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A GENERALIZATION OF THE PEANO KERNEL AND ITS APPLICATIONS

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Abstract. Based on the q-Taylor Theorem, we introduce a more general form of the Peano kernel (q-Peano) which is also applicable to non-differentiable functions. Then we show that quantum B-splines are the q-Peano kernels of divided differences. We also give applications to polynomial interpolation and construct examples in which classical remainder theory fails whereas q-Peano kernel works.

1. Introduction

Recent advances in the quantum B-splines, [4, 6, 17] have given us an opportunity to arise the question if there is a way to link quantum B-splines with a more general Peano kernel. The quantum B-spline functions are piecewise polynomials whose quantum derivatives agree at the joins up to some order. The quantum B-splines are introduced by Simeonov & Goldman [17] to generalize classical B-splines by replacing ordinary derivatives by quantum derivatives. Their work constructs not only a new type of de Boor algorithm but also novel identities via blossoms. Actually the underlying idea in [17] goes back to the work [16] which aimed to find a new form of blossoms to represent q-Bernstein polynomials. Just like classical Bernstein polynomials, the q-Bernstein polynomials possesses remarkable geometric and analytic properties, see [11, 13]. So, our objectives are to extend the Peano kernel and then relate with the quantum B-splines. This extension is important because there are functions whose q-derivatives exist but whose classical derivatives fail to exist. Furthermore it will also lead us to investigate errors in approximations.

The classical Peano kernel theorem provides a useful technique for computing the errors of approximations such as interpolation, quadrature rules and B-splines. The errors are represented by a linear functional that operates on functions

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$f \in C^{n+1}[a,b]$ and annihilates all polynomials of degree at most n.

Namely, if $L(f)=0$ for all $f \in \mathcal{P}_n$, the space of polynomials of degree n, then

$$L(f) = \int_a^b f^{(n+1)}(t)K(x,t)dt,$$

where $K(x,t) = \dfrac{1}{n!}L\left((x-t)_+^n\right)$.

An important application of this result is the Kowalewski's interpolating polynomial remainder. Let $t_0, t_1, \ldots, t_n \in [a,b]$ be fixed and distinct, and

$$L(f) = f(x) - \sum_{k=0}^{n} f(t_k)l_{nk}(x)$$

where $l_{nk}(x) = \prod_{\substack{v=0\\ v\neq k}}^{n} \dfrac{x-t_v}{t_k-t_v}$. If $f \in C^{m+1}[a,b]$, then

$$L(f) = \frac{1}{m!}\sum_{k=0}^{n} l_{nk}(x)\int_{t_k}^{x}(t_k-t)^m f^{(m+1)}(t)dt, \quad \text{for each } \ m = 0,1,\ldots,n$$

is the error functional, see [7].

This paper is organized as follows: We begin with definitions and properties of the quantum calculus needed for this work. In Section 3, we give the q-Taylor theorem and develop a generalization of the Peano kernel (q-Peano kernel). We present a simple way to find $L(f)$ under the condition in which the kernel has no sign change. Moreover, taking $L(f)$ as divided differences we construct a relation between q-B-splines and q-Peano kernel. Section 4 demonstrates how the q-Peano kernel is used to find the error of Lagrange interpolation. Finally, the error bounds of quadrature formula on the remainder involving q-integration is discussed.

2. Preliminaries

Throughout the paper we consider q as a real fixed parameter. Let us give basic definitions and theorems of the q-calculus that are required in the next sections. For a fixed parameter $q \neq 1$, the q-derivatives are defined by,

$$\begin{aligned} D_q f(t) &= \frac{f(qt)-f(t)}{(q-1)t} \\ D_q^n f(t) &= D_q(D_q^{n-1}f(t)), \quad n \geqslant 2. \end{aligned}$$

Note that q-derivatives are approximations to classical derivatives and if f is a differentiable function, then

$$\lim_{q\to 1} D_q f(x) = Df(x).$$

For polynomials the q-derivative is easy to compute. Indeed it follows easily from the definition of the q-derivative that

$$D_q x^n = [n]_q x^{n-1},$$

where the q-integers $[n]_q$ are defined by,

$$[n]_q = \begin{cases} (1-q^n)/(1-q), & q \neq 1, \\ n, & q = 1. \end{cases}$$

Moreover, the q-factorial is defined by

$$[n]_q! = [1]_q \cdots [n]_q.$$

Quantum integrals are the analogues of classical integrals for the quantum calculus. Quantum integrals satisfy a quantum version of the fundamental theorem of calculus, see [9] for details.

