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POTENTIAL OPERATORS ON CARLESON CURVES IN MORREY SPACES

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Abstract. In this paper we study the potential operator $\mathcal{I}^\alpha$ in the Morrey space $L_{p,\lambda}$ and the spaces BMO defined on Carleson curves Γ. We prove that for $0 < \alpha < 1$, $\mathcal{I}^\alpha$ is bounded from the Morrey space $L_{p,\lambda}(\Gamma)$ to $L_{q,\lambda}(\Gamma)$ on simple Carleson curves if (and only if in the infinite simple Carleson curve Γ) $1/p - 1/q = \alpha/(1-\lambda)$, $1 < p < (1-\lambda)/\alpha$, and from the spaces $L_{1,\lambda}(\Gamma)$ to $WL_{q,\lambda}(\Gamma)$ if (and only if in the infinite case) $1 - \frac{1}{q} = \frac{\alpha}{1-\lambda}$.

1. Introduction

Morrey spaces were introduced by C. B. Morrey [11] in 1938 in connection with certain problems in elliptic partial differential equations and calculus of variations. Later, Morrey spaces found important applications to Navier-Stokes and Schrödinger equations, elliptic problems with discontinuous coefficients, and potential theory.

The main purpose of this paper is to establish the boundedness of potential operator $\mathcal{I}^\alpha$ in Morrey spaces $L_{p,\lambda}$ defined on Carleson curves Γ. We prove Sobolev-Morrey inequalities for the operator $\mathcal{I}^\alpha$. In particular, we get the analog of the theorem by D.R. Adams [1] regarding the inequality for the Riesz potentials in Morrey spaces defined on Carleson curves. We emphasize that in the infinite case of Γ the derived conditions are necessary and sufficient for appropriate inequalities.

Note that the results we obtain here the potential operators are valid not only on Carleson curves, but also in a more general context of metric spaces or homogeneous type spaces at least under the condition $\mu(B(x,r)) \sim r^d$ (see [4, 5, 8, 12]).

The paper is organized as follows. In Section 2, we present some definitions and auxiliary results. In Section 3, we establish the main result of the paper: We prove that for $0 < \alpha < 1$, $\mathcal{I}^\alpha$ is bounded from the Morrey space $L_{p,\lambda}(\Gamma)$ to

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$L_{q,\lambda}(\Gamma)$ on simple Carleson curves if (and only if in the infinite simple Carleson curves) $1/p - 1/q = \alpha/(1-\lambda)$, $1 < p < (1-\lambda)/\alpha$, and from the spaces $L_{1,\lambda}(\Gamma)$ to $WL_{q,\lambda}(\Gamma)$ if (and only if in the infinite case) $1 - \frac{1}{q} = \frac{\alpha}{1-\lambda}$.

2. Preliminaries

Let $\Gamma = \{t \in \mathbb{C} : t = t(s),\ 0 \le s \le l \le \infty\}$ be a rectifiable Jordan curve in the complex plane $\mathbb{C}$ with arc-length measure $\nu(t) = s$, here $l = \nu\Gamma =$ lengths of Γ. We denote

$$\Gamma(t,r) = \Gamma \cap B(t,r),\ t \in \Gamma,\ r > 0,$$

where $B(t,r) = \{z \in \mathbb{C} : |z - t| < r\}$.

A rectifiable Jordan curve Γ is called a Carleson curve if the condition

$$\nu\Gamma(t,r) \le c_0 r$$

holds for all $t \in \Gamma$ and $r > 0$, where the constant $c_0 > 0$ does not depend on t and r. Let $L_p(\Gamma)$, $1 \le p < \infty$ be the space of measurable functions on Γ with finite norm

$$\|f\|_{L_p(\Gamma)} = \left(\int_\Gamma |f(t)|^p d\nu(t) \right)^{1/p}.$$

Let $1 \le p < \infty$,$0 \le \lambda \le 1$. We denote by $L_{p,\lambda}(\Gamma)$ the Morrey space as the set of locally integrable functions f on Γ with the finite norm

$$\|f\|_{L_{p,\lambda}(\Gamma)} = \sup_{t\in\Gamma,\ r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(\Gamma(t,r))}.$$

Note that $L_{p,0}(\Gamma) = L_p(\Gamma)$, and if $\lambda < 0$ or $\lambda > 1$, then $L_{p,\lambda}(\Gamma) = \Theta$, where Θ is the set of all functions equivalent to 0 on Γ.

