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SELFADJOINT SINGULAR DIFFERENTIAL OPERATORS FOR FIRST ORDER

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Abstract. The parametrization of all selfadjoint extensions of the minimal operator generated by first order linear symmetric singular differential-operator expression in the Hilbert space of vector-functions defined at the right semi-axis has been given. To this end we use the Calkin-Gorbachuk method. Finally, the structure of spectrum set of such extensions is researched.

1. Introduction

It is known that fundamental question on the parametrization of selfadjoint extensions of the linear closed densely defined with equal deficiency indices symmetric operators in a Hilbert space has been investigated by J. von Neumann [11] and M. H. Stone [10] firstly. Applications of these results to any scaler linear even order symmetric differential operators and representation of all selfadjoint extensions in terms of boundary conditions have been investigated by I. M. Glazman-M. G. Krein- M. A. Naimark (see [5,8]). In mathematical literature there is co-called Calkin-Gorbachuk method (see [6,9]).

The motivation of this paper originates from the interesting researches of W. N. Everitt, L. Markus, A. Zettl, J. Sun, D. O'Regan, R. Agarwal [2,3,4,12] in scaler cases. Throughout this paper A. Zettl's and J. Suns's view about these topics is to be taken into consideration [12]. A selfadjoint ordinary differential operator in a Hilbert space is generated by two things:
(1) a symmetric (formally selfadjoint) differential expression;
(2) a boundary condition which consists selfadjoint differential operators.
And also the geometrical place in plane of the spectrum of given selfadjoint differential operator is one of the important questions of this theory.

In this work in Section 3 the representation of all selfadjoint extensions of the symmetric singular differential operator, generated by first order symmetric

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differential-operator expression (for the definition see [4]) in the Hilbert spaces of vector-functions defined at the semi-axis in terms of boundary conditions are described. In Section 4 the structure of spectrum of these selfadjoint extensions is investigated.

2. Statement of the Problem

Let us H is a separable Hilbert space and $a \in R$. In the Hilbert space $L^2(H,(a,\infty))$ consider the following differential-operator expression in a form (for scaler case see [4])

$$l(u) = i\rho u' + \frac{1}{2} i\rho' u + Au,$$

where:
(1) $\rho : (a,\infty) \to (0,\infty)$;
(2) $\rho \in AC_{loc}(a,\infty)$;
(3) $\int\limits_a^\infty \frac{ds}{\rho(s)} < \infty$;
(4) $A^* = A : D(A) \subset H \to H$.

By standard way the minimal operator L_0 corresponding to differential-operator expression $l(\,\cdot\,)$ in $L^2(H,(a,\infty))$ can be defined (see [7]). The operator $L = (L_0)^*$ is called the maximal operator corresponding to $l(\,\cdot\,)$ in $L^2(H,(a,\infty))$ (see [7]).

It is clear that

$$\begin{aligned} D(L) &= \{u \in L^2(H,(a,\infty)) : l(u) \in L^2(H,(a,\infty)\}, \\ D(L_0) &= \{u \in D(L) : (\sqrt{\rho}u)(a) = (\sqrt{\rho}u)(\infty) = 0\}. \end{aligned}$$

In this case the operator L_0 is symmetric and is not maximal in $L^2(H,(a,\infty))$.

In this paper, firstly the represention of all selfadjoint extensions of the minimal operator L_0 will be described. Secondly, structure of the spectrum of these extensions shall be researched.

In special case when $H = C$ the similar questions was investigated in [4] using the Glazman-Krein-Naimark method.

In left and right semi-infinitive intervals case the similar problems have been surveyed in [1].

3. Description of Selfadjoint Extensions

In this section, the general representation of selfadjoint extensions of the minimal operator L_0 will be investigated by using the Calkin-Gorbachuk method.

Firstly, let us prove the following proposition.

Lemma 1. *The deficiency indices of the operator L_0 is in form* $(m(L_0), n(L_0)) = (dimH, dimH)$.