Definition 1. *Let $0 < a < b$. Then the definite q-integral of a function $f(x)$ is defined by a convergent series*

$$\int_0^b f(x) d_q x = (1-q) b \sum_{i=0}^{\infty} q^i f(q^i b)$$

and

$$\int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x.$$

Theorem 1. *[Fundamental Theorem of q-Calculus]*
If $F(x)$ is continuous at $x = 0$, then

$$\int_a^b D_q F(x) d_q x = F(b) - F(a)$$

where $0 \leqslant a < b \leqslant \infty$.

The work [15] gives the mean value theorem in the q-calculus which will be needed in one of our results.

Theorem 2. *If F is continuous and G is $1/q$-integrable and is non-negative(or non-positive) on $[a, b]$, then there exists $\tilde{q} \in (1, \infty)$ such that for all $q > \tilde{q}$ there exists $\xi \in (a, b)$ for which*

$$\int_a^b F(x) G(x) d_{1/q} x = F(\xi) \int_a^b G(x) d_{1/q} x.$$

We also require a q-Hölder inequality and appropriate notions of distance in q-integrals, see [2, 5, 18].

Definition 2. *We will denote by $L_{p,q}([0,b])$ with $1 \leqslant p < \infty$, the set of all functions f on $[0,b]$ such that*

$$||f||_{p,q} := \left(\int\limits_0^b |f|^p d_{1/q} t \right)^{\frac{1}{p}} < \infty.$$

Furthermore let $L_{\infty,q}([0,b])$ denote the set of all functions f on $[0,b]$ such that

$$||f||_{\infty,q} := \sup_{x \in [0,b]} |f(x)| < \infty.$$

Theorem 3. *Let $x \in [0,b]$, $q \in [1,\infty)$ and $p_1, p_2 > 1$ be such that $\frac{1}{p_1} + \frac{1}{p_2} = 1$. Then*

$$\int\limits_0^x |f(x)||g(x)| d_{1/q} t \leqslant \left(\int\limits_0^x |f(x)|^{p_1} d_{1/q} t \right)^{\frac{1}{p_1}} \left(\int\limits_0^x |g(x)|^{p_2} d_{1/q} t \right)^{\frac{1}{p_2}}. \tag{2.1}$$

3. q-Peano Kernel Theorem

In this section we derive a more general form of the Peano kernel theorem based on a q-Taylor expansion. So we start by giving the q-Taylor Theorem with integral remainder. A detailed treatment of the classical Peano kernel theorem can be found in [7, 12, 14].

We use the notation q-$C^k[a,b]$ to denote the space of bounded functions whose q-derivatives of order up to k are continuous on $[a,b]$.

Theorem 4. *(q-Taylor Theorem) Let f be $n+1$ times $1/q$-differentiable in the closed interval $[a,b]$. Then*

$$f(x) = \sum_{k=0}^{n} q^{k(k-1)/2} \frac{(D_{1/q}^k f)(q^k a)}{[k]_q!} (x-a)^{k,q} + R_n(f), \tag{3.1}$$

where

$$(x-t)^{n,q} = (x - q^{n-1} t) \cdots (x - qt)(x - t)$$

and

$$R_n(f) = \frac{q^{n(n+1)/2}}{[n]_q!} \int_a^x (D_{1/q}^{n+1} f)(q^n t)(x-t)^{n,q} d_{1/q} t.$$

Another way to express the remainder $R_n(f)$ is to employ the truncated power function. That is

$$R_n(f) = \frac{q^{n(n+1)/2}}{[n]_q!} \int_a^b (D_{1/q}^{n+1} f)(q^n t)(x-t)_+^{n,q} d_{1/q} t, \tag{3.2}$$

where

$$(x-t)_+^{n,q} = (x - q^{n-1} t) \cdots (x - qt)(x - t)_+.$$

Here $(x-t)_+$ *is the truncated power function*

$$(x-t)_+ = \begin{cases} x-t, & \text{if} \quad x>t \\ 0, & \text{otherwise.} \end{cases}$$

Although the latter representation of the remainder $R_n(f)$ associated with our results is new, we omit the proof since it can be done in a similar way as in [9]. There are other forms of q-Taylor Theorem, see for example [1, 8, 10]. The work [3] investigates the convergence of q-Taylor series for q-difference operators using q-Cauchy integral formula.

