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A Q-ANALOG OF THE BI-PERIODIC LUCAS SEQUENCE

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Abstract. In this paper, we introduce a q-analog of the bi-periodic Lucas sequence, called as the q-bi-periodic Lucas sequence, and give some identities related to the q-bi-periodic Fibonacci and Lucas sequences. Also, we give a matrix representation for the q-bi-periodic Fibonacci sequence which allow us to obtain several properties of this sequence in a simple way. Moreover, by using the explicit formulas for the q-bi-periodic Fibonacci and Lucas sequences, we introduce q-analogs of the bi-periodic incomplete Fibonacci and Lucas sequences and give a relation between them.

1. Introduction

It is well-known that the classical Fibonacci numbers F_n are defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2 \tag{1.1}$$

with the initial conditions $F_0 = 0$ and $F_1 = 1$. The Lucas numbers L_n, which follows the same recursive pattern as the Fibonacci numbers, but begins with $L_0 = 2$ and $L_1 = 1$. There are a lot of generalizations of Fibonacci and Lucas sequences. In [6], Edson and Yayenie introduced a generalization of the Fibonacci sequence, called as bi-periodic Fibonacci sequence, as follows:

$$q_n = \begin{cases} aq_{n-1} + q_{n-2}, & \text{if } n \text{ is even} \\ bq_{n-1} + q_{n-2}, & \text{if } n \text{ is odd} \end{cases}, \ n \geq 2 \tag{1.2}$$

with initial values $q_0 = 0$ and $q_1 = 1$, where a and b are nonzero numbers. Note that if we take $a = b = 1$ in $\{q_n\}$, we get the classical Fibonacci sequence. These sequences are emerged as denominators of the continued fraction expansion of the quadratic irrational numbers. For detailed information related to these sequences,

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we refer to [6, 19, 8, 11, 12, 17, 18, 15, 16]. Yayenie [19] gave an explicit formula of q_n as:

$$q_n = a^{\xi(n-1)} \sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-1-i}{i} (ab)^{\left\lfloor \frac{n-1}{2} \right\rfloor - i} \tag{1.3}$$

where $\xi(n) = n - 2\left\lfloor \frac{n}{2} \right\rfloor$, i.e., $\xi(n) = 0$ when n is even and $\xi(n) = 1$ when n is odd.

Similar to (1.2), by taking initial conditions $p_0 = 2$ and $p_1 = a$, Bilgici [2] introduced the bi-periodic Lucas numbers as follows:

$$p_n = \begin{cases} bp_{n-1} + p_{n-2}, & \text{if } n \text{ is even} \\ ap_{n-1} + p_{n-2}, & \text{if } n \text{ is odd} \end{cases}, \ n \geq 2. \tag{1.4}$$

It should also be noted that, it gives the classical Lucas sequence in the case of $a = b = 1$ in $\{p_n\}$. In analogy with (1.3), Tan and Ekin [14] gave the explicit formula of the bi-periodic Lucas numbers as:

$$p_n = a^{\xi(n)} \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{n}{n-i} \binom{n-i}{i} (ab)^{\left\lfloor \frac{n}{2} \right\rfloor - i}, \ n \geq 1. \tag{1.5}$$

On the other hand, there are several different q-analogs for the Fibonacci and Lucas sequences [3, 4, 5, 13, 7, 1]. Particularly, Cigler [5] gave the (Carlitz-) q-Fibonacci and q-Lucas polynomials

$$f_n(x,s) = xf_{n-1}(x,s) + q^{n-2} s f_{n-2}(x,s); \ f_0(x,s) = 0, \ f_1(x,s) = 1, \tag{1.6}$$

$$l_n(x,s) = f_{n+1}(x,s) + s f_{n-1}(x,qs); \ l_0(x,s) = 2, \ l_1(x,s) = x, \tag{1.7}$$

respectively.

