Available online: October 09, 2017

Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.
Volume 67, Number 2, Pages 165–177 (2018)
DOI: 10.1501/Commua1_0000000871
ISSN 1303–5991
http://communications.science.ankara.edu.tr/index.php?series=A1

C*-ALGEBRA-VALUED S-METRIC SPACES

MELTEM ERDEN EGE AND CIHANGIR ALACA

Abstract. In this study, we present the concept of a C*-algebra-valued S-metric space. We prove Banach contraction principle in this space. Finally, we prove a common fixed point theorem in C*-algebra-valued S-metric spaces defining new notions such as L-condition and k-contraction.

1. Introduction

As we have known, Banach contraction principle has very useful structure. For this reason, it has been used in various areas such as modern analysis, applied mathematics and fixed point theory. The main goal of researchers is to obtain new results in different metric spaces. On the other hand, coupled fixed point theorems have been given in different metric spaces [12, 23, 31].

The notion of S-metric space was presented by Sedghi et al. [24]. Then, Chouhan [6] proved a common unique fixed point theorem for expansive mappings in S-metric space. Sedghi and Dung [25] proved a general fixed point theorem in S-metric spaces.

Hieu et al. [11] gave a fixed point theorem for a class of maps depending on another map on S-metric spaces. Afra [2] introduced double contractive mappings. For other important papers related to S-metric spaces, see [1, 7, 8, 9, 10, 26].

After studying the operator-valued metric spaces in [17], Ma et al. [18] introduced the concept of C*-valued metric spaces and give a fixed point theorem for C*-valued contraction mappings. In [19], C*-algebra-valued b-metric spaces were presented and some applications related to operator and integral equations were given. Coincidence and common fixed point theorems for two mappings in complete C*-algebra-valued metric spaces were proved in [22].

Batul and Kamran [5] generalized the notion of C*-valued contraction mappings and established a fixed point theorem for such mappings. In [29], Caristi's fixed point theorem was given for C*-algebra-valued metric spaces. Kamran et al. [14]

Received by the editors: November 20, 2016, Accepted: July 19, 2017.
2010 *Mathematics Subject Classification.* Primary 47H10; Secondary 54H25.
Key words and phrases. Fixed point, C*-algebra, S-metric.

gave the Banach contraction principle in C*-algebra-valued b-metric spaces with application. Bai [4] presented coupled fixed point theorems in C*-algebra-valued b-metric spaces. For other works, see [3, 13, 15, 21, 28, 30, 32, 33].

In this work, we introduce C*-algebra-valued S-metric spaces and prove Banach contraction principle. We also prove a coupled fixed point theorem in C*-algebra-valued S-metric spaces. For this purpose, we give some definitions such as coupled fixed point, L-condition and k-contraction.

2. Preliminaries

In this section, we give some basic definitions and theorems from [18] which will be used later. Throughout this paper, $\mathbb{A}$ will denote a unital C*-algebra with a unit I. An involution on $\mathbb{A}$ is a conjugate linear map $a \mapsto a^*$ on $\mathbb{A}$ such that

$$a^{**} = a \quad \text{and} \quad (ab)^* = b^* a^*$$

for all $a, b \in \mathbb{A}$. The pair $(\mathbb{A}, *)$ is called a $*$-algebra. A Banach $*$-algebra is a $*$-algebra $\mathbb{A}$ together with a complete submultiplicative norm such that

$$\|a^*\| = \|a\| \quad (\forall a \in A).$$

A C*-algebra is a Banach $*$-algebra such that $\|a^* a\| = \|a\|^2$.

Set $\mathbb{A}_h = \{x \in \mathbb{A} : x = x^*\}$. An element $x \in \mathbb{A}$ is said to be a positive element, denoted by $x \succeq \theta$, if $x \in \mathbb{A}_h$ and $\sigma(x) \subset \mathbb{R}_+ = [0, \infty)$, where $\sigma(x)$ is the spectrum of x. A partial ordering $\preceq$ on $\mathbb{A}_h$ can be defined with these positive elements as follows:

$$x \preceq y \quad \text{if and only if} \quad y - x \succeq \theta,$$

where θ means the zero element in $\mathbb{A}$. The set $\{x \in \mathbb{A} : x \succeq \theta\}$ will be denoted by $\mathbb{A}_+$.

When $\mathbb{A}$ is a unital C*-algebra, then for any $x \in \mathbb{A}_+$ we have $x \preceq I \ \Leftrightarrow \ \|x\| \leq 1$ and $|x| = (x * x)^{\frac{1}{2}}$.

Definition 2.1. [18]. Let X be a nonempty set. Suppose the mapping $d : X \times X \to \mathbb{A}$ satisfies the following:

(1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta \quad \Leftrightarrow \quad x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a C*-algebra-valued metric on X and $(X, \mathbb{A}, d)$ is called a C*-algebra-valued metric space.