Theorem 5. *Let* $g_t(x) = (x-t)_+^{n,q}$ *and let* L *be a linear functional that commutes with the operation of* q*-integration and also satisfies the conditions:* $L(g_t)$ *exists and* $L(f) = 0$ *for all* $f \in \mathcal{P}_n$*. Then for all* $f \in 1/q - C^{n+1}[a,b]$

$$L(f) = \int_a^b (D_{1/q}^{n+1} f)(q^n t) K(x,t) d_{1/q} t,$$

where

$$K(x,t) = \frac{q^{n(n+1)/2}}{[n]_q!} L(g_t).$$

Proof. Recall that here the function $(x-t)_+^{n,q}$ is a function of t and x behaves as a parameter. When we say $L(g_t)$ we mean that L is applied to the truncated power function, regarded as a function of x with t as a parameter. Hence we find real number that depends on t. We apply L to the equation (3.1). Since L is linear and annihilates polynomials, we have

$$L(f) = \frac{q^{n(n+1)/2}}{[n]_q!} L\left(\int_a^b (D_{1/q}^{n+1} f)(q^n t)(x-t)_+^{n,q} d_{1/q} t \right).$$

Since L commutes with the operation of q-integration,

$$L(f) = \frac{q^{n(n+1)/2}}{[n]_q!} \int_a^b (D_{1/q}^{n+1} f)(q^n t) L\left((x-t)_+^{n,q}\right) d_{1/q} t.$$

$\square$

Corollary 1. *If the conditions in Theorem 5 are satisfied and also the kernel* $K(x,t)$ *does not change sign on* $[a,b]$*, then*

$$L(f) = \frac{\left(D_{1/q}^{n+1} f\right)(\xi)}{[n+1]_q!} q^{n(n+1)/2} L(x^{n+1}).$$

Proof. Since $D_{1/q}^{n+1} f$ is continuous and $K(x,t)$ does not change sign on $[a,b]$, we can apply the Mean Value Theorem 2. Thus we have

$$L(f) = \left(D_{1/q}^{n+1} f\right)(\xi) \int_a^b K(x,t) d_{1/q} t, \quad a<\xi<b.$$

Replacing $f(x)$ by x^{n+1} gives

$$L(x^{n+1}) = \frac{[n+1]_q!}{q^{n(n+1)/2}} \int_a^b K(x,t) d_{1/q}t,$$

so

$$\int_a^b K(x,t) d_{1/q}t = \frac{q^{n(n+1)/2}}{[n+1]_q!} L(x^{n+1}),$$

and this completes the proof. □

We now establish a relation between q-B-splines and q-Peano kernels. Recently q-analogue or quantum B-splines which generalize B-splines have been investigated in several aspects in [4, 6, 17]. The work [4] finds out that q-B-splines are essentially divided differences of q-truncated power functions. That is, the q-B-spline of degree n is given by

$$N_{k,n}(t;q) = (t_{k+n+1} - t_k)[t_k, \ldots, t_{k+n+1}](x-t)_+^{n,q}.$$

Although classical truncated power function has n multiple zero at $t = x$, q-truncated power function has n distinct zeros for $q \neq 0$ or $q \neq 1$. This property drastically alters certain characteristics of the basis functions. For example while the basis functions forms partition of unity, the non-negativity property is lost. On the other hand when q is near one, the additional real parameter q provides extra flexibility to change the shape of basis functions. Sometimes this effect may be useful and practical to match smoothness of piecewise curves and surfaces up to some tolerance, see [6].