Additionally, Ramírez and Sirvent [10] introduced a q-analog of the bi-periodic Fibonacci sequence by

$$F_n^{(a,b)}(q,s) = \begin{cases} aF_{n-1}^{(a,b)}(q,s) + q^{n-2} s F_{n-2}^{(a,b)}(q,s), & \text{if } n \text{ is even} \\ bF_{n-1}^{(a,b)}(q,s) + q^{n-2} s F_{n-2}^{(a,b)}(q,s), & \text{if } n \text{ is odd} \end{cases}, n \geq 2 \tag{1.8}$$

with initial conditions $F_0^{(a,b)}(q,s) = 0$ and $F_1^{(a,b)}(q,s) = 1$. They derived the following equality to evaluate the q-bi-periodic Fibonacci sequence:

$$F_n^{(a,b)}(q,s) = \chi_n F_{n-1}^{(a,b)}(q,qs) - qs F_{n-2}^{(a,b)}\left(q, q^2 s\right), \tag{1.9}$$

where $\chi_n := a^{\xi(n+1)} b^{\xi(n)}$. Also, they gave the relationship between the q-bi-periodic Fibonacci sequence and the (Carlitz-) q-Fibonacci polynomials as:

$$F_n^{(a,b)}(q,s) = \left(\sqrt{\frac{a}{b}}\right)^{\xi(n+1)} f_n\left(\sqrt{ab}, s\right). \tag{1.10}$$

By using (1.10), they obtained the explicit formula of the q-bi-periodic Fibonacci sequence as:

$$F_n^{(a,b)}(q,s) = a^{\xi(n-1)} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \begin{bmatrix} n-k-1 \\ k \end{bmatrix} (ab)^{\lfloor \frac{n-1}{2} \rfloor - k} q^{k^2} s^k, \tag{1.11}$$

where $\begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]_q!}{[k]_q![n-k]_q!}$ is the q-binomial coefficients with $[n]_q := 1 + q + q^2 + \cdots + q^{n-1}$ and $[n]_q! := [1]_q [2]_q \cdots [n]_q$.

Motivated by the Ramirez's results in [10], here we introduce a q-analog of the bi-periodic Lucas sequence, called as the q-bi-periodic Lucas sequence, and give some identities related to the q-bi-periodic Fibonacci and Lucas sequences. Also, we give a matrix representation for the q-bi-periodic Fibonacci sequence which allow us to obtain several properties of this sequence in a simple way. Moreover, by using the explicit formulas for the q-bi-periodic Fibonacci and Lucas sequences, we introduce q-analogs of the bi-periodic incomplete Fibonacci and Lucas sequences and give a relation between them.

2. A q-analog of the bi-periodic Lucas sequence

First, we consider the (Carlitz-) q-Lucas polynomials in (1.7), and define the q-bi-periodic Lucas sequence by means of the (Carlitz-) q-Lucas polynomials.

Definition 1. The q-bi-periodic Lucas sequence is defined by

$$L_n^{(a,b)}(q,s) = \left(\sqrt{\frac{a}{b}}\right)^{\xi(n)} l_n\left(\sqrt{ab}, s\right) \tag{2.1}$$

where $l_n(x,s)$ is the (Carlitz-) q-Lucas polynomials.

The terms of the q-bi-periodic Lucas sequence can be given as:

n	$L_n^{(a,b)}(q,s)$
0	2
1	a
2	$ab + sq + s$
3	$a^2b + as + asq + asq^2$
4	$a^2b^2 + abs + absq + absq^2 + absq^3 + s^2q^2 + s^2q^4$
5	$a^3b^2 + a^2bs + a^2bsq + a^2bsq^2 + a^2bsq^3 + a^2bsq^4$ $+as^2q^2 + as^2q^3 + as^2q^4 + as^2q^5 + as^2q^6$
$\vdots$	$\vdots$

Note that if we take $a = b = x$, we obtain the (Carlitz-) q-Lucas polynomials $l_n(x,s)$.

In the following lemma, we state the q-bi-periodic Lucas sequence in terms of the q-bi-periodic Fibonacci sequence.