It is obvious that C*-algebra-valued metric spaces generalize the concept of metric spaces, replacing the set of real numbers by $\mathbb{A}_+$.

Definition 2.2. [18]. Let $(X, \mathbb{A})$ be a C*-algebra-valued metric space. Suppose that $\{x_n\} \subset X$ and $x \in X$.

(i) If for any $\varepsilon > 0$, there is N such that for all $n > N$, $\|d(x_n, x)\| \leq \varepsilon$, then $\{x_n\}$ is said to be convergent with respect to $\mathbb{A}$ and $\{x_n\}$ converges to x and x is the limit of $\{x_n\}$. We denote it by $\lim_{n\to\infty} x_n = x$.

(ii) If for any $\varepsilon > 0$, there is N such that for all $n, m > N$, $\|d(x_n, x_m)\| \leq \varepsilon$, then $\{x_n\}$ is called a Cauchy sequence with respect to $\mathbb{A}$.

(iii) We say that $(X, \mathbb{A}, d)$ is a complete C*-algebra-valued metric space if every Cauchy sequence with respect to $\mathbb{A}$ is convergent.

Example 2.3. [18]. Let $X = \mathbb{R}$ and $\mathbb{A} = M_2(\mathbb{R})$. Define

$$d(x, y) = diag(|x - y|, \alpha|x - y|),$$

where $x, y \in \mathbb{R}$ and $\alpha \geq 0$ is a constant. d is a C*-algebra-valued metric and $(X, M_2(\mathbb{R}), d)$ is a complete C*-algebra-valued metric space by the completeness of $\mathbb{R}$.

Definition 2.4. [18]. Suppose that $(X, \mathbb{A}, d)$ is a C*-algebra-valued metric space. We call a mapping $T : X \to X$ is a C*-algebra-valued contractive mapping on X, if there exists an $A \in \mathbb{A}$ with $\|A\| < 1$ such that

$$d(Tx, Ty) \preceq A^* d(x, y) A$$

for all $x, y \in A$.

Theorem 2.5. [18]. *If $(X, \mathbb{A}, d)$ is a complete C*-algebra-valued metric space and T is a contractive mapping, there exists a unique fixed point in X.*

Definition 2.6. [18]. Let X be a nonempty set. We call a mapping T is a C*-algebra-valued expansion mapping on X, if $T : X \to X$ satisfies:

(1) $T(X) = X$;
(2) $d(Tx, Ty) \succeq A^* d(x, y) A$, $\forall x, y \in X$,

where $A \in \mathbb{A}$ is an invertible element and $\|A^{-1}\| < 1$.

Theorem 2.7. [18]. *Let $(X, \mathbb{A}, d)$ be a complete C*-algebra-valued metric space. Then for the expansion mapping T, there exists a unique fixed point in X.*

Lemma 2.8. [18]. *Suppose that $\mathbb{A}$ is a unital C*-algebra with a unit I.*

(1) *If $a \in \mathbb{A}_+$ with $\|a\| < \frac{1}{2}$, then $I - a$ is invertible and $\|a(I - a)^{-1}\| < 1$;*
(2) *Suppose that $a, b \in \mathbb{A}$ with $a, b \succeq \theta$ and $ab = ba$, then $ab \succeq \theta$;*
(3) *by $\mathbb{A}'$ we denote the set*

$$\{a \in \mathbb{A} : ab = ba, \ \forall b \in \mathbb{A}\}.$$

Let $a \in \mathbb{A}'$, if $b, c \in \mathbb{A}$ with $b \succeq c \succeq \theta$ and $I - a \in \mathbb{A}'_+$ is an invertible operator, then

$$(I - a)^{-1} b \succeq (I - a)^{-1} c.$$

Theorem 2.9. [18]. *Let $(X, \mathbb{A}, d)$ be a complete C*-valued metric space. Suppose the mapping $T : X \to X$ satisfies for all $x, y \in X$*

$$d(Tx, Ty) \preceq A(d(Tx, y) + d(Ty, x)),$$

where $A \in \mathbb{A}_+^{'}$ and $\|A\| < \frac{1}{2}$. Then there exists a unique fixed point in X.

On the other hand, we need to recall the definition of S-metric spaces.

Definition 2.10. [24]. Let X be a non-empty set. An S-metric on X is a function $S : X^3 \to [0, \infty)$ that satisfies the following conditions, for each $x, y, z, a \in X$,

(i) $S(x, y, z) \geq 0$;
(ii) $S(x, y, z) = 0$ if and only if $x = y = z$;
(iii) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The pair (X, S) is called an S-metric space.

3. Main Results

In this section, we introduce C*-algebra-valued S-metric spaces and give some results on this new space.