Now recall the fact that a divided difference $f[t_0, t_1, \ldots, t_{n+1}]$ can be represented as symmetric sum of $f(t_j)$, see [14],

$$f[t_0, t_1, \ldots, t_{n+1}] = \sum_{i=0}^{n+1} f(t_i) / \prod_{\substack{j=0 \\ j \neq i}}^{n+1} (t_i - t_j). \tag{3.3}$$

Hence we can readily derive

$$N_{k,n}(t;q) = (t_{k+n+1} - t_k) \sum_{i=k}^{k+n+1} (t_i - t)_+^{n,q} / \prod_{\substack{j=k \\ j \neq i}}^{k+n+1} \frac{1}{(t_i - t_j)}.$$

The following theorem shows that q-B-splines are indeed the q-Peano kernels of divided differences.

Theorem 6. *Let $f \in 1/q - C^{n+1}[a,b]$. Then*

$$f[t_0, t_1, \ldots, t_{n+1}] = \frac{q^{n(n+1)/2}}{[n]_q!} \int_a^b \frac{N_{0,n}(t;q)}{t_{n+1} - t_0} \left(D_{1/q}^{n+1} f \right) (q^n t) d_{1/q}t.$$

Proof. We first set L as

$$\begin{aligned} f[t_0, t_1, \ldots, t_{n+1}] =& \sum_{i=0}^{n+1} f(t_i) / \prod_{\substack{j=0 \\ j\neq i}}^{n+1} (t_i - t_j) \\ =& L(f). \end{aligned}$$

We see that for any fixed and distinct points $\{t_i : i = 0, 1, \ldots, n+1\}$, L is a bounded linear operator. From the q-Peano Kernel Theorem 5, we have

$$L(f) = \int_a^b K(x,t)(D_{1/q}^{n+1} f)(q^n t) d_{1/q} t,$$

where

$$K(x,t) = \frac{q^{n(n+1)/2}}{[n]_q!} L\left((x-t)_+^{n,q}\right).$$

This can be written as

$$K(x,t) = \frac{q^{n(n+1)/2}}{[n]_q!} \sum_{i=0}^{n+1} (t_i - t)_+^{n,q} \Big/ \prod_{\substack{j=0 \\ j\neq i}}^{n+1} (t_i - t_j).$$

Thus

$$K(x,t) = \frac{q^{n(n+1)/2}}{[n]_q!} \frac{N_{0,n}(t;q)}{t_{n+1} - t_0}.$$

Combining the last equation with (3.3) we derive

$$f[t_0, t_1, \ldots, t_{n+1}] = \frac{q^{n(n+1)/2}}{[n]_q!} \int_a^b \frac{N_{0,n}(t;q)}{t_{n+1} - t_0} \left(D_{1/q}^{n+1} f\right)(q^n t) d_{1/q} t.$$

□

When $q = 1$, the above Theorem 3.3 reduces to its classical counterpart which can be found in [14]. The work [4] extends several classical formulas of B-splines to quantum B-splines.

4. Application to polynomial interpolation

The main idea in this section is to apply the q-Peano kernel Theorem on the remainder of polynomial interpolation. Findings demonstrate the advantage of using the q-Peano kernel Theorem where the classical theorem does not work. The following theorem has weaker assumption than the classical one and thus gives stronger results.

Theorem 7. *Let $f \in 1/q - C^{n+1}[a,b]$ and suppose $t_0, t_1, \ldots, t_n \in [a,b]$ are distinct points. For a fixed $x \in [a,b]$, define the corresponding error functional by*

$$L(f) = f(x) - \sum_{k=0}^{n} f(t_k) l_{nk}(x).$$

Then

$$L(f) = \frac{q^{m(m+1)/2}}{[m]_q!} \sum_{k=0}^{n} l_{nk}(x) \int_{t_k}^{x} (t_k - t)^{m,q} \left(D_{1/q}^{m+1} f\right)(q^m t) d_{1/q} t, \qquad m = 0, 1, \ldots, n.$$

