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WRONSKIAN SOLUTIONS OF (2+1) DIMENSIONAL NON-LOCAL ITO EQUATION

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Abstract. In this work, the Wronskian determinant technique is performed to (2+1)-dimensional non-local Ito equation in the bilinear form. First, we obtain some sufficient conditions in order to show Wronskian determinant solves the (2+1)-dimensional non-local Ito equation. Second, rational solutions, soliton solutions, positon solutions, negaton solutions and their interaction solutions were deduced by using the Wronskian formulations

1. Introduction

The nonlinear evolution equations (NLEEs) model abundant physical processes which occur in the nature. Therefore, investigating and obtaining solutions of these type equations have an extremely important place in nonlinear science. In this context, in the literature a plenty of analytic and numerical methods were developed such as inverse scattering transform, Hirota bilinear method, the Riccati equation expansion method, the sine–cosine method, the $\tanh-\mathrm{sech}$ method, G'/G expansion method, Adomian decomposition method, He's variational principle, Lie symmetry method and many more ([1],[3]-[6]-[7], [8],[14], [19]-[20], [22]-[23]).

Nowadays, besides to above aforementioned methods, the Wronskian determinant method ([5], [15]) depending upon Hirota bilinear forms has a wide range of impact and applicability on the NLEES. Wronskian determinant technique is a important tool to get exact solutions to the corresponding Hirota bilinear equations of the NLEE equations.

In [11], we observe that there is a bridge between Wronskian solutions and generalized Wronskian solutions. It gives us a way to obtain generalized Wronskian solutions simply from Wronskian determinants. The basic idea was used to generate positons, negatons and their interaction solutions through the Wronskian formulation.

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It is demonstrated in [12] that for each type of Jordan blocks of the coefficient matrix $J(\lambda_{ij})$, there exist special sets of eigenfunctions. These functions were used to generate rational solutions, solitons, positons, negatons, breathers, complexitons and their interaction solutions. The obtained solution formulas of the representative systems allow us to construct more general Wronskian solutions than rational solutions, positons, negatons, complexitons and their interaction solutions.

As stated in [13], integrable equations can have three different kinds of explicit exact transcendental function solutions: negatons, positons and complexitons. Solitons are usually a specific class of negatons. Roughly speaking, negatons and positons are solutions which involve exponential functions and trigonometric functions of space variables, respectively, and they are all associated with real eigenvalues of the associated spectral problems. But complexitons are different solutions which involve both exponential and trigonometric functions of space variables, and they are associated with complex eigenvalues of the associated spectral problems. Interaction solutions among negatons, positons, rational solutions and complexitons are a class of much more general and complicated solutions to soliton equations, in the category of elementary function solutions.

The generalized (2+1) dimensional non-local Ito equation

$$u_{tt} + u_{xxxt} + 3\left(2u_x u_t + u u_{xt}\right) + 3u_{xx}\left(\int u_t \, dx\right) + a u_{yt} + b u_{xt} = 0. \qquad (1)$$

was firstly studied by Ito for generalizing the bilinear Korteweg-de Vries (KdV) equation [9]. To get rid of the integral operator, we use the transformation

$$u = v_x$$

to cast (1) into the following equation

$$v_{xtt} + v_{xxxxt} + 3\left(2v_{xx}v_{xt} + v_x v_{xxt}\right) + 3v_{xxx}v_t + a v_{xyt} + b v_{xxt} = 0. \qquad (2)$$

We observe increasing interest for Eq.(2) in the literature ([2], [4], [18],[21]). For instance in [21], Wazwaz obtains single soliton solutions and periodic solutions of Eq.(2) by tanh-coth method. He also constructs multiple-soliton solutions of sech-squared type by using Hirota bilinear method. In [2], Adem constructs multiple wave solutions of Eq.(2) by exploiting the multiple exp-function algorithm.

To solve Eq.(2) we can get dependent variable v by

$$v = \alpha\left(\ln f\right)_x \sim \begin{cases} v = \alpha w_x \\ w = \ln f \end{cases} \qquad (3)$$

where $f(x, y, t)$ is an unknown real function which will be determined. Substituting Eq.(3) into Eq. (2), we have

$$\alpha w_{xxtt} + \alpha w_{xxxxxt} + 3\left(2\alpha^2 w_{xxx}w_{xxt} + \alpha^2 w_{xx}w_{xxxt}\right) + 3\alpha^2 w_{xxxx}w_{xt} + \alpha a w_{xxyt} + \alpha b w_{xxxt} = 0, \qquad (4)$$

which can be integrated twice with respect to x to give

$$\alpha w_{tt} + \alpha w_{xxxt} + 3\alpha^2 w_{xt} w_{xx} + \alpha a w_{yt} + \alpha b w_{xt} = C, \tag{5}$$

where C is the constant of integration.

