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FIRST AND SECOND ACCELERATION POLES IN LORENTZIAN HOMOTHETIC MOTIONS

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Abstract. In this paper, using matrix methods, we obtained rotation pole in one-parameter motion on the Lorentzian plane homothetic motions and pole orbits, accelerations and combinations of accelerations, first and second in acceleration poles. Moreover, some new theorems are given.

1. Introduction

In Lorentzian plane, a general planar motion as given by

$$\begin{aligned} y_1 &= x\cosh\varphi + y\sinh\varphi + a \\ y_2 &= x\sinh\varphi + y\cosh\varphi + b \end{aligned} \tag{1.1}$$

If θ, a and b are given by the functions of time parameter t, then this motions is called as one parameter motion [2] . One parameter planar motion given by (1.1) can be written in the form

$$\begin{pmatrix} Y \\ 1 \end{pmatrix} = \begin{pmatrix} A & C \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ 1 \end{pmatrix}$$

or

$$Y = AX + C, \quad Y = [y_1\ y_2]^T, \quad X = [x\ y]^T, \quad C = [a\ b]^T \tag{1.2}$$

where $A \in SO(2)$, and Y and X are the position vectors of the same point B, respectively, for the fixed and moving systems, and C is the translation vector [2]. By taking the derivatives with respect to t in (1.2), we get

$$\dot{Y} = \dot{A}X + A\dot{X} + \dot{C} \tag{1.3}$$

$$V_a = V_f + V_r \tag{1.4}$$

where the velocities $V_a = \dot{Y}, V_f = \dot{A}X + \dot{C}, V_r = A\dot{X}$ are called absolute, sliding, and relative velocities of the points B, respectively [1]. the solution of the equation $V_f = 0$ gives us the pole points on the moving plane. The locus of these points is

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called the moving pole curve, and correspondingly the locus of pole points on the fixed plane is called the fixed pole curve [1]. by taking the derivatives with respect to t in (1.3),we get

$$\ddot{Y} = \ddot{A}X + 2\dot{A}\dot{X} + A\ddot{X} + \ddot{C} \tag{1.5}$$

$$b_a = b_r + b_c + b_f, \tag{1.6}$$

where the velocities

$$b_a = \ddot{Y}, \tag{1.7}$$

$$b_f = \ddot{A}X + \ddot{C}, \tag{1.8}$$

$$b_r = A\ddot{X}, \tag{1.9}$$

$$b_c = 2\dot{A}\dot{X}, \tag{1.10}$$

are called absolute acceleration, sliding acceleration, relative acceleration and Coriolis accelerations, respectively [1]. The solution of the equation

$$\ddot{A}X + \ddot{C} = 0 \tag{1.11}$$

gives the acceleration pole of the motion [1]

2. HOMOTHETIC MOTION IN LORENTZIAN PLANE

Definition 2.1. The transformation given by the matrix

$$F = \begin{pmatrix} hA & C \\ 0 & 1 \end{pmatrix}$$

is called Homothetic motion in L^2 here $h = hI_2$ is a scalar matrix, $A \in SO(2)$ and $C \in \mathbb{R}^2_1$ [1].

Definition 2.2. Let $J \subset \mathbb{R}$ be an open interval let $O \in \mathbb{J}$. The transformation $F(t) : L^2 \longrightarrow \mathbb{L}^2$ given by

$$F(t) = \begin{pmatrix} h(t)A & C(t) \\ 0 & 1 \end{pmatrix}$$

is called one-parameter homothetic motion in L^2,where the function $h : J \longrightarrow \mathbb{R}$,the matrix $A \in SO(2)$ and the 2×1 type matrix C are differentiable with respect to [1].Since h is scalar we have $B^{-1} = h^{-1}A^{-1} = \frac{1}{h}A^T$ for $X \in L^2$, the geometric plane of the points is a curve in L^2. We will denote this curve by