Definition 3.1. Let X be a nonempty set. Suppose the mapping $S : X \times X \times X \to \mathbb{A}$ satisfies the following conditions for each $x, y, z, a \in X$:

(i) $S(x, y, z) \succeq \theta$;
(ii) $S(x, y, z) = \theta$ if and only if $x = y = z$;
(iii) $S(x, y, z) \preceq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

Then S is called a C*-algebra-valued S-metric and $(X, \mathbb{A}, S)$ is called a C*-algebra-valued S-metric space.

Example 3.2. Let $\mathbb{A} = M_2(\mathbb{R})$ be all 2×2-matrices with the usual operations of addition, scalar multiplication and matrix multiplication. It is clear that

$$\|A\| = (\sum_{i,j=1}^{2} |a_{i,j}|^2)^{\frac{1}{2}}$$

defines a norm on $\mathbb{A}$ where $A = (a_{ij}) \in \mathbb{A}$. $* : \mathbb{A} \to \mathbb{A}$ defines an involution on $\mathbb{A}$ where $A^* = A$. Then $\mathbb{A}$ is a C*-algebra [27]. For $A = (a_{ij})$ and $B = (b_{ij})$ in $\mathbb{A}$, a partial order on $\mathbb{A}$ can be given as follows:

$$A \preceq B \quad \Leftrightarrow \quad (a_{ij} - b_{ij}) \leq 0 \text{ for all } i, j = 1, 2.$$

If we define on $\mathbb{A}$

$$S(x, y, z) = \begin{bmatrix} d(x, z) + d(y, z) & 0 \\ 0 & d(x, z) + d(y, z) \end{bmatrix},$$

then it is a C*-algebra-valued S-metric space.

Lemma 3.3. *In a C*-algebra-valued S-metric space, we have $S(x, x, y) = S(y, y, x)$.*

Proof. By the condition (iii) of C*-algebra-valued S-metric, we obtain

$$S(x,x,y) \preceq S(x,x,x) + S(x,x,x) + S(y,y,x) = S(y,y,x)$$

and

$$S(y,y,x) \preceq S(y,y,y) + S(y,y,y) + S(x,x,y) = S(x,x,y).$$

Thus we get $S(x,x,y) = S(y,y,x)$. □

Definition 3.4. Let $(X, \mathbb{A}, S)$ be a C*-algebra-valued S-metric space.

(i) A sequence $\{x_n\}$ in X converges to $x \in X$ with respect to $\mathbb{A}$ if and only if $S(x_n, x_n, x) \to 0$ as $n \to \infty$.
(ii) A sequence $\{x_n\}$ in X is called a Cauchy sequence with respect to $\mathbb{A}$ if for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $S(x_n, x_n, x_m) \prec \varepsilon$ for each $n, m \succeq N$.
(iii) We say that $(X, \mathbb{A}, S)$ is a complete C*-algebra-valued S-metric space if every Cauchy sequence with respect to $\mathbb{A}$ is convergent.

Example 3.5. Let $X = \mathbb{R}$, $\mathbb{A} = \mathbb{R}^2$ and $S(x,y,z) = (|x-z| + |y-z|, 0)$ be a C*-algebra valued S-metric space. Consider a sequence $(x_n) = (\frac{1}{n})$. Since

$$S(x_n, x_n, x_m) = (2|\frac{1}{n} - \frac{1}{m}|, 0) \leq (2\big(|\frac{1}{n}| + |\frac{1}{m}|\big), 0) \overset{n,m\to\infty}{\to} (0,0),$$

(x_n) is a Cauchy sequence. On the other hand, (x_n) converges to $0 \in X$ because

$$S(x_n, x_n, 0) = (2|\frac{1}{n}|, 0) \overset{n\to\infty}{\to} (0,0).$$

Definition 3.6. Let $(X, \mathbb{A}, S)$ be a C*-algebra-valued S-metric space. A map $T : X \to X$ is said to be C*-algebra-valued contractive mapping on X, if there exists $A \in \mathbb{A}$ with $\|A\| < 1$ such that

$$S(Tx, Tx, Ty) \preceq A^* S(x,x,y) A \tag{3.1}$$

for all $x, y \in X$.

Example 3.7. Let $X = [0,1]$ and $\mathbb{A} = M_2(\mathbb{R})$ with $\|A\| = \max\{a_1, a_2, a_3, a_4\}$, where a_i's are the entries of A. Then $(X, \mathbb{A}, S)$ is a C*-algebra-valued S-metric space, where

$$S(x,y,z) = \begin{bmatrix} |x-z| + |y-z| & 0 \\ 0 & |x-z| + |y-z| \end{bmatrix},$$

and partial ordering on $\mathbb{A}$ is given by

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \succeq \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \;\Leftrightarrow\; a_i \geq b_i, \text{ for } i = 1,2,3,4.$$

Define a map $T : X \to X$ by $T(x) = \frac{x}{4}$. Since

$$\begin{aligned}
S(Tx, Tx, Ty) &= S(\frac{x}{4}, \frac{x}{4}, \frac{y}{4}) \\
&= \begin{bmatrix} \frac{1}{2}|x-y| & 0 \\ 0 & \frac{1}{2}|x-y| \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} |x-y| & 0 \\ 0 & |x-y| \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \\
&= A^* S(x, x, y) A,
\end{aligned}$$

where $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$ and $\|A\| = \frac{1}{\sqrt{2}} < 1$, T is a C*-algebra-valued contractive mapping.

