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GAUSSIAN PADOVAN AND GAUSSIAN PELL- PADOVAN SEQUENCES

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Abstract. In this paper, we extend Padovan and Pell- Padovan numbers to Gaussian Padovan and Gaussian Pell-Padovan numbers, respectively. Moreover we obtain Binet-like formulas,generating functions and some identities related with Gaussian Padovan numbers and Gaussian Pell-Padovan numbers.

1. Introduction

Horadam [3] in 1963 and Berzsenyi [2] in 1977 defined complex Fibonacci numbers. Horadam introduced the concept the complex Fibonacci numbers as the Gaussian Fibonacci numbers.

Padovan sequence is named after Richard Padovan [7] and Atasonav K., Dimitrov D., Shannon A. and Kritsana S. [1, 4, 5, 6] studied Padovan sequence and Pell-Padovan sequence.

The Padovan sequence is the sequence of integers P_n defined by the initial values $P_0 = P_1 = P_2 = 1$ and the recurrence relation

$$P_n = P_{n-2} + P_{n-3} \quad \text{for all } n \geq 3.$$

The first few values of P_n are $1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37$.

Pell-Padovan sequence is defined by the initial values $R_0 = R_1 = R_2 = 1$ and the recurrence relation

$$R_n = 2R_{n-2} + R_{n-3} \quad \text{for all } n \geq 3.$$

The first few values of Pell-Padovan numbers are $1, 1, 1, 3, 3, 7, 9, 17, 25, 43, 67, 111, 177, 289$.

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2. Gaussian Padovan Sequences

Firstly we give the definition of Gaussian Padovan sequence.

Definition 2.1. The Gaussian Padovan sequence is the sequence of complex numbers GP_n defined by the initial values $GP_0 = 1,\ GP_1 = 1 + i,\ GP_2 = 1 + i$ and the recurrence relation

$$GP_n = GP_{n-2} + GP_{n-3} \quad \text{for all } n \geq 3.$$

The first few values of GP_n are $1,\ 1+i,\ 1+i,\ 2+i,\ 2+2i,\ 3+2i,\ 4+3i, 5+4i,\ 7+5i,\ 9+7i$.

The following theorem is related with the generating function of the Gaussian Padovan sequence.

Theorem 2.2. *The generating function of the Gaussian Padovan sequence is*

$$g(x) = \frac{1 + (1+i)\ x + i\ x^2}{1 - x^2 - x^3}.$$

Proof. Let

$$g(x) = \sum_{n=0}^{\infty} GP_n x^n = GP_0 + GP_1 x + GP_2 x^2 + \cdots + GP_n x^n + \cdots$$

be the generating function of the Gaussian Padovan sequence. On the other hand, since

$$x^2 g(x) = GP_0 x^2 + GP_1 x^3 + GP_2 x^4 + \cdots + GP_{n-2} x^n + \cdots$$

and

$$x^3 g(x) = GP_0 x^3 + GP_1 x^4 + GP_2 x^5 + \cdots + GP_{n-3} x^n + \cdots$$

we write

$$\begin{aligned}(1 - x^2 - x^3) g(x) &= GP_0 + GP_1 x + (GP_2 - GP_0) x^2 + (GP_3 - GP_1 - GP_0) x^3 \\ &\quad + \cdots + (GP_n - GP_{n-2} - GP_{n-3}) x^n + \cdots\end{aligned}$$

Now consider $GP_0 = 1,\ GP_1 = 1 + i,\ GP_2 = 1 + i$ and $GP_n = GP_{n-2} + GP_{n-3}$. Thus, we obtain

$$(1 - x^2 - x^3) g(x) = 1 + (1+i) x + i\ x^2$$

or

$$g(x) = \frac{1 + (1+i)\ x + i\ x^2}{1 - x^2 - x^3}.$$

So, the proof is complete. □

Now we give Binet-like formula for the Gaussian Padovan sequence.

Theorem 2.3. *Binet-like formula for the Gaussian Padovan sequence is*

$$GP_n = \left(a + i\frac{a}{r_1}\right) r_1^n + \left(b + i\frac{b}{r_2}\right) r_2^n + \left(c + i\frac{c}{r_3}\right) r_3^n$$

where

$$a=\frac{(r_2-1)(r_3-1)}{(r_1-r_2)(r_1-r_3)}, b=\frac{(r_1-1)(r_3-1)}{(r_2-r_1)(r_2-r_3)}, c=\frac{(r_1-1)(r_2-1)}{(r_1-r_3)(r_2-r_3)}$$

and r_1, r_2, r_3 *are the roots of the equation* $x^3-x-1=0$.