If we get

$$6\alpha = 3\alpha^2, \alpha = 2,$$

then (5) can be written as

$$w_{tt} + w_{xxxt} + 6 w_{xt} w_{xx} + a w_{yt} + b w_{xt} = C. \tag{6}$$

Substituting $w = \ln f$ into Eq. (6), we get

$$\frac{f_{tt}}{f} - \frac{f_t^2}{f^2} + \frac{f_{xxxt}}{f} - \frac{f_{xxx} f_t}{f^2} - \frac{3 f_{xxt} f_x}{f^2} + \frac{3 f_{xx} f_{xt}}{f^2} + \frac{a f_{yt}}{f} - \frac{a f_y f_t}{f^2} + \frac{b f_{xt}}{f} - \frac{b f_x f_t}{f^2} = C. \tag{7}$$

Substituting $C = 0$ into Eq. (7) and employing Hitora derivative operators [8] we obtain the Hitora bilinear form of Eq.(2) as

$$\left(D_t^2 + D_x^3 D_t + a D_y D_t + b D_x D_t\right) f.f$$

$$= f(f_{xxxt} + f_{tt} + a f_{yt} + b f_{xt}) + 3 f_{xx} f_{xt} - f_t^2 - f_{xxx} f_t - 3 f_{xxt} f_x - a f_y f_t - b f_x f_t. \tag{8}$$

In this work, our intention is to present the generalized Wronskian solutions of the Eq. (2). The generalized Wronskian solutions are obtained through Wronskian solutions. The generalized Wronskian solutions can be viewed as Wronskian solutions. Solitons are examples of Wronskian solutions, and positons and negatons are examples of generalized Wronskian solutions ([11]-[10]).

The paper is organized as follows. In Section 2, the Wronskian determinant solution is deduced for Hirota bilinear form corresponding to Eq. (2). In Section 3, using Wronskian formulation rational solutions, solitons, positons, negatons and their interaction solutions are presented. Lastly, conclusions are given in Section 4.

2. **Wronskian formulation**

We first present notation to be used and recall the definitions and theorems that appear in ([5],[15]-[17]).

The solutions determined by $v = 2\left(\ln f\right)_x$ with $f = |\widehat{N-1}|$ and

$$W(\phi_1, \phi_2, , , , , \phi_n) = (\widehat{N-1}; \Phi) = |\widehat{N-1}| = \begin{vmatrix} \phi_1^{(0)} & \phi_1^{(1)} & .. & \phi_1^{(N-1)} \\ \phi_2^{(0)} & \phi_2^{(1)} & .. & \phi_2^{(N-1)} \\ . & . & . & . \\ . & . & . & . \\ \phi_N^{(0)} & \phi_N^{(1)} & .. & \phi_N^{(N-1)} \end{vmatrix}, \ N \geq 1, \tag{9}$$

where

$$\Phi = (\phi_1, \phi_2, , , , , \phi_n)^T, \quad \phi_i^{(0)} = \phi_i\ , \quad \phi_i^{(j)} = \frac{\partial^j}{\partial x^j} \phi_i\ , \quad j \geq 1\ , \quad 1 \leqslant i \leqslant N\ . \tag{10}$$

to the Eq. (2) will be called Wronskian solutions ([5],[15] and [17]). Now, we give the following important properties on determinants ([17]).

Property 1. If D is $N*(N-2)$ matrix, and a, b, c, d are n-dimensional column vectors then,

$$|D,a,b|\,|D,c,d| - |D,a,c|\,|D,b,d| + |D,a,d|\,|D,b,c| = 0\ . \tag{11}$$

Property 2. If $a_j(j=1,...,n)$ is an n-dimensional column vector, and $b_j(j=1,...,n)$ is a real constant different form zero then

$$\sum_{i=1}^{N} b_i\,|a_1,a_2,....,a_N| = \sum_{j=1}^{N} |a_1,a_2,....,ba_j,....,a_N|\,, \tag{12}$$

where $ba_j = (b_1 a_{1j}, b_2 a_{2j}, ..., b_N a_{Nj})^T$.

Property 3.

$$\begin{aligned}|\widehat{N-1}|\sum_{i=1}^{N}\lambda_{ii}(t)\left(\sum_{i=1}^{N}\lambda_{ii}(t)|\widehat{N-1}|\right) &= |\widehat{N-1}|(|\widehat{N-5},N-3,N-2,N-1,N| \\ &-|\widehat{N-4},N-2,N-1,N+1| - |\widehat{N-3},N-1,N+2|| \\ &+2|\widehat{N-3},N,N+1| + |\widehat{N-2},N+3).\end{aligned} \tag{13}$$

Now, we present a set of sufficient conditions consisting of systems of linear partial differential equations which guarantees that the Wronskian determinant solves the Eq. (2) in the bilinear form (8). Upon solving the linear conditions, the resulting Wronskian formulations bring solution formulas, which can yield rational solutions, solitons, negatons, positons and interaction solutions. Also, positons, negatons and their interaction solutions are called the generalized Wronskian solutions ([11]).

Theorem 1. *Assuming that $\phi_i = \phi_i(x,y,t)$ (where $i = 1,2,...,N$) satisfies the following linear partial differential equations (LPDEs)*

$$\phi_{i,xx} = \sum_{j=1}^{N}\lambda_{ij}(t)\phi_j\ , \tag{14}$$

$$\phi_{i,t} = m\phi_{i,x}\ , \tag{15}$$

$$\phi_{i,y} = n\phi_{i,xxx} + k\phi_{i,x} \tag{16}$$

with

$$n = -\frac{4}{a}\ ,\ m = -(b+ak)$$

then $f = |\widehat{N-1}|$ defined by (9) solves the bilinear Eq. (8).

