/2} u_t^2 dx.\end{aligned}$$

Similarly, we have

$$\int_{L/2}^{L} 2xv_{tt}v_x dx = b^2Lv_x^2(L,t) - b^2\frac{L}{2}v_x^2(L/2,t) - b^2\int_{L/2}^{L} v_x^2 dx,$$

and

$$\int_{L/2}^{L} 2xv_tv_{xt} = Lv_t^2(L,t) - \frac{L}{2}v_t^2(L/2,t) - \int_{L/2}^{L} v_t^2 dx,$$

then

$$\begin{aligned}\frac{dF}{dt} &= \frac{L}{2}\Big(a^2u_x^2(L/2,t) - b^2v_x^2(L/2,t)\Big) + \frac{L}{2}\Big(u_t^2(L/2,t) - v_t^2(L/2,t)\Big)\\ &\quad + L\Big(b^2v_x^2(L,t) + v_t^2(L,t)\Big) - 2\Big[\frac{1}{2}\int_0^{L/2}\Big(a^2u_x^2 + u_t^2\Big)dx + \frac{1}{2}\int_{L/2}^{L}\Big(b^2v_x^2 + v_t^2\Big)dx\Big].\end{aligned}$$

By (1.4)-(1.3), we infer

$$\frac{dF}{dt} = L\Big(1 + \frac{\lambda^2}{b^2}\Big)v_t^2(L,t) - 2E(t),$$

with the fact that

$$\frac{dE_\epsilon}{dt} = \frac{dE}{dt} + \epsilon\frac{dF}{dt}, \quad \text{and} \quad \frac{dE}{dt} = -\lambda\left|v_t(L,t)\right|^2,$$

we get

$$\begin{aligned}\frac{dE_\epsilon(t)}{dt} &= -\lambda v_t^2(L,t) + \epsilon L\Big(1 + \frac{\lambda^2}{b^2}\Big)v_t^2(L,t) - 2\epsilon E(t)\\ &= -2\epsilon E(t) - \lambda\Big[1 - \epsilon\frac{L(b^2+\lambda^2)}{\lambda b^2}\Big]v_t^2(L,t)\\ &\leq -2\epsilon E(t),\end{aligned}$$

for all $0 < \epsilon < \min\Big(\frac{b}{2L}, \frac{\lambda b^2}{L(b^2+\lambda^2)}\Big)$.

It then follows from (2.2) with $b \leq a$ that

$$\begin{aligned}\frac{dE_\epsilon(t)}{dt} &\leq -2\epsilon\Big(1 + \frac{2L\epsilon}{b}\Big)\Big(1 - \frac{2L\epsilon}{b}\Big)E(t)\\ &\leq -2\epsilon\Big(1 - \frac{2L\epsilon}{b}\Big)E_\epsilon(t),\end{aligned}$$

hence

$$E'_\epsilon(t) + \omega E_\epsilon(t) \leq 0, \quad \text{where } \ \omega = 2\epsilon\Big(1 - \frac{2L\epsilon}{b}\Big). \tag{3.1}$$

Multiplying (3.1) by $e^{\omega t}$ and integrating from zero to t, we obtain

$$E_\epsilon(t) \leq E_\epsilon(0)e^{-\omega t},$$

from (2.2) we have

$$\begin{aligned} E(t) &\leq \frac{1}{1-\frac{2L\epsilon}{b}} E_\epsilon(0)e^{-\omega t} \\ &\leq \frac{1}{1-\frac{2L\epsilon}{b}} \Big(1+\frac{2L\epsilon}{b}\Big) E(0)e^{-\omega t} \\ &= \frac{b+2L\epsilon}{b-2L\epsilon} E(0)e^{-\omega t}. \end{aligned}$$

We deduce that

$$E(t) \leq ME(0)e^{-\omega t},$$

where

$$M = \frac{b+2L\epsilon}{b-2L\epsilon}, \quad \omega = 2\epsilon\Big(1-\frac{2L\epsilon}{b}\Big) \text{ such that, } \quad 0<\epsilon<\min\Big(\frac{b}{2L}, \frac{\lambda b^2}{L(b^2+\lambda^2)}\Big).$$

□

Finally, we can say that the wave equation is *exponentially stabilizable* by boundary feedback.

The maximum decay rate: ω represents the decay rate of energy.

Let the functions $\Psi(\lambda) = \dfrac{\lambda b^2}{L(b^2+\lambda^2)}$, and $\omega =: \Phi(\epsilon) = 2\epsilon\Big(1-\frac{2L\epsilon}{b}\Big)$.

Because $\Psi'(\lambda) = \frac{b^2}{L}\Big(\frac{b^2-\lambda^2}{(b^2+\lambda^2)^2}\Big)$, and $\Phi'(\epsilon) = 2 - \frac{8L\epsilon}{b}$, then $\Psi(\lambda)$ attains the maximum $\frac{b}{2L}$, at $\lambda = b$, and $\Phi(\epsilon)$ attains the maximum $\frac{b}{4L}$, at $\epsilon = \frac{b}{4L}$.

We infer that the decay rate ω achieve $\frac{b}{4L}$ when the control gain is $\lambda = b$.

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