Proof. It is easily seen that

$$GP_n = P_n + iP_{n-1}.$$

On the other hand, we know that the Binet-like formula for the Padovan sequence is

$$P_n=\frac{(r_2-1)(r_3-1)}{(r_1-r_2)(r_1-r_3)}r_1^n+\frac{(r_1-1)(r_3-1)}{(r_2-r_1)(r_2-r_3)}r_2^n+\frac{(r_1-1)(r_2-1)}{(r_1-r_3)(r_2-r_3)}r_3^n.$$

So, the proof is easily seen. □

Theorem 2.4.

$$\sum_{j=0}^{n} GP_j = GP_n + GP_{n+1} + GP_{n+2} - 2(1+i).$$

Proof. By the definition of Gaussian Padovan sequence recurrence relation

$$GP_n = GP_{n-2} + GP_{n-3}$$

and

$$\begin{aligned} GP_0 &= GP_2 - GP_{-1} \\ GP_1 &= GP_3 - GP_0 \\ GP_2 &= GP_4 - GP_1 \\ &\vdots \\ GP_{n-2} &= GP_n - GP_{n-3} \\ GP_{n-1} &= GP_{n+1} - GP_{n-2} \\ GP_n &= GP_{n+2} - GP_{n-1} \end{aligned}$$

Then we have

$$\sum_{j=0}^{n} GP_j = GP_n + GP_{n+1} + GP_{n+2} - GP_{-1} - GP_0 - GP_1.$$

Now considering $GP_{-1} = i, GP_0 = 1$ and $GP_1 = 1+i$, we write

$$\sum_{j=0}^{n} GP_j = GP_n + GP_{n+1} + GP_{n+2} - 2 - 2i.$$

and so the proof is complete. □

Now we investigate the new property of Gaussian Padovan numbers in relation with Padovan matrix formula. We consider the following matrices:

$$Q_3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \ K_3 = \begin{bmatrix} 1+i & 1+i & 1 \\ 1+i & 1 & i \\ 1 & i & 1 \end{bmatrix}$$

and

$$M_3^n = \begin{bmatrix} GP_{n+2} & GP_{n+1} & GP_n \\ GP_{n+1} & GP_n & GP_{n-1} \\ GP_n & GP_{n-1} & GP_{n-2} \end{bmatrix}.$$

Theorem 2.5. *For all $n \in Z^+$, we have*

$$Q_3^n K_3 = M_3^n.$$

Proof. The proof is easily seen that using the induction on n. □

Theorem 2.6. *If*

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

then we have

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} 1 \\ 1+i \\ 1+i \end{bmatrix} = \begin{bmatrix} GP_n \\ GP_{n+1} \\ GP_{n+2} \end{bmatrix}.$$

Proof. The proof can be seen by mathematical induction on n. □

3. Gaussian Pell-Padovan Sequence

As well known Pell-Padovan sequence is defined by the recurrence relation

$$R_n = 2R_{n-2} + R_{n-3}, \ n \geq 3$$

and initial values are $R_0 = R_1 = R_2 = 1$.

Now we define Gaussian Pell-Padovan sequence.

Definition 3.1. The Gaussian Pell-Padovan sequence is defined by the recurrence relation

$$GR_n = 2GR_{n-2} + GR_{n-3}, \ n \geq 3$$

and initial values are $GR_0 = 1 - i, GR_1 = 1 + i, GR_2 = 1 + i$.

The first few values of GR_n are $1-i,\ 1+i,\ 1+i,\ 3+i,\ 3+3i,\ 7+3i,\ 9+7i,\ 17+9i$.

Theorem 3.2. *The generating function of Gaussian Pell-Padovan sequence is*

$$f(x) = \frac{1 - i + (1+i)x + (-1+3i)x^2}{1 - 2x^2 - x^3}.$$

Proof. Let

$$f(x)=\sum_{n=0}^{\infty} GR_n x^n$$

be the generating function of the Gaussian Pell-Padovan sequence. In this case, we have

$$2x^2 f(x)=2GR_0x^2+2GR_1x^3+2GR_2x^4+\cdots+2GR_{n-2}x^n+\cdots$$

and

$$x^3 f(x)=GR_0x^3+GR_1x^4+GR_2x^5+\cdots+GR_{n-3}x^n+\cdots$$

so we obtain

$$\begin{aligned}(1-2x^2-x^3)f(x) &= GR_0+GR_1x+(GR_2-2GR_0)x^2+(GR_3-2GR_1-GR_0)x^3\\ &\quad +\cdots+(GR_n-2GR_{n-2}-GR_{n-3})x^n+\cdots .\end{aligned}$$