We now prove the Banach's contraction principle for C*-algebra-valued S-metric spaces.

Theorem 3.8. *Let $(X, \mathbb{A}, S)$ be a complete C*-algebra-valued S-metric space and $T : X \to X$ be a C*-algebra-valued contractive mapping. Then T has a unique fixed point $x_0 \in X$.*

Proof. Let's first prove the existence. We choose $x \in X$ and show that $\{T^n(x)\}$ is a Cauchy sequence with respect to $\mathbb{A}$. Using induction, we obtain the following:

$$\begin{aligned}
S(x_n, x_n, x_{n+1}) &= S(T^n(x), T^n(x), T^{n+1}(x)) \\
&\preceq A^* S(T^{n-1}(x), T^{n-1}(x), T^n(x)) A \\
&\preceq (A^*)^2 S(T^{n-2}(x), T^{n-2}(x), T^{n-1}(x)) A^2 \\
&\vdots \\
&\preceq (A^*)^n S(x, x, T(x)) A^n
\end{aligned}$$

for $n = 0, 1, \ldots$. Therefore for $m > n$, we get

$$\begin{aligned}
S(x_n, x_n, x_m) &= S(T^n(x), T^n(x), T^m(x)) \\
&\preceq 2\sum_{i=n}^{m-2} S(T^i(x), T^i(x), T^{i+1}(x)) + S(T^{m-1}(x), T^{m-1}(x), T^m(x)) \\
&\preceq 2\sum_{i=n}^{m-2} (A^*)^i S(x, x, Tx) A^i + (A^*)^{m-1} S(x, x, T(x)) A^{m-1} \\
&= 2\sum_{i=n}^{m-2} (A^*)^i B^{\frac{1}{2}} B^{\frac{1}{2}} A^i + (A^*)^{m-1} B^{\frac{1}{2}} B^{\frac{1}{2}} A^{m-1} \\
&= 2\sum_{i=n}^{m-2} (B^{\frac{1}{2}} A^i)^* (B^{\frac{1}{2}} A^i) + (B^{\frac{1}{2}} A^{m-1})^* (B^{\frac{1}{2}} A^{m-1}) \\
&= 2\sum_{i=n}^{m-2} |B^{\frac{1}{2}} A^i|^2 + |B^{\frac{1}{2}} A^{m-1}|^2 \\
&\leq \|2\sum_{i=n}^{m-2} |B^{\frac{1}{2}} A^i|^2 + |B^{\frac{1}{2}} A^{m-1}|^2\| I \\
&\leq 2\sum_{i=n}^{m-2} \|B^{\frac{1}{2}}\|^2 \|A^i\|^2 I + \|B^{\frac{1}{2}}\|^2 \|A^{m-1}\|^2 I \\
&\leq 2\|B^{\frac{1}{2}}\|^2 \sum_{i=n}^{m-2} \|A\|^{2i} I + \|B^{\frac{1}{2}}\|^2 \|A\|^{2m-2} I \\
&\leq 2\|B^{\frac{1}{2}}\|^2 \frac{\|A\|^{2n}}{1 - \|A\|} I + \|B^{\frac{1}{2}}\|^2 \|A\|^{2m-2} I \\
&\overset{m,n\to\infty}{\longrightarrow} \theta
\end{aligned}$$

where $B = S(x, x, Tx)$. So $\{T^n(x)\}$ is a Cauchy sequence with respect to $\mathbb{A}$. By the completeness of $(X, \mathbb{A}, S)$, there exists an element $x_0 \in X$ with $\lim_{n\to\infty} T^n(x) = x_0$. Since

$$\begin{aligned}
\theta &\preceq S(Tx_0, Tx_0, x_0) \\
&= S(Tx_0, Tx_0, Tx_n) + S(Tx_0, Tx_0, Tx_n) + S(x_0, x_0, x_n) \\
&\preceq A^* S(x_0, x_0, x_n) A + A^* S(x_0, x_0, x_n) A + S(x_0, x_0, x_n) \\
&\overset{n\to\infty}{\to} \theta,
\end{aligned}$$

we conclude that $Tx_0 = x_0$, i.e., x_0 is a fixed point of T.