Proof. Considering (9), we can obtain the following derivatives

$$f = |\widehat{N-1}|$$

$$f_x = |\widehat{N-2},N|$$

$$f_{xx} = |\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1|$$

$$f_{xxx} = |\widehat{N-4}, N-2, N-1, N| + 2|\widehat{N-3}, N-1, N+1| + |\widehat{N-2}, N+2|.$$

In addition, keeping in mind the conditions of (15)-(16), we can produce that

$$f_t = m|\widehat{N-2}, N|$$

$$f_{xt} = m(|\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1|)$$

$$f_{tt} = m^2(|\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1|)$$

$$f_y = n|\widehat{N-4}, N-2, N-1, N| - n|\widehat{N-3}, N-1, N+1| + n|\widehat{N-2}, N+2| + k|\widehat{N-2}, N|$$

$$f_{yt} = mn|\widehat{N-5}, N-3, N-2, N-1, N| - mn|\widehat{N-3}, N, N+1| + mn|\widehat{N-2}, N+3| + mk|\widehat{N-3}, N-1, N| + mk|\widehat{N-2}, N+1|$$

$$f_{xxt} = m(|\widehat{N-4}, N-2, N-1, N| + 2|\widehat{N-3}, N-1, N+1| + |\widehat{N-2}, N+2|)$$

$$f_{xxxt} = m(|\widehat{N-5}, N-3, N-2, N-1, N| + 3|\widehat{N-4}, N-2, N-1, N+1| + 2|\widehat{N-3}, N, N+1| + 3|\widehat{N-3}, N-1, N+2| + |\widehat{N-2}, N+3|)$$

Therefore, we can compute all terms in Eq.(8) such as

$$\begin{aligned} 3f_{xx}f_{xt} &= 3m(|\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1|)(|\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1|) \\ &= 3m(|\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1|)^2 \\ &= 3m(|\widehat{N-2}, N+1| - |\widehat{N-3}, N-1, N| + 2|\widehat{N-3}, N-1, N|)^2 \\ &= 3m(|\widehat{N-2}, N+1| - |\widehat{N-3}, N-1, N|)^2 + 12m|\widehat{N-3}, N-1, N||\widehat{N-2}, N+1|, \end{aligned} \tag{17}$$

$$ff_{xxxt} = m|\widehat{N-1}|(|\widehat{N-5}, N-3, N-2, N-1, N| + 3|\widehat{N-4}, N-2, N-1, N+1| + 2|\widehat{N-3}, N, N+1| + 3|\widehat{N-3}, N-1, N+2| + |\widehat{N-2}, N+3|),$$

$$ff_{tt} = m^2|\widehat{N-1}(|\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1|),$$

$$aff_{yt} = a|\widehat{N-1}|(mn|\widehat{N-5}, N-3, N-2, N-1, N| - mn|\widehat{N-3}, N, N+1| + mn|\widehat{N-2}, N+3| + mk|\widehat{N-3}, N-1, N| + mk|\widehat{N-2}, N+1|),$$

$$bff_{xt} = bm|\widehat{N-1}|(|\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1|),$$

$$\begin{aligned} f(f_{xxxt} + f_{tt} + af_{yt} + bf_{xt}) &= |\widehat{N-1}|((m + amn)\,|\widehat{N-5}, N-3, N-2, N-1, N| \\ &+ 3m|\widehat{N-4}, N-2, N-1, N+1| + (2m - amn)\,|\widehat{N-3}, N, N+1| \\ &+ 3m|\widehat{N-3}, N-1, N+2| + (m + amn)\,|\widehat{N-2}, N+3| + \left(m^2 + amk + bm\right)|\widehat{N-3}, N-1, N| \\ &+ \left(m^2 + amk + bm\right)|\widehat{N-2}, N+1|). \end{aligned} \tag{18}$$

We can obtain from Eq. (17) and Eq. (18) (Property 3)

$$m + amn = -3m$$

$$n = -\frac{4}{a}$$

and

$$m^2 + amk + bm = 0$$

$$m = -(b + ak).$$

Then, Eq. (8) can be rewritten as the following

$$f(f_{xxxt} + f_{tt} + af_{yt} + bf_{xt}) = -3m|\widehat{N-1}|(|\widehat{N-5}, N-3, N-2, N-1, N|$$

$$-|\widehat{N-4}, N-2, N-1, N+1| - 2|\widehat{N-3}, N, N+1| - |\widehat{N-3}, N-1, N+2|$$
$$+|\widehat{N-2}, N+3|)$$

$$= -3m(|\widehat{N-2}, N+1| - |\widehat{N-3}, N-1, N|)^2 + 12m|\widehat{N-3}, N, N+1||\widehat{N-1}| \quad (19)$$

and

$$-f_t^2 = -m^2|\widehat{N-2}, N|^2$$

$$-f_{xxx}f_t = -m|\widehat{N-2}, N|(|\widehat{N-4}, N-2, N-1, N| + 2|\widehat{N-3}, N-1, N+1| + |\widehat{N-2}, N+2|)$$