Lemma 1. *For any integer $n \geq 0$, we have*

$$L_n^{(a,b)}(q,s) = F_{n+1}^{(a,b)}(q,s) + sF_{n-1}^{(a,b)}(q,qs). \tag{2.2}$$

Proof. By using the definition of the q-bi-periodic Lucas sequence and the relations (1.7) and (1.10), we have

$$\begin{aligned} L_n^{(a,b)}(q,s) &= \left(\sqrt{\frac{a}{b}}\right)^{\xi(n)} l_n\left(\sqrt{ab},s\right) \\ &= \left(\sqrt{\frac{a}{b}}\right)^{\xi(n)} \left(f_{n+1}\left(\sqrt{ab},s\right) + sf_{n-1}\left(\sqrt{ab},qs\right)\right) \\ &= \left(\sqrt{\frac{a}{b}}\right)^{\xi(n)} \left(\sqrt{\frac{b}{a}}\right)^{\xi(n)} \left(F_{n+1}^{(a,b)}(q,s) + sF_{n-1}^{(a,b)}(q,qs)\right) \end{aligned}$$

which gives the desired result. □

Now we give an another relation between the q-bi-periodic Fibonacci sequence and q-bi-periodic Lucas sequence.

Theorem 1. *For any integer $n \geq 0$, we have*

$$\chi_n L_n^{(a,b)}(q,qs) = F_{n+2}^{(a,b)}(q,s) - q^{n+1}s^2F_{n-2}^{(a,b)}\left(q,q^2s\right) \tag{2.3}$$

where $\chi_n := a^{\xi(n+1)}b^{\xi(n)}$.

Proof. By using the definition of the q-bi-periodic Fibonacci sequence in (1.8) and the relations (2.2) and (1.9), we get

$$\begin{aligned} \chi_n L_n^{(a,b)}(q,qs) &= \chi_n\left(F_{n+1}^{(a,b)}(q,qs) + qsF_{n-1}^{(a,b)}\left(q,q^2s\right)\right) \\ &= F_{n+2}^{(a,b)}(q,s) - qsF_n^{(a,b)}\left(q,q^2s\right) + \chi_n qsF_{n-1}^{(a,b)}\left(q,q^2s\right) \\ &= F_{n+2}^{(a,b)}(q,s) - qs\left(F_n^{(a,b)}\left(q,q^2s\right) - \chi_n F_{n-1}^{(a,b)}\left(q,q^2s\right)\right) \\ &= F_{n+2}^{(a,b)}(q,s) - q^{n+1}s^2F_{n-2}^{(a,b)}\left(q,q^2s\right). \end{aligned}$$

□

If we take $a = b = x$ in (2.3), it reduces to the relation between q-bi-periodic Fibonacci sequence and Lucas polynomials

$$xl_n(x,qs) = f_{n+2}(x,s) - q^{n+1}s^2f_{n-2}\left(x,q^2s\right)$$

which can be found in [5, Equation (3.15)].

In the following theorem, we give the explicit expression of the q-bi-periodic Lucas sequence $L_n^{(a,b)}(q,s)$. Since we define the incomplete sequences by using its explicit formula, the following theorem play a key role for our further study in the next section.

Theorem 2. *For any integer $n \geq 0$, we have*

$$L_n^{(a,b)}(q,s) = a^{\xi(n)} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{[n]}{[n-k]} \left[\begin{array}{c} n-k \\ k \end{array} \right] (ab)^{\lfloor \frac{n}{2} \rfloor - k} q^{k^2-k} s^k. \tag{2.4}$$

Proof. By using the relations (2.2) and (1.11), we have

$$\begin{aligned} L_n^{(a,b)}(q,s) &= F_{n+1}^{(a,b)}(q,s) + sF_{n-1}^{(a,b)}(q,qs) \\ &= a^{\xi(n)} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left[\begin{array}{c} n-k \\ k \end{array} \right] (ab)^{\lfloor \frac{n}{2} \rfloor - k} q^{k^2} s^k \\ &\quad + a^{\xi(n-2)} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - 1} \left[\begin{array}{c} n-2-k \\ k \end{array} \right] (ab)^{\lfloor \frac{n}{2} \rfloor - 1 - k} q^{k^2+k} s^{k+1} \\ &= a^{\xi(n)} \left(\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left[\begin{array}{c} n-k \\ k \end{array} \right] (ab)^{\lfloor \frac{n}{2} \rfloor - k} q^{k^2} s^k \right. \\ &\quad \left. + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \left[\begin{array}{c} n-k-1 \\ k-1 \end{array} \right] (ab)^{\lfloor \frac{n}{2} \rfloor - k} q^{k^2-k} s^k \right) \\ &= a^{\xi(n)} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left(q^k \left[\begin{array}{c} n-k \\ k \end{array} \right] + \left[\begin{array}{c} n-k-1 \\ k-1 \end{array} \right] \right) (ab)^{\lfloor \frac{n}{2} \rfloor - k} q^{k^2-k} s^k. \end{aligned}$$