Finally we show the uniqueness. Assume that there exists $u, v \in X$ with $u = T(u)$ and $v = T(v)$. Since T is a C*-algebra-valued contractive mapping, we have

$$\theta \preceq S(u, u, v) = S(Tu, Tu, Tv) \preceq A^* S(u, u, v) A.$$

On the other hand, since $\|A\| < 1$, we obtain

$$\begin{aligned} 0 &\leq \|S(u,u,v)\| = \|S(Tu,Tu,Tv)\| \\ &\leq \|A^*S(u,u,v)A\| \\ &\leq \|A^*\|\|S(u,u,v)\|\|A\| \\ &= \|A\|^2\|S(u,u,v)\| \\ &< \|S(u,u,v)\|. \end{aligned}$$

But this is impossible. So $S(u,u,v) = \theta$ and $u = v$ which implies that the fixed point is unique. □

Example 3.9. Let X, $\mathbb{A}$, S and T be as in Example 3.7. T satisfies the hypothesis of Theorem 3.8. So 0 is the unique fixed point of T.

Definition 3.10. Let $(X, \mathbb{A}, S)$ be a C*-algebra-valued S-metric space. An element $(x,y) \in X \times X$ is called a coupled fixed point of a mapping $F : X \times X \to X$ if $F(x,y) = x$ and $F(y,x) = y$.

Definition 3.11. Let $(X, \mathbb{A}, S)$ be a C*-algebra-valued S-metric space. An element $(x,y) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if $F(x,y) = gx$ and $F(y,x) = gy$.

Definition 3.12. [1]. Let X be a nonempty set. We say the mappings $F : X \times X \to X$ and $g : X \to X$ satisfy the L-condition if $gF(x,y) = F(gx,gy)$ for all $x,y \in X$.

Definition 3.13. Let $(X, \mathbb{A}, S)$ be a C*-algebra-valued S-metric space. We say the mappings $F : X \times X \to X$ and $g : X \to X$ satisfy the k-contraction if

$$S(F(x,y), F(x,y), F(z,w)) \preceq kA^*[S(gx,gx,gz) + S(gy,gy,gw)]A \tag{3.2}$$

with respect to $\mathbb{A}$ for all $x,y,z,w,u,v \in X$.

Lemma 3.14. *Let $(X, \mathbb{A}, S)$ be a C*-algebra-valued S-metric space. Suppose that $F : X \times X \to X$ and $g : X \to X$ satisfies the k-contraction for $k \in (0, \frac{1}{2})$. If (x,y) is a coupled coincidence point of the mappings F and g, then*

$$F(x,y) = gx = gy = F(y,x).$$

Proof. We have $gx = F(x,y)$ and $gy = F(y,x)$ because (x,y) is the coupled coincidence point of the mappings F and g. If we assume $gx \neq gy$, then we obtain

$$\begin{aligned} S(gx,gx,gy) &= S(F(x,y), F(x,y), F(y,x)) \\ &\preceq kA^*[S(gx,gx,gy) + S(gy,gy,gx)]A \\ &= 2kA^*S(gx,gx,gy)A \end{aligned}$$

and

$$\begin{aligned} \|S(gx,gx,gy)\| &\leq 2k\|A\|^2\|S(gx,gx,gy)\| \\ &< \|S(gx,gx,gy)\| \end{aligned}$$

by (3.2) and Lemma 3.3. But it is a contradiction. Therefore $gx = gy$ and

$$F(x, y) = gx = gy = F(y, x).$$

□

Theorem 3.15. *Let $(X, \mathbb{A}, S)$ be a C*-algebra-valued S-metric space. Suppose that $F : X \times X \to X$ and $g : X \to X$ are mappings satisfying k-contraction for $k \in (0, \frac{1}{2})$ and L-condition. If $g(X)$ is continuous with closed range such that $F(X \times X) \subset g(X)$, then there is a unique x in X such that $gx = F(x, x) = x$.*

Proof. Let $x_0, y_0 \in X$. By the fact that $F(X \times X) \subseteq g(X)$, two elements x_1, y_1 could be chosen as follows:

$$gx_1 = F(x_0, y_0) \quad \text{and} \quad gy_1 = F(y_0, x_0).$$

Starting from the pair (x_1, y_1), two sequences $\{x_n\}$ and $\{y_n\}$ in X can be obtained such that

$$gx_{n+1} = F(x_n, y_n) \quad \text{and} \quad gy_{n+1} = F(y_n, x_n).$$

The inequality (3.2) gives the following for $n \in \mathbb{N}$:

$$S(gx_{n-1}, gx_{n-1}, gx_n) \preceq kA^*[S(gx_{n-2}, gx_{n-2}, gx_{n-1}) + S(gy_{n-2}, gy_{n-2}, gy_{n-1})]A. \tag{3.3}$$