$$-3f_{xxt}f_x = -3m|\widehat{N-2}, N|(|\widehat{N-4}, N-2, N-1, N| + 2|\widehat{N-3}, N-1, N+1| + |\widehat{N-2}, N+2|)$$

$$-af_yf_t = -am|\widehat{N-2}, N|(n|\widehat{N-4}, N-2, N-1, N| - n|\widehat{N-3}, N-1, N+1| + n|\widehat{N-2}, N+2|$$

$$+k|\widehat{N-2}, N|) - bf_xf_t = -bm|\widehat{N-2}, N||\widehat{N-2}, N| = -bm|\widehat{N-2}, N|^2$$

$$-f_t^2 - f_{xxx}f_t - 3f_{xxt}f_x - af_yf_t - bf_xf_t = -12m|\widehat{N-3}, N-1, N+1||\widehat{N-2}, N| \quad (20)$$

After substituting of the Eq. (17),(19) and (20) into (8) we obtain the following Plücker relation:

$$\left(D_t^2 + D_x^3D_t + aD_yD_t + bD_xD_t\right)ff = 12m|\widehat{N-3}, N-1, N||\widehat{N-2}, N+1|$$

$$+12m|\widehat{N-3}, N, N+1||\widehat{N-1}| - 12m|\widehat{N-3}, N-1, N+1||\widehat{N-2}, N|$$

As result of Property 1, we get

$$12m|\widehat{N-3}, N-1, N||\widehat{N-2}, N+1| + 12m|\widehat{N-3}, N, N+1||\widehat{N-1}|$$
$$- 12m|\widehat{N-3}, N-1, N+1||\widehat{N-2}, N| = 0.$$

□

This demonstrates that $f = |\widehat{N-1}|$ solves the bilinear Eq. (8). The corresponding solution of Eq. (2) is

$$v = 2\left(\ln f\right)_x = \frac{2f_x}{f} = 2\frac{|\widehat{N-2}, N|}{|\widehat{N-1}|}$$

3. Wronskian solutions of Eq.(2)

In this section, new exact solutions including rational solutions, soliton solutions, positon solutions, negaton solutions and their interaction solutions are formally derived to Eq.(8) ([11]-[10]).

The Jordan form of a real matrix

$$A=\begin{bmatrix} J(\lambda_1) & & & & 0 \\ 1 & J(\lambda_2) & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ 0 & & & 1 & J(\lambda_m) \end{bmatrix}_{n\mathrm{x}n}$$

has the following type of block:

$$J(\lambda_i)=\begin{bmatrix} \lambda_i & & & & 0 \\ 1 & \lambda_i & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ 0 & & & 1 & \lambda_i \end{bmatrix}_{k_i\mathrm{x}k_i}$$

This type of block has the real eigenvalue λ_i.

3.1. Rational solutions.

Let's assume that $J(\lambda_1)$ is

$$J(\lambda_1)=\begin{bmatrix} \lambda_1 & & & & 0 \\ 1 & \lambda_1 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ 0 & & & 1 & \lambda_1 \end{bmatrix}_{k_1\mathrm{x}k_1}$$

If the eigenvalue $\lambda_1=0$, then $J(\lambda_1)$ becomes to the following form:

$$\begin{bmatrix} 0 & & & & 0 \\ 1 & 0 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ 0 & & & 1 & 0 \end{bmatrix}_{k_1\mathrm{x}k_1}$$

Then the conditions (14)-(16), convert to

$$\phi_{1,xx}=0\ ,\ \phi_{i+1,xx}=\phi_i\ ,\ \phi_{i,t}=-(b+ak)\phi_{i,x},$$
$$\phi_{i,y}=-\frac{4}{a}\phi_{i,xxx}+k\phi_{i,x}\ ,\ i\geq 1 \tag{21}$$

If we can obtain the functions of $\phi_i (i \geq 1)$ from Eq.(21) then

$$v = 2\partial_x \ln W(\phi_1, \phi_2, ..., \phi_{k_1})$$

is called a rational Wronskian solution of order k_1.

After solving

$$\phi_{1,xx} = 0 \ , \quad \phi_{1,t} = -(b+ak)\phi_{1,x} \ , \quad \phi_{1,y} = -\frac{4}{a}\phi_{1,xxx} + k\phi_{1,x}$$

we get

$$\phi_1 = c_1 \left(x + ky - (b+ak)t\right) + c_2.$$

where c_1, c_2 and $k \neq 0$ are all real constants.

Similarly, by solving

$$\phi_{i+1,xx} = \phi_i \ , \ \phi_{i+1,t} = -(b+ak)\phi_{i+1,x} \ , \quad \phi_{i+1,y} = -\frac{4}{a}\phi_{i+1,xxx} + k\phi_{i+1,x} \ , \ i \geq 1 \ ,$$

then zero,first and second order rational solutions can be achieved.