By using the identity

$$q^k \left[\begin{array}{c} n-k \\ k \end{array} \right] + \left[\begin{array}{c} n-k-1 \\ k-1 \end{array} \right] = \frac{[n]}{[n-k]} \left[\begin{array}{c} n-k \\ k \end{array} \right],$$

we obtain the desired result. □

If we take $a = b = x$ in the above theorem, it reduces to the (Carlitz-) q-Lucas polynomials

$$l_n(x,s) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{[n]}{[n-k]} \left[\begin{array}{c} n-k \\ k \end{array} \right] q^{k^2-k} s^k x^{n-2k}$$

which can be found in [5, Equation (3.14)].

Now we give a matrix representation for the q-bi-periodic Fibonacci sequence which can be proven by induction. By using matrix formula, one can obtain several properties of this sequence.

Theorem 3. *For $n \geq 1$, let define the matrix* $C(\chi_n, s) := \begin{pmatrix} 0 & 1 \\ s & \chi_n \end{pmatrix}$. *Then we have*

$$M_n(\chi_n, s) := C\left(\chi_n, q^{n-1}s\right) C\left(\chi_{n-1}, q^{n-2}s\right) \cdots C(\chi_2, qs)\, C(\chi_1, s)$$
$$= \begin{pmatrix} sF_{n-1}^{(a,b)}(q, qs) & \left(\frac{b}{a}\right)^{\xi(n+1)} F_n^{(a,b)}(q, s) \\ sF_n^{(a,b)}(q, qs) & \left(\frac{b}{a}\right)^{\xi(n)} F_{n+1}^{(a,b)}(q, s) \end{pmatrix}. \tag{2.5}$$

In the following theorem, we give the q-Cassini formula for the q-bi-periodic Fibonacci sequence by taking the determinant of the both sides of the equation (2.5).

Theorem 4. *For any integer $n > 0$, we have*

$$\left(\frac{b}{a}\right)^{\xi(n)} F_{n-1}^{(a,b)}(q, qs)\, F_{n+1}^{(a,b)}(q, s) - \left(\frac{b}{a}\right)^{\xi(n+1)} F_n^{(a,b)}(q, s)\, F_n^{(a,b)}(q, qs)$$
$$= (-1)^n s^{n-1} q^{\frac{n(n-1)}{2}}. \tag{2.6}$$

Note that by taking $a = b = x$, we obtain the result in [5, Equation (3.12)].

Theorem 5. *For any integer $n > 0$, we have*

$$F_{2n}^{(a,b)}(q, s) = \left(\frac{a}{b}\right)^{\xi(n)} q^n s F_{n-1}^{(a,b)}\left(q, q^{n+1}s\right) F_n^{(a,b)}(q, s) + F_n^{(a,b)}(q, q^n s)\, F_{n+1}^{(a,b)}(q, s). \tag{2.7}$$

Proof. Since $M_{m+n}(\chi_n, s) = M_m(\chi_n, q^n s)\, M_n(\chi_n, s)$, if we equate the corresponding entries of each matrices and take $m = n$ in the resulting equality, we get the desired result. □

One can get several properties of the q-bi-periodic Fibonacci sequence by taking proper powers of the matrix in (2.5).

3. $q-$bi-periodic incomplete Fibonacci and Lucas sequences

In this section, we define q-bi-periodic incomplete Fibonacci and Lucas sequences. Let n be a positive integer and l be an integer.