$$Y(t) = B(t)X(t) + C(t) \tag{2.1}$$

differentiating with respect to t we obtain

$$\frac{dY}{dt} = \frac{dB}{dt}X + B\frac{dX}{dt} + \frac{dC}{dt}. \tag{2.2}$$

Definition 2.3. Equation of the general motion in L^2

$$Y(t) = B(t)X(t) + C(t) \tag{2.3}$$

where $A = A(t) \in SO(2)$ and $C = C(t) \in \mathbb{R}^2_1$ [1].Differentiating this equation with respect to t we have

$$\frac{dY}{dt} = \frac{dB}{dt}X + B\frac{dX}{dt} + \frac{dC}{dt}. \tag{2.4}$$

Here $V_a = \frac{dY}{dt}$, $V_r = B\frac{dX}{dt}$ and $V_f = \frac{dB}{dt}X + \frac{dC}{dt}$ are called absolute velocity, relative velocity and sliding velocity of the motion, respectively[3]. We denote motions in L^2 by $\frac{L}{\acute{L}}$ where $\acute{L}$ is fixed plane and L is the moving plane with respect to $\acute{L}$. If the matrix A and C are the functions of the parameter $t \in \mathbb{R}$ this motion is called a one parameter motion and denoted by $B_1 = \frac{L}{\acute{L}}$ [1].

Definition 2.4. The velocity vector of the point X with respect to the Lorentzian plane L (moving space) i.e. the vectorial velocity of X while it is drawing its orbit in L is called relative velocity of the point X and denoted by V_r [1].

Definition 2.5. The velocity vector of the point X with respect to the fixed plane $\acute{L}$ is called the absolute velocity of X and denoted by V_a. Thus we obtain the relation

$$V_a = V_f + V_r$$

If X is a fixed point in the moving plane L , since $V_r = 0$, then we have $V_a = V_f$. The quality (**??**) is said to be the velocity law the motion $B_1 = \frac{L}{\acute{L}}$ [1].

3. POLES OF ROTATING AND ORBIT

The point in which the sliding velocity V_f at each moment t of a fixed point X in L in the one-parameter homothetic motion $B_1 = \frac{L}{\acute{L}}$ are fixed points in moving and fixed plane. These points are called the pole points of the motion.

Theorem 3.1. *In a motion $B_1 = \frac{L}{\acute{L}}$ whose angular velocity is non zero, there exists a unique point which is fixed in both planes at every moment t.*

Proof. Since the point $X \in L$ is fixed in L then $V_r = 0$ and since X is also fixed in $\acute{L}$ then $V_f = 0$. Hence for this type of points if $V_f = 0$ then

$$\dot{B}X + \dot{C} = 0 \tag{3.1}$$

and

$$X = -\dot{B}^{-1}\dot{C} \tag{3.2}$$

Indeed,since

$$B = \begin{pmatrix} h\cosh\varphi & h\sinh\varphi \\ h\sinh\varphi & h\cosh\varphi \end{pmatrix}$$

and

$$\dot{B}=\begin{pmatrix} \dot{h}\cosh\varphi+h\dot{\varphi}\sinh\varphi & \dot{h}\sinh\varphi+h\dot{\varphi}\cosh\varphi \\ \dot{h}\sinh\varphi+h\dot{\varphi}\cosh\varphi & \dot{h}\cosh\varphi+h\dot{\varphi}\sinh\varphi \end{pmatrix}$$

then

$$C=[a\ b]^T, \tag{3.3}$$

implies that

$$\dot{C}=[\dot{a}\ \dot{b}]^T \tag{3.4}$$

and

$$det\dot{B}=\dot{h}^2-h^2\dot{\varphi}^2\neq 0. \tag{3.5}$$

Thus $\dot{B}$ is regular and

$$\dot{B}^{-1}=\frac{1}{\dot{h}^2-h^2\dot{\varphi}^2}\begin{pmatrix} \dot{h}\cosh\varphi+h\dot{\varphi}\sinh\varphi & -(\dot{h}\sinh\varphi+h\dot{\varphi}\cosh\varphi) \\ -(\dot{h}\sinh\varphi+h\dot{\varphi}\cosh\varphi) & \dot{h}\cosh\varphi+h\dot{\varphi}\sinh\varphi \end{pmatrix}$$