On the other hand, we get

$$\begin{aligned} F(y_{n-2}, x_{n-2}) = S(gy_{n-1}, gy_{n-1}, gy_n) \preceq & kA^*[S(gy_{n-2}, gy_{n-2}, gy_{n-1}) \\ & + S(gx_{n-2}, gx_{n-2}, gx_{n-1})]A. \end{aligned} \tag{3.4}$$

If we sum (3.3) and (3.4), we get

$$\begin{aligned} S(gx_{n-1}, gx_{n-1}, gx_n) + S(gy_{n-1}, gy_{n-1}, gy_n) \preceq & 2kA^*[S(gx_{n-2}, gx_{n-2}, gx_{n-1}) \\ & + S(gy_{n-2}, gy_{n-2}, gy_{n-1})]A \end{aligned}$$

for all $n \in \mathbb{N}$. If (3.2) is applied adequately,

$$\begin{aligned} S(gx_n, gx_n, gx_{n+1}) &\preceq 2k^2(A^*)^2[S(gx_{n-2}, gx_{n-2}, gx_{n-1}) + S(gy_{n-2}, gy_{n-2}, gy_{n-1})]A^2 \\ &\cdots \\ &\preceq \frac{1}{2}k^n(\sqrt{2}A^*)^n[S(gx_0, gx_0, gx_1) + S(gy_0, gy_0, gy_1)](\sqrt{2}A)^n. \end{aligned}$$

Using the definition of C*-algebra-valued S-metric space and Lemma 3.3,

$$\begin{aligned} S(gx_n, gx_n, gx_m) &\preceq 2\sum_{i=n}^{m-2} S(gx_i, gx_i, gx_{i+1}) + S(gx_{m-1}, gx_{m-1}, gx_m) \\ &\preceq 2\sum_{i=n}^{m-2} \frac{1}{2}k^i(\sqrt{2}A^*)^i[S(gx_0, gx_0, gx_1) + S(gy_0, gy_0, gy_1)](\sqrt{2}A)^i \\ &+ \frac{1}{2}k^{m-1}(\sqrt{2}A^*)^{m-1}[S(gx_0, gx_0, gx_1) + S(gy_0, gy_0, gy_1)](\sqrt{2}A)^{m-1}, \end{aligned}$$

where $m, n \in \mathbb{N}$, $m > n+2$, then we conclude that

$$\begin{aligned}\|S(gx_n, gx_n, gx_m)\| &\leq \sum_{i=n}^{m-2} k^i \|\sqrt{2}A\|^{2i}[S(gx_0, gx_0, gx_1) + S(gy_0, gy_0, gy_1)] \\ &+ \frac{1}{2}k^{m-1}\|\sqrt{2}A\|^{2m-2}[S(gx_0, gx_0, gx_1) + S(gy_0, gy_0, gy_1)].\end{aligned}$$

Since $\|A\| < \frac{1}{\sqrt{2}}$, when $n, m \to \infty$, we get $\|S(gx_n, gx_n, gx_m)\| \to 0$. So $\{gx_n\}$ is a Cauchy sequence. In a similar way, $\{gy_n\}$ is a Cauchy sequence. From the closedness of $g(X)$, $\{gx_n\}$ and $\{gy_n\}$ are convergent to $x \in X$ and $y \in X$. Since g is continuous, $\{g(gx_n)\}$ is convergent to gx and $\{g(gy_n)\}$ is convergent to gy. Since F and g satisfy the L-condition, we get

$$\begin{aligned} g(gx_{n+1}) &= g(F(x_n, y_n)) = F(gx_n, gy_n) \\ g(gy_{n+1}) &= g(F(y_n, x_n)) = F(gy_n, gx_n). \end{aligned}$$

This shows that the following inequalities:

$$S(g(gx_{n+1}), g(gx_{n+1}), F(x,y)) \preceq kA^*[S(g(gx_n), g(gx_n), gx) + S(g(gy_n), g(gy_n), gy)]A$$

and

$$\|S(g(gx_{n+1}), g(gx_{n+1}), F(x,y))\| \leq k\|A\|^2\|S(g(gx_n), g(gx_n), gx) + S(g(gy_n), g(gy_n), gy)\|.$$

If we take the limit as $n \to \infty$,

$$\|S(gx, gx, F(x,y))\| \leq k\|A\|^2\|S(gx, gx, gx)\| + \|S(gy, gy, gy)\| = 0.$$