Proof. For the simplicity of calculations it will be taken $A = 0$. It is clear that the general solutions of following differential equations

$$i\rho(t)u'_{\pm}(t) + \frac{1}{2}i\rho'(t)u_{\pm}(t) \pm iu_{\pm}(t) = 0,$$

in the $L^2(H,(a,\infty))$ are in forms

$$u_{\pm}(t) = exp\left(\mp\int\limits_c^t \frac{2 \pm \rho'(s)}{2\rho(s)}ds\right) f,\ f \in H,\ t > a,\ c > a.$$

From these representations, we have

$$\begin{aligned}
\|u_+\|^2_{L^2(H,(a,\infty))} &= \int\limits_a^\infty \|u_+(t)\|^2_H dt \\
&= \int\limits_a^\infty exp\left(-\int\limits_c^t \frac{2+\rho'(s)}{\rho(s)}ds\right) dt\|f\|^2_H \\
&= \int\limits_a^\infty \frac{\rho(c)}{\rho(t)} exp\left(-\int\limits_c^t \frac{2}{\rho(s)}ds\right) dt\|f\|^2_H \\
&= \frac{\rho(c)}{2}\int\limits_a^\infty exp\left(-\int\limits_c^t \frac{2}{\rho(s)}ds\right) d\left(\int\limits_c^t \frac{2}{\rho(s)}ds\right)\|f\|^2_H \\
&= \frac{\rho(c)}{2}\left[exp\left(-\int\limits_c^a \frac{2}{\rho(s)}ds\right) - exp\left(-\int\limits_c^\infty \frac{2}{\rho(s)}ds\right)\right]\|f\|^2_H < \infty.
\end{aligned}$$

Consequently $m(L_0) = dim\ ker(L + iE) = dimH$.

On the other hand it is clear that for any $f \in H$ the solution

$$\begin{aligned}
\|u_-\|^2_{L^2(H,(a,\infty))} &= \int_a^\infty \|u_-(t)\|^2_H dt \\
&= \int_a^\infty exp\left(\int_c^t \frac{2-\rho'(s)}{\rho(s)}ds\right) dt\|f\|^2_H \\
&= \int_a^\infty \frac{\rho(c)}{\rho(t)} exp\left(\int_c^t \frac{2}{\rho(s)}ds\right) dt\|f\|^2_H \\
&= \frac{\rho(c)}{2}\int_a^\infty exp\left(\int_c^t \frac{2}{\rho(s)}ds\right) d\left(\int_c^t \frac{2}{\rho(s)}ds\right)\|f\|^2_H \\
&= \frac{\rho(c)}{2}\left[exp\left(\int_c^\infty \frac{2}{\rho(s)}ds\right) - exp\left(\int_c^a \frac{2}{\rho(s)}ds\right)\right]\|f\|^2_H < \infty.
\end{aligned}$$

It follows from that $n(L_0) = dim\ ker(L - iE) = dim H$. This completes the proof of theorem consequently, the minimal operator L_0 has at least one selfadjoint extensions (see [6]).

Definition 1. *Let H be any Hilbert space and $S : D(S) \subset H \to H$ be a closed densely defined symmetric operator in the Hilbert space H having equal finite or infinite deficiency indices. A triplet (H, γ_1, γ_2), where H is a Hilbert space, γ_1 and γ_2 are linear mappings from $D(S^*)$ into H, is called a space of boundary values for the operator S if for any $f, g \in D(S^*)$*

$$(S^* f, g)_{\mathcal{H}} - (f, S^* g)_{\mathcal{H}} = (\gamma_1(f), \gamma_2(g))_{\mathbf{H}} - (\gamma_2(f), \gamma_1(g))_{\mathbf{H}}$$

while for any $F_1, F_2 \in H$, there exists an element $f \in D(S^*)$ such that $\gamma_1(f) = F_1$ and $\gamma_2(f) = F_2$.