Hence there exists a unique solution X of the equation $V_f=0$. This point X is called pole point in moving plane. For this reason (3.2) leads to

$$X=-\dot{B}^{-1}\dot{C} \tag{3.6}$$

$$P=X=\frac{1}{h^2\dot{\varphi}^2-\dot{h}^2}\begin{pmatrix} \dot{a}(\dot{h}\cosh\varphi+h\dot{\varphi}\sinh\varphi)-\dot{b}(\dot{h}\sinh\varphi+h\dot{\varphi}\cosh\varphi) \\ -\dot{a}(\dot{h}\sinh\varphi+h\dot{\varphi}\cosh\varphi)+\dot{b}(\dot{h}\cosh\varphi+h\dot{\varphi}\sinh\varphi) \end{pmatrix}$$

$$P=\frac{1}{M}\begin{pmatrix} (\dot{a}\dot{h}-\dot{b}h\dot{\varphi})\cosh\varphi+(\dot{a}h\dot{\varphi}-\dot{b}\dot{h})\sinh\varphi \\ (-\dot{a}h\dot{\varphi}+\dot{b}\dot{h})\cosh\varphi)+(-\dot{a}\dot{h}+\dot{b}h\dot{\varphi})\sinh\varphi \end{pmatrix}$$

where $h^2\dot{\varphi}^2-\dot{h}^2=M$ and the pole point in the fixed plane is

$$\acute{P}=BP+C$$

setting these values in their planes and calculating we have

$$Y=\acute{P}=\frac{1}{M}\begin{pmatrix} h\dot{h}\dot{a}-h^2\dot{b}\dot{\varphi} \\ h\dot{h}\dot{b}-h^2\dot{a}\dot{\varphi} \end{pmatrix}+\begin{pmatrix} a \\ b \end{pmatrix}$$

or as a vector

$$Y=\acute{P}=(\frac{1}{M}(h\dot{h}\dot{a}-h^2\dot{b}\dot{\varphi})+a,\frac{1}{M}(h\dot{h}\dot{b}-h^2\dot{a}\dot{\varphi})+b) \tag{3.7}$$

Here we assume that $\dot{\varphi(t)}\neq 0$ for all t. That is, angular velocity is not zero. In this case there exists a unique pole points in each of the moving and fixed planes of each moment t.

Corollary 1. *If $\varphi(t)=t$, then we obtain*

$$X=P=\frac{1}{h^2-\dot{h}^2}\begin{pmatrix} (\dot{a}\dot{h}-\dot{b}h)\cosh\varphi+(\dot{a}h-\dot{b}\dot{h})\sinh\varphi) \\ (-\dot{a}h+\dot{b}\dot{h})\cosh\varphi)+(-\dot{a}\dot{h}+\dot{b}h)sinh\varphi) \end{pmatrix}$$

Corollary 2. *If $\varphi(t) = t$ and $h(t) = 1$,then we obtain*

$$X = P = \begin{pmatrix} \dot{a}\sinh\varphi - \dot{b}\cosh\varphi) \\ -\dot{a}\cosh\varphi + \dot{b}\sinh\varphi) \end{pmatrix}$$

Corollary 3. *If $\varphi(t) = t$,then we obtain*

$$\acute{P} = (\frac{1}{h^2 - \dot{h}^2}(h\dot{h}\dot{a} - h^2\dot{b}\dot{\varphi}) + a, \frac{1}{h^2 - \dot{h}^2}(h\dot{h}\dot{b} - h^2\dot{a}\dot{\varphi}) + b) \quad (3.8)$$

Corollary 4. *If $\varphi(t) = t$ and $h(t) = 1$, then we obtain*

$$\acute{P} = (-\dot{b} + a, -\dot{a} + b) \quad (3.9)$$

Definition 3.2. The point $P = (p_1, p_2)$ is called the instantaneous rotation center or the pole at moment t of the one parameter Euclidean motion $B_1 = \frac{L}{\dot{L}}$ [2]