Proof. Since $\sum_{k=0}^{n} l_{nk}(x) = 1$, by the q-Peano kernel Theorem 5 we get,

$$\begin{aligned} \frac{[m]_q!}{q^{m(m+1)/2}} K(x,t) = L\left((x-t)_+^{m,q}\right) &= (x-t)_+^{m,q} - \sum_{k=0}^{n} (t_k - t)_+^{m,q} l_{nk}(x) \\ &= \sum_{k=0}^{n} \left[(x-t)_+^{m,q} - (t_k - t)_+^{m,q}\right] l_{nk}(x). \end{aligned}$$

It follows that

$$\begin{aligned} &\frac{[m]_q!}{q^{m(m+1)/2}} \int_a^b K(x,t) \left(D_{1/q}^{m+1} f\right)(q^m t) d_{1/q} t = \\ &\quad \int_a^x \left\{ \left(D_{1/q}^{m+1} f\right)(q^m t) \sum_{k=0}^{n} \left[(x-t)^{m,q} - (t_k - t)^{m,q}\right] l_{nk}(x) \right\} d_{1/q} t \\ &\qquad + \sum_{k=0}^{n} l_{nk}(x) \int_{t_k}^{x} (t_k - t)^{m,q} \left(D_{1/q}^{m+1} f\right)(q^m t) d_{1/q} t. \end{aligned}$$

For each $m \leqslant n$, since the interpolation operator is a projection operator, it reproduces polynomials and hence the term in the first summation of the last equation vanishes for $f(x) = (x-t)^{m,q}$. Accordingly,

$$\begin{aligned} L(f) &= \int_a^b K(x,t) \left(D_{1/q}^{m+1} f\right)(q^m t) d_{1/q} t \\ &= \frac{q^{m(m+1)/2}}{[m]_q!} \sum_{k=0}^{n} l_{nk}(x) \int_{t_k}^{x} (t_k - t)^{m,q} \left(D_{1/q}^{m+1} f\right)(q^m t) d_{1/q} t \end{aligned}$$

for each $m = 0, 1, \ldots, n$. □

Now, we give examples that show how we can find the q-Peano kernel.

Example: Let

$$f(x)=\begin{cases}\dfrac{q^3x^3}{6}, & 0\leqslant x<1\\[2mm] \dfrac{1}{6}\left(4-4[3]_qx+4q[3]_qx^2-3q^3x^3\right), & 1\leqslant x<2\\[2mm] \dfrac{1}{6}\left(-44+20[3]_qx-8q[3]_qx^2+3q^3x^3\right), & 2\leqslant x<3\\[2mm] -\dfrac{1}{6}(-4+x)(-4+qx)(-4+q^2x), & 3\leqslant x<4\\[2mm] 0, & \text{otherwise.}\end{cases}$$

It is obvious that for $q\neq 1$, $f\in C[0,4]$ but $f\notin C^1[0,4]$. However, one may check that $f\in 1/q-C^2[0,4]$. Classical error functionals cannot work but we may find the error via the q-Peano kernel theorem. Let $t_0=0$, $t_1=2$ and $t_2=4$. Then it is appropriate to take the error functional

$$L(f)=q\sum_{k=0}^{2}l_{2k}(x)\int_{t_k}^{x}(t_k-t)\left(D^2_{1/q}f\right)(qt)d_{1/q}t,$$

where $l_{20}(x)=\frac{1}{8}(x-2)(x-4)$, $l_{21}(x)=-\frac{1}{4}x(x-4)$ and $l_{22}(x)=\frac{1}{8}x(x-2)$. Hence

$$\begin{aligned}\frac{1}{q}L(f) &= l_{20}(x)\int_0^x(-t)\left(D^2_{1/q}f\right)(qt)d_{1/q}t+l_{21}(x)\int_2^x(2-t)\left(D^2_{1/q}f\right)(qt)d_{1/q}t\\ &+ l_{22}(x)\int_4^x(4-t)\left(D^2_{1/q}f\right)(qt)d_{1/q}t.\end{aligned}$$

Now we will find the kernel. If $0\leqslant x<2$, then

$$K(x,t)=\begin{cases}-l_{20}(x)t, & 0\leqslant t<x\\[2mm] l_{21}(x)(2-t)-l_{22}(x)(4-t), & x\leqslant t<2\\[2mm] -l_{22}(x)(4-t), & 2\leqslant t<4.\end{cases}$$