Ramirez [9] defined the bi-periodic incomplete Fibonacci numbers by using the explicit formula of the bi-periodic Fibonacci sequences in (1.3) as:

$$q_n(l) = a^{\xi(n-1)} \sum_{i=0}^{l} \binom{n-1-i}{i} (ab)^{\lfloor \frac{n-1}{2} \rfloor - i}, \; 0 \leq l \leq \left\lfloor \frac{n-1}{2} \right\rfloor$$

Similarly, by using the explicit formula of the bi-periodic Lucas sequence in (1.5), Tan and Ekin [14] defined the bi-periodic incomplete Lucas numbers as:

$$p_n(l) = a^{\xi(n)} \sum_{i=0}^{l} \frac{n}{n-i} \binom{n-i}{i} (ab)^{\lfloor \frac{n}{2} \rfloor - i}, \; 0 \leq l \leq \left\lfloor \frac{n}{2} \right\rfloor.$$

Analogously, by using the explicit formulas of the q-bi-periodic Fibonacci sequence in (1.11) and the q-bi-periodic Lucas sequence in (2.4), we define the q-bi-periodic incomplete Fibonacci and Lucas sequences as follows.

Definition 2. For any non negative integer n, the q-bi-periodic incomplete Fibonacci and Lucas sequences are defined by

$$F_{n,l}^{(a,b)}(q,s) = a^{\xi(n-1)} \sum_{k=0}^{l} \begin{bmatrix} n-1-k \\ k \end{bmatrix} (ab)^{\lfloor \frac{n-1}{2} \rfloor - k} q^{k^2} s^k, \ 0 \le l \le \left\lfloor \frac{n-1}{2} \right\rfloor \tag{3.1}$$

and

$$L_{n,l}^{(a,b)}(q,s) = a^{\xi(n)} \sum_{k=0}^{l} \frac{[n]}{[n-k]} \begin{bmatrix} n-k \\ k \end{bmatrix} (ab)^{\lfloor \frac{n}{2} \rfloor - k} q^{k^2-k} s^k, \ 0 \le l \le \left\lfloor \frac{n}{2} \right\rfloor, \tag{3.2}$$

respectively.

If we take $l = \lfloor \frac{n-1}{2} \rfloor$ in (3.1), we obtain the q-bi-periodic Fibonacci sequence, and if we take $l = \lfloor \frac{n}{2} \rfloor$ in (3.2), we obtain the q-bi-periodic Lucas sequence.

Next, we give non-homogenous recurrence relation for the q-bi-periodic incomplete Fibonacci sequence.

Theorem 6. *For $0 \le l \le \frac{n-2}{2}$, the non-linear recurrence relation of the q-bi-periodic incomplete Fibonacci sequence is*

$$F_{n+2,l+1}^{(a,b)}(q,s) = \begin{cases} aF_{n+1,l+1}^{(a,b)}(q,s) + q^n s F_{n,l}^{(a,b)}(q,s), & \text{if } n \text{ is even} \\ bF_{n+1,l+1}^{(a,b)}(q,s) + q^n s F_{n,l}^{(a,b)}(q,s), & \text{if } n \text{ is odd} \end{cases}. \tag{3.3}$$

The relation (3.3) can be transformed into the non-homogeneous recurrence relation

$$F_{n+2,l}^{(a,b)}(q,s) = aF_{n+1,l}^{(a,b)}(q,s) + q^n s F_{n,l}^{(a,b)}(q,s) - a \begin{bmatrix} n-1-l \\ l \end{bmatrix} (ab)^{\lfloor \frac{n-1}{2} \rfloor - l} q^{n+l^2} s^{l+1} \tag{3.4}$$

for even n, and

$$F_{n+2,l}^{(a,b)}(q,s) = bF_{n+1,l}^{(a,b)}(q,s) + q^n s F_{n,l}^{(a,b)}(q,s) - \begin{bmatrix} n-1-l \\ l \end{bmatrix} (ab)^{\lfloor \frac{n-1}{2} \rfloor - l} q^{n+l^2} s^{l+1} \tag{3.5}$$

for odd n.