Theorem 3.3. *The following relation exists between the pole ray from the pole P to the point X, and the sliding velocity vector V_f at each moment t.*

$$h < V_f, \acute{P}Y >= \dot{h}\|\acute{P}Y\| \quad (3.10)$$

Proof. The pole point in the moving plane

$$Y = BX + C. \quad (3.11)$$

implies that

$$X = B^{-1}(Y - C) \quad (3.12)$$

$$V_f = \dot{B}X + \dot{C} \quad (3.13)$$

and

$$\dot{B}X + \dot{C} = 0, \quad (3.14)$$

Leads to

$$X = P = -\dot{B}^{-1}\dot{C}, \quad (3.15)$$

Now Let's find pole points in the fixed plane. Then we have from equation $Y = BX + C$

$$Y = BX + C, \quad (3.16)$$

$$Y = \acute{P} = B(-\dot{B}^{-1}\dot{C}) + C, \quad (3.17)$$

Hence, we get

$$\acute{P} - C = -B\dot{B}^{-1}\dot{C}, \quad (3.18)$$

$$\dot{C} = -\dot{B}B^{-1}(\acute{P} - C). \quad (3.19)$$

If we substitute this values in the equation $V_f = \dot{B}X + \dot{C}$, we have $V_f = \dot{B}B^{-1}\acute{P}Y$. Now let us calculate the value of $\dot{B}B^{-1}\acute{P}Y$ here since $\acute{P}Y = (y_1 - p_1, y_2 - p_2)$ then

$$V_f = (\frac{\dot{h}}{h}(y_1 - p_1) - \dot{\varphi}(y_2 - p_2), \dot{\varphi}(y_1 - p_1) + \frac{\dot{h}}{h}(y_2 - p_2)), \quad (3.20)$$

hence we obtain

$$< V_f, \acute{P}Y >= \frac{\dot{h}}{h}[(y_1 - p_1)^2 - (y_2 - p_2)^2], \tag{3.21}$$

$$< V_f, \acute{P}Y >= \frac{\dot{h}}{h}\|\acute{P}Y\|^2, \tag{3.22}$$

on the other hand we know that

$$h < V_f, \acute{P}Y >= \dot{h}\|\acute{P}Y\|^2 \tag{3.23}$$

Corollary 5. *The pole ray from the pole P to the point X, when the scalar matrix h is constant, is perpendicular to the sliding velocity vector V_f at each instant moment t.*

Corollary 6. *There is a relation among the pole ray from the pole P to the point X, the sliding velocity vector V_f, and angular velocity $\dot{\varphi(t)} \neq 0$ at each moment t.*

$$h(t) = exp\left(\int \frac{< V_f, \acute{P}Y >}{\|\acute{P}Y\|}dt\right). \tag{3.24}$$

Theorem 3.4. *The length of the sliding velocity vector V_f is*

$$\|V_f\| = \sqrt{|(\left(\frac{\dot{h}}{h}\right)^2 - \dot{\varphi}^2)|}\|P'Y\| \tag{3.25}$$

Proof.

$$V_f = (\frac{\dot{h}}{h}(y_1 - p_1) + \dot{\varphi}(y_2 - p_2), \dot{\varphi}(y_1 - p_1) + \frac{\dot{h}}{h}(y_2 - p_2)), \tag{3.26}$$

hence

$$\|V_f\| = \sqrt{|(\left(\frac{\dot{h}}{h}\right)^2 - \dot{\varphi}^2)|}\|P'Y\|. \tag{3.27}$$

Corollary 7. *If the scalar matrix is h is constant, then the length of the sliding velocity vector is*

$$\|V_f\| = |\dot{\varphi}|\|P'Y\| \tag{3.28}$$

Corollary 8. *There is a relation among the pole ray from the pole P to the point X, the sliding velocity vector V_f, and angular velocity $\dot{\varphi(t)} \neq 0$ at each moment t.*

$$h(t) = exp\left(\int \sqrt{|(\left(\frac{\|V_f\|}{|\|P'Y\|}\right)^2 + \dot{\varphi}^2)|dt}\right). \tag{3.29}$$