Similarly, for $2\leqslant x<4$,

$$K(x,t)=\begin{cases}-l_{20}(x)t, & 0\leqslant t<2\\[2mm] -l_{20}(x)t+l_{21}(x)(2-t), & 2\leqslant t<x\\[2mm] l_{21}(x)(2-t)-l_{22}(x)(4-t), & x\leqslant t<4.\end{cases}$$

One may notice that the function $f(x)$ given above is indeed a cubic q-B-spline. A more recent work [6] on the q-B-splines demonstrates that these functions prove useful in several aspects including geometric modelling.

4.1. **Trapezoidal rule in q-integration.** Consider the $1/q$-integral of a function f on the interval $[a, b]$. We want to evaluate the q-integral approximately using linear interpolation formula. Let us define the operator L as

$$L(f) = \int_a^b f(x) d_{1/q}x - \frac{b - aq}{[2]_q} f(a) - \frac{bq - a}{[2]_q} f(b).$$

Since $L(f) = 0$ for all functions $f \in \mathcal{P}_1$ and for all $f \in 1/q - C^2[a, b]$, we have

$$L(f) = \int_a^b \left(D^2_{1/q} f \right) (qt) K(x, t) d_{1/q}t$$

and

$$K(x, t) = qL((x - t)_+).$$

Thus,

$$K(x, t) \quad = \quad \frac{q}{[2]_q} (b - t)(a - t), \qquad a \leqslant t \leqslant b.$$

Notice that $K(x, t) < 0$ on $[a, b]$. Then by applying Mean Value Theorem 2 we have

$$L(f) = \frac{D^2_{1/q} f(\xi)}{[2]_q!} qL(x^2),$$

where

$$L(x^2) \quad = \quad \frac{-(b - a)(bq - a)(b - aq)}{[3]_q!}$$

Therefore we find that

$$L(f) \quad = \quad \frac{-q(b - a)(bq - a)(b - aq)}{[3]_q! [2]_q!} D^2_{1/q} f(\xi), \quad a < \xi < b.$$

While $q = 1$ reduces the above $L(f)$ to the classical error functional of the trapezoidal rule, on the other hand it provides extra flexibility on the control of error functional by changing the parameter q appropriately.

4.2. **The remainder on quadrature.** We now discuss error bounds of quadrature formulas on remainders given by

$$R_n(f; q) = \int_0^b f(x) d_{1/q}x - \sum_{k=0}^{n} \gamma_{nk} f(t_{nk}).$$

For $q = 1$, this formula is well-known in the context of numerical integration. Assuming $f \in 1/q - C^{m+1}[0, b]$ and $R_n(f; q) = 0$ for all $f \in \mathcal{P}_m, m = 0, 1, \ldots, n$, we can apply the q-Peano kernel theorem. Hence

$$R_n(f;q) = \int_0^b K(x,t) \left(D_{1/q}^{m+1} f\right)(q^m t) d_{1/q} t.$$

By applying the q-Hölder inequality (2.1), we have

$$|R_n(f;q)| \leqslant \left[\int_0^b \left|\left(D_{1/q}^{m+1} f\right)(q^m t)\right|^{p_1} d_{1/q} t\right]^{\frac{1}{p_1}} \left[\int_0^b |K(x,t)|^{p_2}\, d_{1/q} t\right]^{\frac{1}{p_2}}$$

for all $1 \leqslant p_1, p_2 \leqslant \infty$ and $\frac{1}{p_1} + \frac{1}{p_2} = 1$. Since the second integral in the last inequality is independent of f, by choosing coefficients and nodes appropriately we can minimize the remainder $R_n(f; q)$.