We denote by $WL_{p,\lambda}(\Gamma)$ the weak Morrey space as the set of locally integrable functions f with finite norm

$$\|f\|_{WL_{p,\lambda}(\Gamma)} = \sup_{\beta>0} \beta \sup_{r>0,\, t\in\Gamma} \left(r^{-\lambda} \int_{\{\tau\in\Gamma(t,r):\ |f(\tau)|>\beta\}} d\nu(\tau) \right)^{1/p}.$$

Let $f \in L_1^{loc}(\Gamma)$. The maximal operator $\mathcal{M}$ and the potential operator $\mathcal{I}^\alpha$ on Γ are defined by

$$\mathcal{M}f(t) = \sup_{t>0} |\Gamma(t,r)|^{-1} \int_{\Gamma(t,r)} |f(\tau)| d\nu(\tau),$$

and

$$\mathcal{I}^\alpha f(t) = \int_\Gamma \frac{f(\tau) d\nu(\tau)}{|t-\tau|^{1-\alpha}}, \quad 0 < \alpha < 1,$$

respectively.

Maximal operators and potential operators in various spaces defined on Carleson curves has been widely studied by many authors (see, for example [2, 3, 6, 7, 8, 9, 10, 12]).

N. Samko [12] studied the boundedness of the maximal operator $\mathcal{M}$ defined on quasimetric measure spaces, in particular on Carleson curves in Morrey spaces $L_{p,\lambda}(\Gamma)$:

Theorem A. *Let Γ be a Carleson curve, $1 < p < \infty$, $0 < \alpha < 1$ and $0 \le \lambda < 1$. Then $\mathcal{M}$ is bounded from $L_{p,\lambda}(\Gamma)$ to $L_{p,\lambda}(\Gamma)$.*

V. Kokilashvili and A. Meskhi [9] studied the boundedness of the potential operator defined on quasimetric measure spaces, in particular on Carleson curves in Morrey spaces and proved the following:

Theorem B. *Let Γ be a Carleson curve, $1 < p < q < \infty$, $0 < \alpha < 1$, $0 < \lambda_1 < \frac{p}{q}$, $\frac{\lambda_1}{p} = \frac{\lambda_2}{q}$ and $\frac{1}{p} - \frac{1}{q} = \alpha$. Then the operator $\mathcal{I}^\alpha$ is bounded from the spaces $L_{p,\lambda_1}(\Gamma)$ to $L_{q,\lambda_2}(\Gamma)$.*

3. Sobolev-Morrey inequality for potential operator on Carleson curves

In this section we prove Sobolev-Morrey inequalities for the potential operators in Morrey space defined on Carleson curves.

Theorem 1. *Let Γ be a simple Carleson curve, $0 < \alpha < 1$, $0 \le \lambda < 1 - \alpha$ and $1 \le p < \frac{1-\lambda}{\alpha}$.*

1) If $1 < p < \frac{1-\lambda}{\alpha}$, then the condition $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{1-\lambda}$ is sufficient and in the infinite case also necessary for the boundedness of $\mathcal{I}^\alpha$ from $L_{p,\lambda}(\Gamma)$ to $L_{q,\lambda}(\Gamma)$.

2) If $p = 1$, then the condition $1 - \frac{1}{q} = \frac{\alpha}{1-\lambda}$ is sufficient and in the infinite case also necessary for the boundedness of $\mathcal{I}^\alpha$ from $L_{1,\lambda}(\Gamma)$ to $WL_{q,\lambda}(\Gamma)$.

Proof. 1) *Sufficiency.* Let Γ be a simple Carleson curve, $0 < \alpha < 1$, $0 \le \lambda < 1 - \alpha$, $f \in L_{p,\lambda}(\Gamma)$ and $1 < p < \frac{1-\lambda}{\alpha}$. Then

$$\mathcal{I}^\alpha f(t) = \left(\int_{\Gamma(t,r)} + \int_{\Gamma \setminus \Gamma(t,r)} \right) f(\tau) |t-\tau|^{\alpha-1} d\nu(\tau) \equiv A(t,r) + C(t,r). \tag{1}$$