Lemma 2. *The triplet (H, γ_1, γ_2),*

$$\gamma_1 : D(L) \to H,\ \gamma_1(u) = \frac{1}{\sqrt{2}}((\sqrt{\rho}u)(\infty) - (\sqrt{\rho}u)(a)),$$
$$\gamma_2 : D(L) \to H,\ \gamma_2(u) = \frac{1}{i\sqrt{2}}((\sqrt{\rho}u)(\infty) + (\sqrt{\rho}u)(a)),\ u \in D(L)$$

is a space of boundary values of the minimal operator L_0 in $L^2(H, (a, \infty))$.

Proof. In this case the direct calculations show for arbitrary $u, v \in D(L)$ that

$$\begin{aligned}
(Lu,v)_{L^2(H,(a,\infty))} - (u,Lv)_{L^2(H,(a,\infty))} &= (i\rho u' + \frac{1}{2}i\rho' u + Au, v)_{L^2(H,(a,\infty))} \\
&\quad -(u, i\rho v' + \frac{1}{2}i\rho' v + Av)_{L^2(H,(a,\infty))} \\
&= (i\rho u', v)_{L^2(H,(a,\infty))} + \frac{1}{2}(i\rho' u, v)_{L^2(H,(a,\infty))} \\
&\quad -(u, i\rho v')_{L^2(H,(a,\infty))} - (u, \frac{1}{2}i\rho' v)_{L^2(H,(a,\infty))} \\
&= i\Big[(\rho u', v)_{L^2(H,(a,\infty))} + (\rho' u, v)_{L^2(H,(a,\infty))} \\
&\quad +(\rho u, v')_{L^2(H,(a,\infty))}\Big] \\
&= i\Big[((\rho u)', v)_{L^2(H,(a,\infty))} + (\rho u, v')_{L^2(H,(a,\infty))}\Big] \\
&= i\,((\rho u, v))'_{L^2(H,(a,\infty))} \\
&= i\,((\sqrt{\rho}u, \sqrt{\rho}v))'_{L^2(H,(a,\infty))} \\
&= i\,[((\sqrt{\rho}u)(\infty), (\sqrt{\rho}v)(\infty))_H \\
&\quad - ((\sqrt{\rho})u(a), (\sqrt{\rho})v(a))_H] \\
&= (\gamma_1(u), \gamma_2(v))_H - (\gamma_2(u), \gamma_1(v))_H .
\end{aligned}$$

Now for any given elements $f,\ g \in H$, let us find the function $u \in D(L)$ satisfying

$$\gamma_1(u) = \frac{1}{\sqrt{2}}((\sqrt{\rho}u)(\infty) - (\sqrt{\rho}u)(a)) = f \text{ and } \gamma_2(u) = \frac{1}{i\sqrt{2}}((\sqrt{\rho}u)(\infty) + (\sqrt{\rho}u)(a)) = g.$$

From this

$$(\sqrt{\rho}u)(\infty) = (ig+f)/\sqrt{2} \text{ and } (\sqrt{\rho}u)(a) = (ig-f)/\sqrt{2}$$

is obtained.

If we choose the function u in following form

$$u(t) = \frac{1}{\sqrt{\rho(t)}}(1 - e^{a-t})(ig+f)/\sqrt{2} + \frac{1}{\sqrt{\rho(t)}}e^{a-t}(ig-f)/\sqrt{2},$$

$u \in D(L),\ \gamma_1(u) = f$ and $\gamma_2(u) = g$.

Finally, using the method given in [6], we can introduce the following result.

Theorem 1. *If $\widetilde{L}$ is a selfadjoint extension of the minimal operator L_0 in $L^2(H,(a,\infty))$, then it is generated by the differential-operator expression $l(\cdot)$ and boundary condition*

$$(\sqrt{\rho}u)(\infty) = W(\sqrt{\rho}u)(a),$$

where $W : H \to H$ is a unitary operator. Moreover, the unitary operator W in H is determined uniquely by the extension $\widetilde{L}$, i.e. $\widetilde{L} = L_W$ and vice versa.