Definition 3.5. In Lorentzian motion $B_1 = \frac{L}{\acute{L}}$, the geometric place of the pole points P in the moving plane L is called the moving pole curve of the motion $B_1 = \frac{L}{\acute{L}}$ and is denoted by (P). the geometric place of the pole points P in the fixed plane $\acute{L}$ is called fixed and is denoted by $\acute{P}$ [2] .

Theorem 3.6. *The velocity on the curve (P) and $(\acute{P})$ of every moment t of the rotating pol P which draws the pole curves in the fixed and moving planes are equal to each other. In other words, two curves are always tangent to each other* [2] *.*

Proof. The velocity of the point $X \in L$ while drawing the curve (P) is V_r and the velocity of this point while drawing the curve $(\acute{P})$ is V_a. Since $V_f = 0$ then $V_a = V_r$.

Theorem 3.7. *If two curves α and $\acute{\alpha}$ are tangent to each other of each moment t and if length of the ways ds and ds' of the point drawing these two curves at moment dt on these curves are the same then α and $\acute{\alpha}$ are said to be revolving by sliding on each other. Herehis the coefficient of rolling* [2].

Theorem 3.8. *In the one parameter planer Lorentzian motion $B_1 = \frac{L}{\acute{L}}$ the moving pole curve (P) of the plane L revolves by sliding on the fixed pole curve $(\acute{P})$ of the plane $\acute{L}$* [1] *.*

Proof. Acording to the definition of ray element of a curve ray of (P) is $ds = \|V_r\|$ and those of (P) is $ds' = \|V_a\|$ .Since for (P) and $(\acute{P})$, $V_a = V_r$ then $ds = hds'$. According to this theorem we way define a Lorentzian motion without mentioning the time. A Lorentzian motion $B_1 = \frac{L}{\acute{L}}$ is obtained by a moving pol curve (P) of L revolving without sliding on a fixed pol curve $(\acute{P})$.

Definition 3.9. Absolute acceleration vector of the point X with respect to the fixed Lorentzian plane $\acute{L}$ is V_a. This vector is denoted by b_a. Since $V_a = \dot{Y}$ then $b_a = \dot{V} = \ddot{Y}$ [2].

Definition 3.10. Let X be a fixed point the moving Lorentzian plane L. The acceleration vector of the point X with respect to the fixed Lorentzian plane $\acute{L}$ is called as sliding acceleration vector and denoted by b_f. Since in the acceleration of the sliding acceleration X is a fixed point of E,then $b_f = \dot{V}_f = \ddot{B}X + \ddot{C}$ [2].

4. ACCELERATIONS AND UNION OF ACCELERATIONS

Assume that the Minkowski motion $B_1 = \frac{L}{\acute{L}}$ of the moving Lorentzian plane L with respect to the fixed Lorentzian plane $\acute{L}$ exists. In this motion, let us consider a point X moving with respect to the plane L,and thus moving respect to the plane $\acute{L}$. We had obtained the velocity formulas concerning the motion of X, now we will obtain the acceleration formulas the acceleration of the point X.

Definition 4.1. The vector $b_r = \dot{V}_r = \ddot{B}X$ which is obtained by differentiating the relative velocity vector $V_r = B\dot{X}$ of the point X with respect to the moving plane L is called the relative acceleration vector of X in L and denote by b_r.Since when taking the derivative X is considered as a moving point in L,the matrix A is taken as constant [2].

Theorem 4.2. *Let X be a point in the moving Lorentzian plane which moves with respect to a parameter t. Hence we have that*

Theorem 4.3.
$$b_a = b_f + b_c + b_r \tag{4.1}$$
Here $b_c = 2\dot{B}\dot{X}$ is called Corilois acceleration [1].