1) Zero-order: When $c_1 = 1, c_2 = 0, \phi_1 = x + ky - (b+ak)t$, we have the corresponding Wronskian determinant $f = W(\phi_1) = x + ky - (b+ak)t$, and the associated rational Wronskian solution of zero-order:

$$v = 2\partial_x \ln W(\phi_1) = \frac{2}{x + ky - (b+ak)t} \tag{22}$$

2) First-order: When $c_1 = 1, c_2 = 0, \phi_1 = x + ky - (b+ak)t$, we have $\phi_2 = \frac{(x+ky-(b+ak)t)^3}{6} - \frac{4y}{a}$ and the corresponding Wronskian determinant $f = W(\phi_1, \phi_2) = \frac{(x+ky-(b+ak)t)^3}{3} + \frac{4y}{a}$, and the associated rational Wronskian solution of first-order

$$v = 2\partial_x \ln W(\phi_1, \phi_2) = \frac{2\left(x + ky - (b+ak)t\right)^2}{\frac{(x+ky-(b+ak)t)^3}{3} + \frac{4y}{a}} \tag{23}$$

3) Second-order: When $\phi_1 = x + ky - (b+ak)t, \ \phi_2 = \frac{(x+ky-(b+ak)t)^3}{6} - \frac{4y}{a}$, we have $\phi_3 = \frac{(x+ky-(b+ak)t)^5}{120} - \frac{2y(x+ky-(b+ak)t)^2}{a}$ and the corresponding Wronskian determinant $f = W(\phi_1, \phi_2, \phi_3) = \frac{(x+ky-(b+ak)t)^6}{45} + \frac{4y(x+ky-(b+ak)t)^3}{3a} - \frac{16y^2}{a^2}$, and the associated rational Wronskian solution of second-order

$$v = 2\partial_x \ln W(\phi_1, \phi_2, \phi_3) = \frac{\frac{4(x+ky-(b+ak)t)^5}{15} + \frac{8y(x+ky-(b+ak)t)^2}{a}}{\frac{(x+ky-(b+ak)t)^6}{45} + \frac{4y(x+ky-(b+ak)t)^3}{3a} - \frac{16y^2}{a^2}} \tag{24}$$

3.2. **Solitons, negatons and positons.** If the eigenvalue $\lambda_1 \neq 0$, $J(\lambda_1)$ becomes to the following form

$$\begin{bmatrix} \lambda_1 & & & & 0 \\ 1 & \lambda_1 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ 0 & & & 1 & \lambda_1 \end{bmatrix}_{k_1 \mathrm{x} k_1}$$

We start from the eigenfunction $\phi_1(\lambda_1)$, which is determined by

$$(\phi_1(\lambda_1))_{xx} = \lambda_1 \phi_1(\lambda_1)\ ,\ (\phi_1(\lambda_1))_t = -(b+ak)\,(\phi_1(\lambda_1))_x\ ,$$
$$(\phi_1(\lambda_1))_y = -\frac{4}{a}\,(\phi_1(\lambda_1))_{xxx} + k\,(\phi_1(\lambda_1))_x \tag{25}$$

General solutions to this system in two cases of $\lambda_1 > 0$ and $\lambda_1 < 0$ are

$$\begin{aligned} \phi_1(\lambda_1) &= C_1 \sinh\left(\sqrt{\lambda_1}\left(x+ky-(b+ak)\,t-\frac{4y\lambda_1}{a}\right)\right) \\ &\quad + C_2 \cosh\left(\sqrt{\lambda_1}\left(x+ky-(b+ak)\,t-\frac{4y\lambda_1}{a}\right)\right) \end{aligned} \tag{26}$$

when $\lambda_1 > 0$,

$$\begin{aligned} \phi_1(\lambda_1) &= C_3 \cos\left(\sqrt{-\lambda_1}\left(x+ky-(b+ak)\,t-\frac{4y\lambda_1}{a}\right)\right) \\ &\quad - C_4 \sin\left(\sqrt{-\lambda_1}\left(x+ky-(b+ak)\,t-\frac{4y\lambda_1}{a}\right)\right) \end{aligned} \tag{27}$$

when $\frac{ak}{4} < \lambda_1 < 0$ respectively, where C_1, C_2, C_3 and C_4 are arbitrary real constants.

1) Solitons: The $n-$soliton solution is a special $n-$negaton:

$$v = 2\partial_x \ln W(\phi_1, \phi_2, ..., \phi_n)$$

with ϕ_i given by

$$\phi_i = \cosh\left(\sqrt{\lambda_i}\left(x+ky-(b+ak)\,t-\frac{4y\lambda_i}{a}\right)+\gamma_i\right),\quad i \text{ odd},$$
$$\phi_i = \sinh\left(\sqrt{\lambda_i}\left(x+ky-(b+ak)\,t-\frac{4y\lambda_i}{a}\right)+\gamma_i\right),\quad i \text{ even},$$

where $0 < \lambda_1 < \lambda_2.... < \lambda_n$ and γ_i $(1 \leq i \leq n)$ are arbitrary real constants.