Proof. It is known from [6] or [9] that all selfadjoint extensions of the minimal operator L_0 are described by differential-operator expression $l(\cdot)$ and the boundary condition

$$(V-E)\gamma_1(u) + i(V+E)\gamma_2(u) = 0,$$

where $V : H \to H$ is a unitary operator. So from Lemma 2, we have

$$(V - E)\left((\sqrt{\rho}u)(\infty) - (\sqrt{\rho}u)(a)\right) + (V + E)\left((\sqrt{\rho}u)(\infty) + (\sqrt{\rho}u)(a)\right) = 0.$$

Hence, we obtain

$$(\sqrt{\rho}u)(a) = -V(\sqrt{\rho}u)(\infty).$$

Choosing $W = -V^{-1}$ in last boundary condition, we have

$$(\sqrt{\rho}u)(\infty) = W(\sqrt{\rho}u)(a).$$

4. The Spectrum of the Selfadjoint Extensions

In this section the structure of the spectrum of the selfadjoint extensions L_W of the minimal operator L_0 in $L^2(H,(a,\infty))$ will be investigated.

First of all let us prove the following result.

Theorem 2. *The spectrum of any selfadjoint extension L_W is in form*

$$\sigma(L_W) = \left\{\lambda \in \mathbb{C} : \lambda = \left(\int\limits_a^\infty \frac{ds}{\rho(s)}\right)^{-1} (2n\pi - arg\mu),\ n \in \mathbb{Z},\ \mu \in \sigma\left(Wexp\left(-iA\int\limits_a^\infty \frac{ds}{\rho(s)}\right)\right)\right\}.$$

Proof. Consider the following problem to spectrum of the extension L_W

$$l(u) = \lambda u + f, \quad u,\ f \in L^2(H,(a,\infty)),\ \lambda \in \mathbb{R},$$

$$(\sqrt{\rho}u)(\infty) = W(\sqrt{\rho}u)(a),$$

that is,

$$i\rho(t)u'(t) + \frac{1}{2}i\rho'(t)u(t) + Au(t) = \lambda u(t) + f(t),\ t > a,$$

$$(\sqrt{\rho}u)(\infty) = W(\sqrt{\rho}u)(a).$$

The general solution of the last differential equation is in the following form

$$\begin{aligned} u(t;\lambda) &= \sqrt{\frac{\rho(c)}{\rho(t)}} exp\left(i(A - \lambda E)\int\limits_c^t \frac{ds}{\rho(s)}\right) f_\lambda \\ &+ \frac{i}{\sqrt{\rho(t)}} \int\limits_t^\infty \exp\left(i(A - \lambda E)\int\limits_s^t \frac{d\tau}{\rho(\tau)}\right) \frac{f(s)}{\sqrt{\rho(s)}} ds,\ f_\lambda \in H,\ t > a,\ c > a. \end{aligned}$$

In this case

$$\left\|\sqrt{\frac{\rho(c)}{\rho(t)}} exp\left(i(A - \lambda E)\int\limits_c^t \frac{ds}{\rho(s)}\right) f_\lambda\right\|^2_{L^2(H,(a,\infty))} = \rho(c)\int\limits_a^\infty \frac{dt}{\rho(t)} \|f_\lambda\|_H^2 < \infty$$