Corollary 9. *If a point $X \in L$ is constant,then the sliding acceleration of the point X is equal to the absolute acceleration of X.*

Proof. Note that
$$V_a = \dot{B}X + B\dot{X} + \dot{C} \tag{4.2}$$
differentiating the both sides we have
$$\dot{V_a} = \ddot{B}X + 2\dot{B}\dot{X} + B\ddot{X} + \dot{C} \tag{4.3}$$
since the point X is constant its derivatives zero. Hence
$$\dot{V_a} = \ddot{B}X + \ddot{C} = b_f. \tag{4.4}$$

Theorem 4.4. *We have the following relation between the Coriolis acceleration vector b_c and relative velocity vector V_r.*
$$< b_c, V_r >= 2h\dot{h}(\dot{x_1}^2 - \dot{x_2}^2) \tag{4.5}$$

Proof. Since $b_c = 2\dot{B}\dot{X} =$, $V_r = B\dot{X}$. Then
$$< b_c, V_r >= 2h\dot{h}(\dot{x_1}^2 - \dot{x_2}^2) \tag{4.6}$$

Corollary 10. *If h is a constant,then Coriolis acceleration b_c is perpendicular to the relative velocity vector V_r at each instant moment t.*

5. FIRST AND SECOND ACCELERATION POLES

The solution of the equation $\dot{V_f} = 0$ gives the first order acceleration pole. $V_f = \ddot{B}X + \ddot{C} = 0$ implies $X = -\ddot{B}^{-1}\ddot{C}$. Now calculating the matrices $-\ddot{B}^{-1}$ and $\ddot{C}$ and setting these in $X = P_1 = -\ddot{B}^{-1}\ddot{C}$ we obtain

$$X = P_1 = \frac{-1}{k}\begin{pmatrix} \ddot{a}(m\cosh\varphi + n\sinh\varphi) - \ddot{b}(m\sinh\varphi + n\cosh\varphi) \\ -\ddot{a}(m\sinh\varphi + n\cosh\varphi) + \ddot{b}(m\cosh\varphi + n\sinh\varphi) \end{pmatrix}$$

Let $k = (\ddot{h} + h\dot{\varphi}^2)^2 - (2\dot{h}\dot{\varphi} + h\ddot{\varphi})^2$, $k \neq 0$, $m = \ddot{h} + h\dot{\varphi}^2$, $n = 2\dot{h}\dot{\varphi} + h\ddot{\varphi}$. Here P_1 is called first order pole curve in the moving plane. Denoting the pole curve in the fixed plane by $\acute{P_1}$ we get
$$\acute{P_1} = BP_1 + C \tag{5.1}$$
Hence
$$\acute{P_1} = (\frac{1}{k}(-\ddot{a}hm + \ddot{b}hn) + a, \frac{1}{k}(\ddot{a}hn - \ddot{b}hm) + b) \tag{5.2}$$

Corollary 11. *If $\varphi(t) = t$,then we obtain*

$$X = P_1 = \frac{-1}{(\ddot{h}+h)^2 - 4(\dot{h})^2}\begin{pmatrix} (\ddot{a}\ddot{h} - 2\ddot{b}\dot{h} + \ddot{a}h)\cosh\varphi - (\ddot{b}\ddot{h} - 2\ddot{a}\dot{h} + \ddot{b}h)\sinh\varphi) \\ (\ddot{b}\ddot{h} - 2\ddot{a}\dot{h} + \ddot{b}h)\cosh\varphi) - (\ddot{a}\ddot{h} - 2\ddot{b}\dot{h} + \ddot{a}h)sinh\varphi) \end{pmatrix}$$

Corollary 12. *If $\varphi(t) = t$ and $h(t) = 1$, then we obtain*

$$P_1 = (-\ddot{a}\cosh\varphi + \ddot{b}\sinh\varphi, -\ddot{b}\cosh\varphi + \ddot{a}\sinh\varphi) \tag{5.3}$$