For $A(t,r)$ we have

$$\begin{aligned} |A(t,r)| &\le \int_{\Gamma(t,r)} |f(\tau)| |t-\tau|^{\alpha-1} d\nu(\tau) \\ &\le \sum_{j=1}^\infty \left(2^{-j} r\right)^{\alpha-1} \int_{\Gamma(t,2^{-j+1}r) \setminus \Gamma(t,2^{-j}r)} |f(\tau)| d\nu(\tau) \\ &\le \sum_{j=1}^\infty \left(2^{-j} r\right)^{\alpha-1} \nu\Gamma(t, 2^{-j+1} r)\, \mathcal{M} f(t) \\ &\le 2 c_0 r^\alpha\, \mathcal{M} f(t) \sum_{j=1}^\infty 2^{-j\alpha}. \end{aligned}$$

Hence

$$|A(t,r)| \leq C_1 r^{\alpha} \mathcal{M}f(t) \qquad \text{with} \qquad C_1 = \frac{2c_0}{2^{\alpha}-1}. \tag{2}$$

For $C(t,r)$ by the Hölder's inequality we have

$$|C(t,r)| \leq \left(\int_{\Gamma \setminus \Gamma(t,r)} |t-\tau|^{-\beta} |f(\tau)|^p d\nu(\tau) \right)^{1/p}$$

$$\times \left(\int_{\Gamma \setminus \Gamma(t,r)} |t-\tau|^{\left(\frac{\beta}{p}+\alpha-1\right)p'} d\nu(\tau) \right)^{1/p'} = J_1 \cdot J_2.$$

Let $\lambda < \beta < 1-\alpha p$. For J_1 we get

$$J_1 = \Big(\sum_{j=0}^{\infty} \int_{\Gamma(t,2^{j+1}r)\setminus \Gamma(t,2^j r)} |f(\tau)|^p |t-\tau|^{-\beta} d\nu(\tau) \Big)^{1/p}$$

$$\leq 2^{\frac{\lambda}{p}} r^{\frac{\lambda-\beta}{p}} \|f\|_{L_{p,\lambda}(\Gamma)} \Big(\sum_{j=0}^{\infty} 2^{(\lambda-\beta)j} \Big)^{1/p} = C_2 r^{\frac{\lambda-\beta}{p}} \|f\|_{L_{p,\lambda}(\Gamma)}, \tag{3}$$

where $C_2 = \left(\frac{2^{\beta}}{2^{\beta-\lambda}-1} \right)^{1/p}$.

For J_2 we obtain

$$J_2 = \left(\sum_{j=1}^{\infty} \int_{\Gamma(t,2^{j+1}r)\setminus \Gamma(t,2^j r)} |t-\tau|^{\left(\frac{\beta}{p}+\alpha-1\right)p'} d\nu(\tau) \right)^{1/p'}$$

$$\leq \left(\sum_{j=1}^{\infty} \left(2^j r\right)^{\left(\frac{\beta}{p}+\alpha-1\right)p'} \nu\Gamma(t,2^{j+1}r) \right)^{1/p'}$$

$$\leq \left(c_0 \sum_{j=1}^{\infty} \left(2^j r\right)^{\left(\frac{\beta}{p}+\alpha-1\right)p'+1} \right)^{1/p'} \leq C_3 r^{\frac{\beta}{p}+\alpha-\frac{1}{p}}, \tag{4}$$

where $C_3 = \frac{c_0^{\frac{1}{p'}}}{1-2^{\frac{1-\beta}{p}-\alpha}}$.

Then from (3) and (4) we have

$$|C(t,r)| \leq C_4 r^{\frac{\lambda-Q}{p}+\alpha} \|f\|_{L_{p,\lambda}(\Gamma)}, \tag{5}$$

where $C_4 = C_2 \cdot C_3$.

Thus, from (2) and (5) we have

$$|\mathcal{I}^{\alpha} f(t)| \leq C_1 r^{\alpha} \mathcal{M}f(t) + C_4 r^{\frac{\lambda-1}{q}} \|f\|_{L_{p,\lambda}(\Gamma)}.$$

Minimizing with respect to r, at $t = \left[(\mathcal{M}f(t))^{-1} \|f\|_{L_{p,\lambda}} \right]^{p/(1-\lambda)}$ we arrive at

$$|\mathcal{I}^\alpha f(t)| \le C_5 \left(\mathcal{M}f(t)\right)^{p/q} \|f\|_{L_{p,\lambda}(\Gamma)}^{1-p/q},$$

where $C_5 = C_1 + C_4$.