and

$$
\begin{aligned}
& \| \frac{i}{\sqrt{\rho(t)}} \int_t^\infty \exp\left(i(A-\lambda E) \int_s^t \frac{d\tau}{\rho(\tau)} \right) \frac{f(s)}{\sqrt{\rho(s)}} ds \|^2_{L^2(H,(a,\infty))} \\
= & \int_a^\infty \frac{1}{\rho(t)} \| \int_t^\infty \exp\left(i(A-\lambda E) \int_s^t \frac{d\tau}{\rho(\tau)} \right) \frac{f(s)}{\sqrt{\rho(s)}} ds \|^2_H dt \\
\leq & \int_a^\infty \frac{1}{\rho(t)} \left[\int_t^\infty \| \exp\left(i(A-\lambda E) \int_s^t \frac{d\tau}{\rho(\tau)} \right) \|_H \frac{\|f(s)\|_H}{\sqrt{\rho(s)}} ds \right]^2 dt \\
\leq & \int_a^\infty \frac{1}{\rho(t)} \left(\int_t^\infty \frac{ds}{\rho(s)} \right) \left(\int_t^\infty \|f(s)\|^2_H ds \right) dt \\
\leq & \int_a^\infty \frac{1}{\rho(t)} \left(\int_a^\infty \frac{ds}{\rho(s)} \right) \left(\int_a^\infty \|f(s)\|^2_H ds \right) dt \\
= & \int_a^\infty \frac{dt}{\rho(t)} \int_a^\infty \frac{ds}{\rho(s)} \|f(s)\|^2_{L^2(H,(a,\infty))} ds \\
= & \left(\int_a^\infty \frac{dt}{\rho(t)} \right)^2 \|f\|^2_{L^2(H,(a,\infty))} < \infty.
\end{aligned}
$$

Hence for $u(\cdot, \lambda) \in L^2(H,(a,\infty))$ for $\lambda \in R$. From this and boundary condition, we have

$$
\begin{aligned}
& \left(exp\left(-i\lambda \int_a^\infty \frac{ds}{\rho(s)} \right) - W exp\left(-iA \int_a^\infty \frac{ds}{\rho(s)} \right) \right) exp\left(iA \int_c^\infty \frac{ds}{\rho(s)} \right) exp\left(-i\lambda \int_c^a \frac{ds}{\rho(s)} \right) f_\lambda \\
= \quad & \frac{i}{\sqrt{\rho(c)}} W \int_a^\infty \exp\left(i(A-\lambda) \int_s^a \frac{d\tau}{\rho(\tau)} \right) \frac{f(s)}{\sqrt{\rho(s)}} ds
\end{aligned}
$$

In order to get $\lambda \in \sigma(L_W)$, the necessary and sufficient condition is

$$
exp\left(-i\lambda \int_a^\infty \frac{ds}{\rho(s)} \right) = \mu \in \sigma\left(W exp\left(-iA \int_a^\infty \frac{ds}{\rho(s)} \right) \right)
$$

Consequently,

$$
\lambda \int_a^\infty \frac{ds}{\rho(s)} = 2n\pi - arg\mu, \ n \in \mathbb{Z},
$$

that is,

$$\lambda = \left(\int_a^{\infty} \frac{ds}{\rho(s)} \right)^{-1} (2n\pi - arg\mu), \ n \in \mathbb{Z}.$$

This completes proof of theorem.

Example. All selfadjoint extensions L_{φ} of the minimal operator L_0 generated by differential expression

$$l(u) = it^2 \frac{\partial u(t,x)}{\partial t} + itu(t,x) + Au,$$

$$A : D(A) \subset L^2(0,1) \to L^2(0,1),$$

where $Av(t) = -\dfrac{\partial^2 v(t)}{\partial t^2}$,

$$D(A) = \left\{ u \in W_2^2(0,1) : \ v(0) = v(1), \ v'(0) = v'(1) \right\},$$

in the Hilbert space $L^2((1,\infty)\times(0,1))$ in terms of boundary conditions are described by following form

$$(tu(t,x))(\infty) = e^{i\varphi}(tu(t,x))(1), \ \varphi \in [0,2\pi), \ x \in (0,1).$$

Moreover, the spectrum of such extension is

$$\sigma(L_{\varphi}) = \{\lambda \in \mathbb{C} : \lambda = 2n\pi + (\varphi - \alpha), \ n \in \mathbb{Z}, \ \alpha \in \sigma(A)\}.$$

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