On the other hand, since $GR_0=1-i,\ GR_1=1+i,\ GR_2=1+i$ and $GR_n=2GR_{n-2}+GR_{n-3}$, then we have

$$f(x)=\frac{1-i+(1+i)x+(-1+3i)x^2}{1-2x^2-x^3}$$

which is desired. □

Theorem 3.3. *The Binet-like formula of Gaussian Pell-Padovan sequence is*

$$GR_n=\frac{2}{\sqrt{5}}\left[\alpha-1+i\left(1-\frac{1}{\alpha}\right)\right]\alpha^n-\frac{2}{\sqrt{5}}\left[\beta-1+i\left(1-\frac{1}{\beta}\right)\right]\beta^n+(i-1)\gamma^n$$

where

$$\alpha=\frac{1+\sqrt{5}}{2},\beta=\frac{1-\sqrt{5}}{2},\gamma=1$$

are roots of the equation $x^3-2x-1=0$.

Proof. The Binet-like formula of Pell-Padovan sequence is given by

$$R_n=2\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}-2\frac{\alpha^n-\beta^n}{\alpha-\beta}+\gamma^{n+1}.$$

Now consider

$$GR_n=R_n+iR_{n-1}$$

so the proof is easily seen. □

Theorem 3.4. $\sum_{j=0}^{n} GR_j=\frac{1}{2}\left[(-1-3i)-GR_{n+1}+GR_{n+2}+GR_{n+3}\right].$

Proof. We find that

$$\sum_{j=0}^{n} R_j=\frac{1}{2}(-1-R_{n+1}+R_{n+2}+R_{n+3})$$

and

$$\sum_{j=0}^{n} R_{j-1} = \frac{1}{2}(-3 - 2R_n - R_{n+1} + R_{n+2} + R_{n+3}).$$

Since

$$GR_n = R_n + iR_{n-1}$$

we have

$$\sum_{j=0}^{n} GR_j = \sum_{j=0}^{n} R_j + i\sum_{j=0}^{n} R_{j-1}$$

So the theorem is proved. □

Theorem 3.5. $\sum_{j=1}^{n} GR_{2j} = R_{2n+1} + iR_{2n} - (n+1) + i(n-1)$.

Proof. If we consider the following equalities, then the proof is seen:

$$\begin{aligned} \sum_{j=1}^{n} R_{2j} &= R_{2n+1} - (n+1) \\ \sum_{j=1}^{n} R_{2j-1} &= R_{2n} + (n-1) \end{aligned}$$

and

$$\sum_{j=1}^{n} GR_{2j} = \sum_{j=1}^{n} R_{2j} + i\sum_{j=1}^{n} R_{2j-1}$$

□

Theorem 3.6. $\sum_{j=1}^{n} \binom{n}{j} GR_j = GR_{2n} + (1-i)$.

Proof. Considering the following equalities:

$$\begin{aligned} \sum_{j=1}^{n} \binom{n}{j} R_j &= R_{2n} + 1 \\ \sum_{j=1}^{n} \binom{n}{j} R_{j-1} &= R_{2n-1} - 1 \end{aligned}$$

and

$$\sum_{j=1}^{n} \binom{n}{j} GR_j = \sum_{j=1}^{n} \binom{n}{j} R_j + i\sum_{j=1}^{n} \binom{n}{j} R_{j-1}$$

then the proof is easily seen. □

Now we shall give the new properties of Gaussian Pell-Padovan numbers relation with Pell-Padovan matrix.

Theorem 3.7. *If we take the following matrices*

$$Q_3 = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \ K_3 = \begin{bmatrix} 1+i & 1+i & 1-i \\ 1+i & 1-i & -1+3i \\ 1-i & -1+3i & 3-5i \end{bmatrix}$$

and

$$S_3^n = \begin{bmatrix} GR_{n+2} & GR_{n+1} & GR_n \\ GR_{n+1} & GR_n & GR_{n-1} \\ GR_n & GR_{n-1} & GR_{n-2} \end{bmatrix}.$$

then

$$Q_3^n.K_3 = S_3^n \text{ for all } n \in \mathbb{Z}^+.$$

Theorem 3.8. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}^n \begin{bmatrix} 1-i \\ 1+i \\ 1+i \end{bmatrix} = \begin{bmatrix} GR_n \\ GR_{n+1} \\ GR_{n+2} \end{bmatrix}$ *for all* $n \in Z^+$.

We note that for the proofs Theorem 3.7 and Theorem 3.8 are used induction on n.

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