Proof. If n is even, then $\lfloor \frac{n+1}{2} \rfloor = \lfloor \frac{n}{2} \rfloor$. By using the Definition (3.1), we can write the RHS of (3.3) as

$$a^{1+\xi(n)} \sum_{k=0}^{l+1} \begin{bmatrix} n-k \\ k \end{bmatrix} (ab)^{\lfloor \frac{n}{2} \rfloor - k} q^{k^2} s^k$$

$$+q^n s a^{\xi(n-1)} \sum_{k=0}^{l} \begin{bmatrix} n-1-k \\ k \end{bmatrix} (ab)^{\lfloor \frac{n-1}{2} \rfloor - k} q^{k^2} s^k$$

$$= a\sum_{k=0}^{l+1}\begin{bmatrix} n-k \\ k \end{bmatrix}(ab)^{\lfloor \frac{n}{2}\rfloor -k}q^{k^2}s^k + q^n\ a\sum_{k=0}^{l}\begin{bmatrix} n-1-k \\ k \end{bmatrix}(ab)^{\lfloor \frac{n-1}{2}\rfloor -k}q^{k^2}s^{k+1}$$

$$= a\sum_{k=0}^{l+1}\begin{bmatrix} n-k \\ k \end{bmatrix}(ab)^{\lfloor \frac{n}{2}\rfloor -k}q^{k^2}s^k$$

$$+q^n\ a\sum_{k=1}^{l+1}\begin{bmatrix} n-k \\ k-1 \end{bmatrix}(ab)^{\lfloor \frac{n}{2}\rfloor -k}q^{(k-1)^2}s^k$$

$$= a\sum_{k=0}^{l+1}\left(\begin{bmatrix} n-k \\ k \end{bmatrix} + q^{n-2k+1}\begin{bmatrix} n-k \\ k-1 \end{bmatrix}\right)(ab)^{\lfloor \frac{n}{2}\rfloor -k}q^{k^2}s^k\ \ (ab)^{\lfloor \frac{n}{2}\rfloor -k} - 0$$

$$= a\sum_{k=0}^{l+1}\begin{bmatrix} n-k+1 \\ k \end{bmatrix}(ab)^{\lfloor \frac{n}{2}\rfloor -k}q^{k^2}s^k\ \ (ab)^{\lfloor \frac{n}{2}\rfloor -k}$$

$$= F_{n+2,l+1}^{(a,b)}(q,s).$$

Also from equation (3.3), we have

$$\begin{aligned}
F_{n+2,l}^{(a,b)}(q,s) &= aF_{n+1,l}^{(a,b)}(q,s) + q^n sF_{n,l-1}^{(a,b)}(q,s) \\
&= aF_{n+1,l}^{(a,b)}(q,s) + q^n sF_{n,l}^{(a,b)}(q,s) + q^n s(F_{n,l-1}^{(a,b)}(q,s) - F_{n,l}^{(a,b)}(q,s)) \\
&= aF_{n+1,l}^{(a,b)}(q,s) + q^n sF_{n,l}^{(a,b)}(q,s) - a\begin{bmatrix} n-1-l \\ l \end{bmatrix}(ab)^{\lfloor \frac{n-1}{2}\rfloor -l}q^{n+l^2}s^{l+1}.
\end{aligned}$$

If n is odd, the proof is completely analogous. □

Note that the q-bi-periodic Lucas sequence does not satisfy a recurrence like (3.3), since $F_{n+1}^{(a,b)}(q,s)$ and $F_{n+1}^{(a,b)}(q,qs)$ do not satisfy the same recurrence relation.

Finally we give the relationship between the q-bi-periodic incomplete Fibonacci and Lucas sequences as follows:

Theorem 7. *For* $0 \le l \le \lfloor \frac{n}{2} \rfloor$, *we have*

$$L_{n,l}^{(a,b)}(q,s) = F_{n+1,l}^{(a,b)}(q,s) + F_{n-1,l-1}^{(a,b)}(q,qs). \tag{3.6}$$

Proof. It can be proved easily by using the definitions (3.1) and (3.2). □

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