So $gx = F(x,y)$. Similarly, $gy = F(y,x)$. From Lemma 3.14, (x,y) is a coupled coincidence point of the mappings F and g. So $gx = F(x,y) = F(y,x) = gy$. Since

$$\begin{aligned} S(gx_{n+1}, gx_{n+1}, gx) &= S(F(x_n, y_n), F(x_n, y_n), F(x,y)) \\ &\preceq kA^*(S(gx_n, gx_n, gx) + S(gy_n, gy_n, gy))A \end{aligned}$$

and

$$S(gy_{n+1}, gy_{n+1}, gy) \preceq kA^*(S(gy_n, gy_n, gy) + S(gx_n, gx_n, gx))A,$$

we have

$$S(gx_{n+1}, gx_{n+1}, gx) + S(gy_{n+1}, gy_{n+1}, gy) \preceq 2kA^*(S(gx_n, gx_n, gx) + S(gy_n, gy_n, gy))A$$

and

$$\|S(gx_{n+1}, gx_{n+1}, gx) + S(gy_{n+1}, gy_{n+1}, gy)\| \leq 2k\|A^*\|\|S(gx_n, gx_n, gx) + S(gy_n, gy_n, gy)\|\|A\|.$$

Taking the limit as $n \to \infty$, we obtain the following:

$$\begin{aligned} \|S(x,x,gx) + S(y,y,gy)\| &\leq 2k\|A^*\|\|S(x,x,gx) + S(y,y,gy)\|\|A\| \\ &= 2k\|A\|^2\|S(x,x,gx) + S(y,y,gy)\|. \end{aligned}$$

Since $2k < 1$ and $\|A\| < \frac{1}{\sqrt{2}}$, we have $S(x,x,gx) = 0$ and $S(y,y,gy) = 0$. So $gx = x$ and $gy = y$, that is, $gx = gy = x = y$. As a result, we have $gx = F(x,x) = x$.

To show the uniqueness, assume that there is an element $z \neq x$ in X such that $z = gz = F(z, z)$. We have

$$\begin{aligned} S(x,x,z) &= S(F(x,x), F(x,x), F(z,z)) \\ &\preceq 2kA^* S(gx, gx, gz)A \\ &= 2kA^* S(x,x,z)A. \end{aligned}$$

Since $2k < 1$, $\|A\| < \frac{1}{\sqrt{2}}$ and

$$\|S(x,x,z)\| \leq 2k\|A\|^2 \|S(x,x,z)\|,$$

we conclude that $S(x,x,z) = 0$, that is, $x = z$. □

The following corollary can be easily deduced from the Theorem 3.15.

Corollary 3.16. *Let $(X, \mathbb{A}, S)$ be a C*-algebra-valued S-metric space. If a mapping $F : X \times X \to X$ satisfies the following condition*

$$S(F(x,y), F(u,v), F(z,w)) \preceq kA^*[S(x,u,z) + S(y,v,w)]A$$

with respect to $\mathbb{A}$ for all $x, y, z, u, v, w \in X$ and $k \in (0, \frac{1}{2})$, then there exists a unique element $x \in X$ such that $F(x,x) = x$.

4. Conclusion

In this work, we investigate whether there are correspondences of some metric and fixed point properties in S-metric spaces taking the domain set of S-metric function as $\mathbb{A}$ which is a C*-algebra-valued set, and first present C*-algebra-valued S-metric space on the set having this structure using properties of this algebraic notion. This given structure is important in terms of integrating some metric constructions of algebraic topology and fixed point theory.

5. Acknowledgements

The authors express their sincere gratitude to the anonymous referees for their careful reading and suggestions that improved the presentation of this paper.

References

[1] Afra, J.M., Fixed point type theorem in S-metric spaces, *Middle-East Journal of Scientific Research*, 22(6), (2014) 864–869.
[2] Afra, J.M., Double contraction in S-metric spaces, *International Journal of Mathematical Analysis*, 9(3), (2015), 117–125.
[3] Alsulami, H.H., Agarwal, R.P., Karapinar E. and Khojasteh, F., A short note on C^*-valued contraction mappings, *Journal of Inequalities and Applications*, (2016), 50.
[4] Bai, C., Coupled fixed point theorems in C*-algebra-valued b-metric spaces with application, *Fixed Point Theory and Applications*, (2016) , 70.
[5] Batul S. and Kamran, T., C*-valued contractive type mappings, *Fixed Point Theory and Applications*, (2015), 142.
[6] Chouhan, P., A common unique fixed point theorem for expansive type mappings in S-metric spaces, *International Mathematical Forum*, 8, (2013) 1287–1293.