Hence, by Theorem B, we have

$$\begin{aligned} \int_{\Gamma(t,r)} |\mathcal{I}^\alpha f(t)|^q \, d\nu(\tau) &\le C_5 \|f\|_{L_{p,\lambda}(\Gamma)}^{q-p} \int_{\Gamma(t,r)} \left(\mathcal{M}f(t)\right)^p d\nu(\tau) \\ &\le C_5 C_{p,\lambda} r^\lambda \|f\|_{L_{p,\lambda}(\Gamma)}^{q-p} \|f\|_{L_{p,\lambda}(\Gamma)}^{p} = C_6 r^\lambda \|f\|_{L_{p,\lambda}(\Gamma)}^{q}, \end{aligned}$$

where $C_6 = C_5 \cdot C_{p,\lambda}$.

Therefore $\mathcal{I}^\alpha f \in L_{q,\lambda}(\Gamma)$ and

$$\|\mathcal{I}^\alpha f\|_{L_{q,\lambda}(\Gamma)} \le C_6 \|f\|_{L_{p,\lambda}(\Gamma)}.$$

Necessity. Let Γ be an infinite simple Carleson curve, $1 < p < \frac{1-\lambda}{\alpha}$ and $\mathcal{I}^\alpha$ bounded from $L_{p,\lambda}(\Gamma)$ to $L_{q,\lambda}(\Gamma)$.

Define $f_r(\tau) =: f(r\tau)$. Then

$$\|f_r\|_{L_{p,\lambda}(\Gamma)} = r^{-\frac{1}{p}} \sup_{r_1>0,\, \tau\in\Gamma} \left(r_1^{-\lambda} \int_{\Gamma(t, rr_1)} |f(\tau)|^p d\nu(\tau) \right)^{1/p} = r^{-\frac{1-\lambda}{p}} \|f\|_{L_{p,\lambda}(\Gamma)}$$

and

$$\mathcal{I}^\alpha f_r(t) = r^{-\alpha} \mathcal{I}^\alpha f(rt),$$

$$\begin{aligned} \|\mathcal{I}^\alpha f_r\|_{L_{q,\lambda}(\Gamma)} &= r^{-\alpha} \sup_{r_1>0,\, t\in\Gamma} \left(r_1^{-\lambda} \int_{\Gamma(t,r_1)} |\mathcal{I}^\alpha f(rt)|^q \, d\nu(t) \right)^{1/q} \\ &= r^{-\alpha-\frac{1}{q}} \sup_{r_1>0,\, t\in\Gamma} \left(r_1^{-\lambda} \int_{\Gamma(t,rr_1)} |\mathcal{I}^\alpha f(t)|^q \, d\nu(t) \right)^{1/q} \\ &= r^{-\alpha-\frac{1-\lambda}{q}} \|\mathcal{I}^\alpha f\|_{L_{q,\lambda}(\Gamma)}. \end{aligned}$$

By the boundedness $\mathcal{I}^\alpha$ from $L_{p,\lambda}(\Gamma)$ to $L_{q,\lambda}(\Gamma)$

$$\|\mathcal{I}^\alpha f\|_{L_{q,\lambda}(\Gamma)} \le C_{p,q,\lambda} r^{\alpha+\frac{1-\lambda}{q}-\frac{1-\lambda}{p}} \|f\|_{L_{p,\lambda}(\Gamma)},$$

where $C_{p,q,\lambda}$ depends only on p, q and λ.

If $\frac{1}{p} < \frac{1}{q} + \frac{\alpha}{1-\lambda}$, then for all $f \in L_{p,\lambda}(\Gamma)$, we have $\|\mathcal{I}^\alpha f\|_{L_{q,\lambda}} = 0$ as $r \to 0$.

Similarly, if $\frac{1}{p} > \frac{1}{q} + \frac{\alpha}{1-\lambda}$, then for all $f \in L_{p,\lambda}(\Gamma)$, we obtain $\|\mathcal{I}^\alpha f\|_{L_{q,\lambda}(\Gamma)} = 0$ as $r \to \infty$

Therefore $\frac{1}{p} = \frac{1}{q} + \frac{\alpha}{1-\lambda}$.