Zero-order:

$$\begin{aligned} v &= 2\partial_x \ln W(\phi_1) = 2\partial_x \ln\left(\cosh\left(\sqrt{\lambda_1}\left(x+ky-(b+ak)\,t-\frac{4y\lambda_1}{a}\right)+\gamma_1\right)\right) \\ &= 2\sqrt{\lambda_1}\tanh(\theta_1) \end{aligned} \tag{28}$$

$$\begin{aligned} v &= 2\partial_x \ln W(\phi_1) = 2\partial_x \ln\left(\sinh\left(\sqrt{\lambda_1}\left(x+ky-(b+ak)\,t-\frac{4y\lambda_1}{a}\right)+\gamma_1\right)\right) \\ &= 2\sqrt{\lambda_1}\coth(\theta_1) \end{aligned} \tag{29}$$

where $\theta_1 = \sqrt{\lambda_1}\left(x+ky-(b+ak)\,t-\frac{4y\lambda_1}{a}\right)+\gamma_1, \quad \lambda_1>0$

First-order:

$$\begin{aligned} v &= 2\partial_x \ln W(\cosh(\phi_1),\sinh(\phi_2)) \\ &= \frac{2\left(\lambda_1-\lambda_2\right)\left(\sinh\left(\theta_1+\theta_2\right)-\sinh\left(\theta_1-\theta_2\right)\right)}{\left(\sqrt{\lambda_1}-\sqrt{\lambda_2}\right)\cosh\left(\theta_1+\theta_2\right)-\left(\sqrt{\lambda_1}+\sqrt{\lambda_2}\right)\cosh\left(\theta_1-\theta_2\right)} \end{aligned} \tag{30}$$

where $\theta_i = \sqrt{\lambda_i}\left(x+ky-(b+ak)\,t-\frac{4y\lambda_i}{a}\right)+\gamma_i, \quad \lambda_i>0, \quad i=1,2.$

2) Positons: We obtain two special positon solutions as the following

$$v = 2\partial_x \ln W(\phi,\partial_\lambda\phi,.....,\partial_\lambda^{k-1}\phi)$$

$$\phi(\lambda) = \cos\left(\sqrt{-\lambda}\left(x+ky-(b+ak)\,t-\frac{4y\lambda}{a}\right)+\gamma\right) \qquad \lambda<0,$$

$$\phi(\lambda) = \sin\left(\sqrt{-\lambda}\left(x+ky-(b+ak)\,t-\frac{4y\lambda}{a}\right)+\gamma\right) \qquad \lambda<0.$$

Zero-order:

$$\begin{aligned} v &= 2\partial_x \ln W(\phi_1) = 2\partial_x \ln\left(\cos\left(\sqrt{-\lambda_1}\left(x+ky-(b+ak)\,t-\frac{4y\lambda_1}{a}\right)+\gamma_1\right)\right) \\ &= -2\sqrt{-\lambda_1}\tan(\theta_3) \end{aligned} \tag{31}$$

$$\begin{aligned} v &= 2\partial_x \ln W(\phi_1) = 2\partial_x \ln\left(\sin\left(\sqrt{-\lambda_1}\left(x+ky-(b+ak)\,t-\frac{4y\lambda_1}{a}\right)+\gamma_1\right)\right) \\ &= 2\sqrt{-\lambda_1}\cot(\theta_3) \end{aligned} \tag{32}$$

where $\theta_3 = \sqrt{-\lambda_1}\left(x+ky-(b+ak)\,t-\frac{4y\lambda_1}{a}\right)+\gamma_1$

First-order:

$$v = 2\partial_x \ln W\left(\cos(\theta),\partial_{\lambda_1}\cos(\theta)\right) = \frac{4\sqrt{-\lambda_1}\left(1+\cos(2\theta)\right)}{2\sqrt{-\lambda_1}\left(x+ky-(b+ak)\,t-\frac{12y\lambda_1}{a}\right)+\sin(2\theta)} \tag{33}$$

where $\theta = \sqrt{-\lambda_1}\left(x+ky-(b+ak)\,t-\frac{4y\lambda_1}{a}\right)+\gamma_1.$

3) Negatons: We obtain two special negaton solutions as the following

$$v = 2\partial_x \ln W(\phi,\partial_\lambda\phi,.....,\partial_\lambda^{k-1}\phi)$$

$$\phi = \cosh\left(\sqrt{\lambda}\left(x+ky-(b+ak)\,t-\frac{4y\lambda}{a}\right)+\gamma\right)$$

$$\phi = \sinh\left(\sqrt{\lambda}\left(x + ky - (b + ak)\,t - \frac{4y\lambda}{a}\right) + \gamma\right)$$

where $\lambda > 0$ and γ is an arbitrary constant.

First-order:

$$v = 2\partial_x \ln W\left(\cosh(\theta), \partial_{\lambda_1}\cosh(\theta)\right) = \frac{4\sqrt{\lambda_1}\,(1 + \cosh(2\theta))}{2\sqrt{\lambda_1}\left(x + ky - (b + ak)\,t - \frac{12y\lambda_1}{a}\right) + \sinh(2\theta)} \tag{34}$$

where $\theta = \sqrt{\lambda_1}\left(x + ky - (b + ak)\,t - \frac{4y\lambda_1}{a}\right) + \gamma_1$

3.3. **Interaction solutions.** A Wronskian solution $v = 2\partial_x \ln W(\phi_1(\lambda), \phi_2(\lambda), ..., \phi_k(\lambda); \psi_1(\mu), ..., \psi_l(\mu))$ will be called as Wronskian interaction solution between two solutions determined by the two sets of eigenfunctions

$$(\phi_1(\lambda), \phi_2(\lambda), ..., \phi_k(\lambda); \psi_1(\mu), ..., \psi_l(\mu)) \tag{35}$$

Moreover, one can generate more general Wronskian interaction solutions for instance using the rational solutions, negatons and positons.