Corollary 13. *If $\varphi(t) = t$, then we obtain*

$$\acute{P_1} = \frac{-1}{(\ddot{h}+h)^2 - 4(\dot{h})^2}(-\ddot{a}h(\ddot{h}+h) + \ddot{b}h(2\dot{h}), \ddot{a}h(2\ddot{h}) - \ddot{b}h(\ddot{h}+h)) + (a,b) \tag{5.4}$$

Corollary 14. *If $\varphi(t) = t$ and $h(t) = 1$, then we obtain*

$$\acute{P_1} = (-\ddot{a} + a, -\ddot{b} + b) \tag{5.5}$$

The solution of the equation $\ddot{V_f} = 0$ gives the second order acceleration pole. $\ddot{V_f} = \dddot{B}X + \dddot{C} = 0$ implies $X = -\dddot{B}^{-1}\dddot{C}$. Now calculating the matrices $\dddot{B}^{-1}$ and $\dddot{C}$ and setting these in $X = -\dddot{B}^{-1}\dddot{C}$ we get

$$X = P_2 = \frac{-1}{A^2 - B^2}\begin{pmatrix} \dddot{a}(A\cosh\varphi + B\sinh\varphi) - \dddot{b}(A\sinh\varphi + B\cosh\varphi) \\ -\dddot{a}(A\sinh\varphi + B\cosh\varphi) + \dddot{b}(A\cosh\varphi + B\sinh\varphi) \end{pmatrix}$$

The pole curve in the fixed plane is obtained as

$$\acute{P_2} = (\frac{-1}{A^2 - B^2}(\dddot{a}hA - \dddot{b}hB) + a, \frac{-1}{A^2 - B^2}(-\dddot{a}hB + \dddot{b}hA) + b) \tag{5.6}$$

Let us

$$A = (3h\dot{\varphi}\ddot{\varphi} + 3\dot{h}\dot{\varphi}^2 + \dddot{h}), B = (h\dot{\varphi}^3 + 3\dot{h}\ddot{\varphi} + h\dddot{\varphi} + 3\ddot{h}\dot{\varphi}) \tag{5.7}$$

Corollary 15. *If $\varphi(t) = t$, then we obtain*

$$X = P_2 = \frac{-1}{T}\begin{pmatrix} (\dddot{a}\dddot{h} - 3\dddot{b}\ddot{h} + 3\dddot{a}\dot{h} - \dddot{b}h)\cosh\varphi + (-\dddot{b}\dddot{h} + 3\dddot{a}\ddot{h} - 3\dddot{b}\dot{h} + \dddot{a}h)\sinh\varphi \\ (-\dddot{a}\dddot{h} + 3\dddot{b}\ddot{h} - 3\dddot{a}\dot{h} + \dddot{b}h)\sinh\varphi + (\dddot{b}\dddot{h} - 3\dddot{a}\ddot{h} + 3\dddot{b}\dot{h} - \dddot{a}h)\cosh\varphi) \end{pmatrix}$$

where $T = (3\dot{h} + \dddot{h})^2 - (h + 3\ddot{h})^2$.

Corollary 16. *If $\varphi(t) = t$ and $h(t) = 1$, then we obtain*

$$P_2 = (-\dddot{b}\cosh\varphi + \dddot{a}\sinh\varphi, \dddot{b}\sinh\varphi - \dddot{a}\cosh\varphi) \tag{5.8}$$

Corollary 17. *If $\varphi(t) = t$, then we obtain*

$$\acute{P_2} = (\frac{-1}{T}(\dddot{a}h(3\dot{h} + \dddot{h}) - \dddot{b}h(h + 3\ddot{h}), -\dddot{a}h(h + 3\ddot{h}) + \dddot{b}h(3\dot{h} + \dddot{h})) + (a,b) \tag{5.9}$$

where $T = (3\dot{h} + \dddot{h})^2 - (h + 3\ddot{h})^2$.

Corollary 18. *If $\varphi(t) = t$ and $h(t) = 1$, then we obtain*

$$\acute{P_2} = (-\dddot{b} + a, -\dddot{a} + b) \tag{5.10}$$

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