2) *Sufficiency.* Let $f \in L_{1,\lambda}(\Gamma)$. We have

$$\nu\{\tau \in \Gamma(t,r) : |\mathcal{I}^\alpha f(\tau)| > 2\beta\} \le \nu\{\tau \in \Gamma(t,r) : |A(\tau,r)| > \beta\} \\ + \nu\{\tau \in \Gamma(t,r) : |C(\tau,r)| > \beta\}.$$

Taking into account inequality (2) and Theorem A we have

$$\nu\{\tau \in \Gamma(t,r)\ :\ |A(\tau,r)| > \beta\} \le \nu\left\{\tau \in \Gamma(t,r)\ :\ \mathcal{M}f(\tau) > \frac{\beta}{C_1 r^\alpha}\right\} \\ \le \frac{C_7 r^\alpha}{\beta} \cdot r^\lambda \|f\|_{L_{1,\lambda}(\Gamma)},$$

where $C_7 = C_1 \cdot C_{1,\lambda}$ and thus if $C_4 r^{\frac{\lambda-1}{q}} \|f\|_{L_{1,\lambda}(\Gamma)} = \beta$, then $|C(\tau,r)| \le \beta$ and consequently, $|\{\tau \in \Gamma(t,r)\ :\ |C(\tau,r)| > \beta\}| = 0$.

Finally

$$\nu\{\tau \in \Gamma(t,r)\ :\ |\mathcal{I}^\alpha f(\tau)| > 2\beta\} \le \frac{C_7}{\beta} r^\lambda r^\alpha \|f\|_{L_{1,\lambda}(\Gamma)} = C_8 r^\lambda \left(\frac{\|f\|_{L_{1,\lambda}(\Gamma)}}{\beta}\right)^q,$$

where $C_8 = C_7 \cdot C_4^{q-1}$.

Necessity. Let $\mathcal{I}^\alpha$ bounded from $L_{1,\lambda}(\Gamma)$ to $WL_{q,\lambda}(\Gamma)$. We have

$$\|\mathcal{I}^\alpha f_r\|_{WL_{q,\lambda}} = \sup_{\beta>0} \beta \sup_{r_1>0,\, \tau\in\Gamma} \left(r_1^{-\lambda} \int_{\{\tau\in\Gamma(t,r_1)\ :\ |\mathcal{I}^\alpha f_r(\tau)|>\beta\}} d\nu(\tau)\right)^{1/q}$$

$$= r^{-\alpha} \sup_{\beta>0} \beta r^\alpha \sup_{r_1>0,\, \tau\in\Gamma} \left(\tau^{-\lambda} \int_{\{\tau\in\Gamma(t,r_1)\ :\ |\mathcal{I}^\alpha f(r\tau)|>\beta r^\alpha\}} d\nu(\tau)\right)^{1/q}$$

$$= r^{-\alpha-\frac{1}{q}} \sup_{\beta>0} \beta r^\alpha \sup_{r_1>0,\, \tau\in\Gamma} \left(r^\lambda (r_1 r)^{-\lambda} \int_{\{\tau\in\Gamma(t,rr_1)\ :\ |\mathcal{I}^\alpha f(\tau)|>\beta r^\alpha\}} d\nu(\tau)\right)^{1/q}$$

$$= r^{-\alpha-\frac{1-\lambda}{q}} \|\mathcal{I}^\alpha f\|_{WL_{q,\lambda}}.$$

By the boundedness $\mathcal{I}^\alpha$ from $L_{1,\lambda}(\Gamma)$ to $WL_{q,\lambda}(\Gamma)$

$$\|\mathcal{I}^\alpha f\|_{WL_{q,\lambda}} \le C_{1,q,\lambda} r^{\alpha+\frac{1-\lambda}{q}-(1-\lambda)} \|f\|_{L_{1,\lambda}(\Gamma)},$$

where $C_{1,q,\lambda}$ depends only on q and λ.

If $1 < \frac{1}{q} + \frac{\alpha}{1-\lambda}$, then for all $f \in L_{1,\lambda}(\Gamma)$, we have $\|\mathcal{I}^\alpha f\|_{WL_{q,\lambda}} = 0$ as $r \to 0$.

Similarly, if $1 > \frac{1}{q} + \frac{\alpha}{1-\lambda}$, then for all $f \in L_{1,\lambda}(\Gamma)$, we obtain $\|\mathcal{I}^\alpha f\|_{WL_{q,\lambda}} = 0$ as $r \to \infty$. Therefore $1 = \frac{1}{q} + \frac{\alpha}{1-\lambda}$.

Thus Theorem 1 is proved. □

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