Now, our aim is to demonstrate some special Wronskian interaction solutions. First, we consider the following eigenfunctions:

$$\phi_{\text{rational}} = x + ky - (b + ak)\,t$$

$$\phi_{\text{soliton}} = \cosh\left(\sqrt{\lambda_1}\left(x + ky - (b + ak)\,t - \frac{4y\lambda_1}{a}\right)\right)$$

$$\phi_{\text{positon}} = \cos\left(\sqrt{-\lambda_2}\left(x + ky - (b + ak)\,t - \frac{4y\lambda_2}{a}\right)\right)$$

where $\lambda_1 > 0$, $\lambda_2 < 0$ are constants.

We get the following Wronskian interaction determinants using the rational, a single soliton and a single positon solutions

$$W(\phi_{\text{rational}}, \phi_{\text{soliton}}) = \sqrt{\lambda_1}\,(x + ky - (b + ak)\,t)\sinh(\theta_1) - \cosh(\theta_1) \tag{36}$$

$$W(\phi_{\text{rational}}, \phi_{\text{positon}}) = -\sqrt{-\lambda_2}\,(x + ky - (b + ak)\,t)\sin(\theta_2) - \cos(\theta_2) \tag{37}$$

$$W(\phi_{\text{soliton}}, \phi_{\text{positon}}) = -\sqrt{-\lambda_2}\cosh(\theta_1)\sin(\theta_2) - \sqrt{\lambda_1}\sinh(\theta_1)\cos(\theta_2) \tag{38}$$

where $\theta_1 = \sqrt{\lambda_1}\left(x + ky - (b + ak)\,t - \frac{4y\lambda_1}{a}\right)$, $\theta_2 = \sqrt{-\lambda_2}\left(x + ky - (b + ak)\,t - \frac{4y\lambda_2}{a}\right)$

Then, the corresponding Wronskian interaction solutions are

$$v = 2\partial_x \ln W(\phi_{\text{rational}}, \phi_{\text{soliton}}) = \frac{2\sqrt{\lambda_1}\,(x + ky - (b + ak)\,t)\cosh(\theta_1)}{\sqrt{\lambda_1}\,(x + ky - (b + ak)\,t)\sinh(\theta_1) - \cosh(\theta_1)} \tag{39}$$

$$v = 2\partial_x \ln W(\phi_{\text{rational}}, \phi_{\text{positon}}) = \frac{-2\lambda_2\,(x + ky - (b + ak)\,t)\cos(\theta_2)}{\sqrt{-\lambda_2}\,(x + ky - (b + ak)\,t)\sin(\theta_2) + \cos(\theta_2)} \tag{40}$$

$$v = 2\partial_x \ln W(\phi_{\text{soliton}}, \phi_{\text{positon}}) = \frac{2\,(\lambda_1 - \lambda_2)\cosh(\theta_1)\cos(\theta_2)}{\sqrt{-\lambda_2}\cosh(\theta_1)\sin(\theta_2) + \sqrt{\lambda_1}\sinh(\theta_1)\cos(\theta_2)} \tag{41}$$

where $\theta_1 = \sqrt{\lambda_1}\left(x + ky - (b+ak)\,t - \frac{4y\lambda_1}{a}\right)$, $\theta_2 = \sqrt{-\lambda_2}\left(x + ky - (b+ak)\,t - \frac{4y\lambda_2}{a}\right)$

The following is one Wronskian interaction determinant and solution involving the three eigenfunctions.

$$\begin{aligned} W(\phi_{\text{rational}}, \phi_{\text{soliton}}, \phi_{\text{positon}}) &= (x + ky - (b+ak)\,t) \\ &\quad \times \left(\lambda_2\sqrt{\lambda_1}\sinh(\theta_1)\cos(\theta_2) + \lambda_1\sqrt{-\lambda_2}\cosh(\theta_1)\sin(\theta_2)\right) \\ &\quad + (\lambda_1 - \lambda_2)\cosh(\theta_1)\cos(\theta_2) = p \end{aligned} \tag{42}$$

$$v = 2\partial_x \ln W(\phi_{\text{rational}}, \phi_{\text{soliton}}, \phi_{\text{positon}}) = \frac{2q}{p} \tag{43}$$

where

$$q = (x + ky - (b+ak)\,t)\,\sqrt{-\lambda_1\lambda_2}\,(\lambda_1 - \lambda_2)\sinh(\theta_1)\sin(\theta_2) + \lambda_1\sqrt{\lambda_1}\sinh(\theta_1)\cos(\theta_2) + \lambda_2\sqrt{-\lambda_2}\cosh(\theta_1)\sin(\theta_2)$$

$$\theta_1 = \sqrt{\lambda_1}\left(x + ky - (b+ak)\,t - \frac{4y\lambda_1}{a}\right),\ \theta_2 = \sqrt{-\lambda_2}\left(x + ky - (b+ak)\,t - \frac{4y\lambda_2}{a}\right)$$

4. Conclusions

In summary, based on Hirota's bilinear method, we have used Wronskian determinant method to construct exact solutions of (2+1) dimensional nonlocal Ito equation. The performance of this method is reliable and effective and gives more important physical solutions including solitons, negatons and positons. Some of the results are in agreement with the results obtained in the previous literature, and also new results are formally developed. We hope that the obtained solutions can be used in numerical schemes as initial values and they may be of significant importance for the explanation of some special physical phenomenas.