[7] Dung, N.V., Hieu N.T. and Radejovic, S., Fixed point theorems for g-monotone maps on partially ordered S-metric spaces, *Filomat*, 28(9), (2014) 1885–1898.
[8] Ege M.E. and Alaca, C., Some properties of modular S-metric spaces and its fixed point results, *Journal of Computational Analysis and Applications*, 20(1), (2016) 24–33.
[9] Gupta, A., Cyclic contraction on S-metric space, *International Journal of Analysis and Applications*, 3(2), (2013) 119–130.
[10] Gupta V. and Deep, R., Some coupled fixed point theorems in partially ordered S-metric spaces, *Miskolc Mathematical Notes*, 16(1), (2015) 181–194.
[11] Hieu, N.T., Ly N.T.T. and Dung, N.V., A generalization of Ciric quasi-contractions for maps on S-metric spaces, *Thai Journal of Mathematics*, 13(2), (2015) 369–380.
[12] Isik H. and Turkoglu, D., Coupled fixed point theorems for new contractive mixed monotone mappings and applications to integral equations, *Filomat*, 28(6), (2014) 1253–1264.
[13] Kadelburg Z. and Radenovic, S., Fixed point results in C^*-algebra-valued metric spaces are direct consequences of their standard metric counterparts, *Fixed Point Theory and Applications*, (2016) , 53.
[14] Kamran, T., Postolache, M., Ghiura, A., Batul S. and Ali, R., The Banach contraction principle in C*-algebra-valued b-metric spaces with application, *Fixed Point Theory and Applications*, (2016) ,10.
[15] Klineam C. and Kaskasem, P., Fixed point theorems for cyclic contractions in C*-algebra-valued b-metric spaces, *Journal of Function Spaces*, Article ID 7827040 (2016).
[16] Lakshmikantham V. and Ciric, L., Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, *Nonlinear Analysis*, 70, (2009) 4341–4349.
[17] Ma, Z., Jiang L. and Xin, Q., Fixed point theorems on operator-valued metric space, *Trans. Beijing Inst. Techol.*, 34(10), (2014) 1078–1080.
[18] Ma, Z., Jiang L. and Sun, H., C*-algebra-valued metric spaces and related fixed point theorems, *Fixed Point Theory and Applications* (2014), 2014:206.
[19] Ma Z., and Jiang, L., C*-algebra-valued b-metric spaces and related fixed point theorems, *Fixed Point Theory and Applications*, (2015), 2015:222.
[20] Murphy, G.J., C*-Algebras and Operator Theory, Academic Press, London (1990).
[21] Ozer O. and Omran, S., Common fixed point in C*-algebra b-valued metric space, AIP Conference Proceedings, (2016), 1173, 050005.
[22] Qiaoling, X., Lining J. and Ma, Z., Common fixed point theorems in C*-algebra valued metric spaces, *Journal of Nonlinear Science and Applications*, 9, (2016) 4617–4627.
[23] Sedghi, S., Altun I. and Shobe, N., Coupled fixed point theorems for contractions in fuzzy metric spaces, *Nonlinear Analysis: Theory, Methods and Applications*, 72(3-4), (2010) 1298–1304.
[24] Sedghi, S., Shobe N. and Aliouche, A., A generalization of fixed point theorem in S-metric spaces, *Mat. Vesnik*, 64, (2012), 258–266.
[25] Sedghi S. and Dung, N.V., Fixed point theorems on S-metric spaces, *Mat. Vesnik*, 66, (2014), 113–124.
[26] Sedghi, S., Altun, I., Shobe, N. and Salahshour, M.A., Some properties of S-metric spaces and fixed point results, *Kyungpook Mathematical Journal*, 54(1), (2014) 113–122.
[27] Shateri, T.L., C*-algebra-valued modular spaces and fixed point theorems, *Journal of Fixed Point Theory and Applications*, 19(2), (2017) 1551–1560.
[28] Shehwar Dur-E. and Kamran, T., C*-valued G-contractions and fixed points, *Journal of Inequalities and Applications,* (2015), 2015:304.
[29] Shehwar, Dur-E, Batul, S., Kamran T. and Ghiura, A., Caristi's fixed point theorem on C*-algebra valued metric spaces, *Journal of Nonlinear Science and Applications,* 9, (2016) 584–588.
[30] Tianqing, C., Some coupled fixed point theorems in C*-algebra-valued metric spaces, (2016) arXiv:1601.07168v1.

[31] Turkoglu D. and Binbasioglu, D., Coupled coincidence point theorems for compatible mappings in ordered uniform space, *Miskolc Mathematical Notes*, 16(1), (2015) 527–541.

[32] Zada A. and Saifullah, S., Fixed point theorem for integral type C*-valued contraction, *Journal of Analysis and Number Theory*, 4(2), (2016) 101–104.

[33] Zada, A., Saifullah S. and Ma, Z., Common fixed point theorems for G-contraction in C*-algebra-valued metric spaces, *International Journal of Analysis and Applications*, (2016) 11(1), 23–27.

Current address: Meltem ERDEN EGE (Corresponding author): Manisa Celal Bayar University, Institute of Natural and Applied Sciences, Department of Mathematics, Yunusemre 45140 Manisa, Turkey

E-mail address: mltmrdn@gmail.com

ORCID Address: http://orcid.org/0000-0002-4519-9506

Current address: Cihangir ALACA: Manisa Celal Bayar University, Faculty of Science and Arts, Department of Mathematics, Yunusemre 45140 Manisa, Turkey

E-mail address: cihangiralaca@yahoo.com.tr

ORCID Address: http://orcid.org/0000-0001-7777-8839