References

[1] Ablowitz, M. J., Segur, H., Solitons and the inverse scattering transform (Vol. 4, pp. x+-425). Philadelphia: Siam, 1981.

[2] Adem, A. R., The generalized (1 + 1)-dimensional and (2 + 1)-dimensional Ito equations: multiple exp-function algorithm and multiple wave solutions. *Computers & Mathematics with Applications.* (2016), 71(6), 1248-1258.

[3] Adomian, G., Solving frontier problems of physics: the decomposition method. Vol. 60. Springer Science & Business Media, 2013.

[4] Ebadi, G., Kara, A. H., Petkovic, M. D., Yildirim, A. and Biswas, A. Solitons and conserved quantities of the Ito equation. P*roceedings of the Romanian Academy, Series A.* (2012), 13(3), 215-224.

[5] Freeman N.C and Nimmo J.J.C., Soliton solutions of the Korteweg-de Vries and Kadomtsev-Petviashvili equations: the Wronskian technique. *Phys. Lett. A.* (1983), 95, 1-3.

[6] Gandarias, M. L., and Chaudry M. K., Symmetries, solutions and conservation laws of a class of nonlinear dispersive wave equations. *Communications in Nonlinear Science and Numerical Simulation* 32 (2016): 114-121.

[7] He, J.H., Variational iteration method–a kind of non-linear analytical technique: some examples. *International journal of non-linear mechanics.* (1999), 34.4, 699-708.

[8] Hirota, R., The direct method in soliton theory. Vol. 155. Cambridge University Press, 2004.

[9] Ito M., An extension of nonlinear evolution equations of the K-dV (mK-dV) type to higher order. *J. Phys. Soc. Japan.* (1980), 49 (2), 771–778.

[10] Li C.X., Ma W.X., Liu X.J. and Zeng Y.B., Wronskian solutions of the Boussinesq equation solitons, negatons, positons and complexitons, *Inverse Problems* 23 (2007) 279 296.

[11] Ma, W. X., Wronskians, generalized Wronskians and solutions to the Korteweg–de Vries equation. *Chaos, Solitons & Fractals,* (2004) 19(1), 163-170.

[12] Ma, W. X. and You, Y., Solving the Korteweg-de Vries equation by its bilinear form: Wronskian solutions. *Transactions of the American mathematical society*, (2005) 357(5), 1753-1778.

[13] Ma, W. X. and Maruno, K. I., Complexiton solutions of the Toda lattice equation. *Physica A: Statistical Mechanics and its Applications,* (2004) 343, 219-237.

[14] Momani, S and Salah A., Application of He's variational iteration method to Helmholtz equation. *Chaos, Solitons & Fractals.* (2006), 27.5, 1119-1123.

[15] Nimmo J.J.C. and Freeman N.C., A method of obtaining the N-soliton solution of the Boussinesq equation in terms of a Wronskian. *Phys. Lett. A* . (1983), 95, 4-6.

[16] Su, J., New exact solutions for the $(3+1)$-dimensional generalized BKP equation, preprint.

[17] Tang, Y., Ma, W. X., Xu, W. and Gao, L., Wronskian determinant solutions of the $(3+1)$-dimensional Jimbo–Miwa equation. *Applied Mathematics and Computation.* (2011), 217(21), 8722-8730.

[18] Tian, S. F. and Zhang, H. Q., Riemann theta functions periodic wave solutions and rational characteristics for the $(1+1)$-dimensional and $(2+1)$-dimensional Ito equation. *Chaos, Solitons & Fractals.* (2013), 49 (2), 27-41.

[19] Wazwaz, A.M., A sine-cosine method for handlingnonlinear wave equations. *Mathematical and Computer modelling.* (2004), 40.5 499-508.

[20] Wazwaz, A.M., The extended tanh method for new compact and noncompact solutions for the KP–BBM and the ZK–BBM equations. *Chaos, Solitons & Fractals.* (2008), 38.5, 1505-1516.

[21] Wazwaz A. M., Multiple-soliton solutions for the generalized $(1+1)$-dimensional and the generalized $(2{+}1)$-dimensional Ito equations, *Applied Mathematics and Computation.* (2008), 202, 840–849.

[22] Yong, C., Li B. and Zhang H. Q., Generalized Riccati equation expansion method and its application to the Bogoyavlenskii's generalized breaking soliton equation. *Chinese Physics.* (2003), 12.9, 940.

[23] Zayed, E. M. E. and Khaled A. G., The $(G\prime/G)$-expansion method for finding traveling wave solutions of nonlinear partial differential equations in mathematical physics. *Journal of Mathematical Physics.* (2009), 50.1 013502. APA.

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