(i) For $p_1 = \infty$ and $p_2 = 1$,

$$|R_n(f;q)| \leqslant ||D_{1/q}^{m+1} f||_\infty \int_0^b |K(x,t)| d_{1/q} t.$$

(ii) For $p_1 = p_2 = 2$,

$$|R_n(f;q)| \leqslant ||D_{1/q}^{m+1} f||_2 \left[\int_0^b |K(x,t)|^2 d_{1/q} t\right]^{\frac{1}{2}}.$$

The q-Peano kernel $K(x, t)$ in the latter inequality can be written as

$$K(x,t) = q^{m(m+3)/2} \frac{(b - \frac{t}{q})^{m+1,q}}{[m+1]_q!} - s(t;q),$$

where $s(t;q) = \dfrac{q^{m(m+1)/2}}{[m]_q!} \displaystyle\sum_{k=0}^{n} \gamma_{nk}(t_{nk} - t)_+^{m,q}$ is a quantum spline with the knot sequence $\{t_{n0}, \ldots, t_{nn}\}$. Eventually, the problem of minimizing the q-integral

$$\left[\int_0^b |K(x,t)|^{p_1} d_{1/q} t\right]^{\frac{1}{p_1}}$$

is equivalent to finding the best approximation of the polynomial

$$q^{m(m+3)/2} \frac{(b - \frac{t}{q})^{m+1,q}}{[m+1]_q!}$$

in t by a quantum spline with respect to the norm $||.||_{p_1}$.

5. Conclusion

In this work, we investigated a generalization of classical Peano Kernel theorem via quantum calculus. Applications to polynomial interpolation, q-integration and quantum spline functions, and best approximation were also presented. In the future, we aim to establish relations between q-Peano kernels and Green's functions using q-difference equations and quantum calculus.

References

[1] Al-Salam W.A., Verma, A., A Fractional Leibniz q-Formula, *Pasific Journal of Mathematics* (1972), 60(2).

[2] Anastassiou G. A., Intelligent Mathematics: Computational Analysis. Springer-Verlag, Berlin Heidelberg, 2010.

[3] Annaby M.H., Mansour Z.S., q-Taylor and interpolation series for Jackson q-difference operators, *Journal of Mathematical Analysis and Applications* 344 (2008), 472-483.

[4] Budakçı G., Dişibüyük Ç., Goldman R., Oruç H., Extending Fundamental Formulas from Classical B-Splines to Quantum B-Splines, *Journal of Computational and Applied Mathematics* 282 (2015), 17-33.

[5] Gauchman H., Integral Inequalities in q-Calculus, *Computers and Mathematics with Applications* 47 (2004), 281-300.

[6] Goldman, R., Simeonov, P. Generalized quantum splines, *Computer Aided Geometric Design* (2016), http://dx.doi.org/10.1016/j.cagd.2016.02.019.

[7] Hammerlin G.,Hoffmann K., Numerical Mathematics. Springer-Verlag, New York, 1991.

[8] Ismail M.E.H., Stanton D., Applications of q-Taylor theorems, *Journal of Computaional and Applied Mathematics* 153 (2003), 259-272.

[9] Kac V., Cheung P., Quantum Calculus. Universitext Series, IX, Springer Verlag, 2002.

[10] Pashaev O.K., Nalci S., q-analytic functions, fractals and generalized analytic functions, *Journal of Physics a-Mathematical and Theoretical* 47(4) (2014), 045204.

[11] Oruç, H. & Phillips, G.M. q-Bernstein polynomials and Bèzier curves. *Journal of Computational and Applied Mathematics*, 151 (2003) 1-12.

[12] Phillips G.M., Interpolation and Approximation by Polynomials. Springer-Verlag, New York, 2003.

[13] Phillips, G.M., Survey of results on the q-Bernstein polynomials. *IMA Journal of Numerical Analysis*, 30(1) (2010), 277-288.

[14] Powell M.J.D., Approximation Theory and Methods. Cambridge University Press, 1981.

[15] Rajković P. M., Stanković M. S.,Marinković S. D., Mean value theorems in q-calculus, *Proceedings of the 5th International Symposium on Mathematical Analysis and its Applications, Mat. Vesnik* 54(3-4) (2002), 171–178.

[16] Simeonov P. , Zafiris V., Goldman R., q-Blossoming: A new approach to algorithms and identities for q-Bernstein bases and q-Bézier curves, *Journal of Approximation Theory* 164(1) (2012), 77-104.

[17] Simeonov P., Goldman R, Quantum B-splines, *BIT Numerical Mathematics* Vol. 53 (2013), pp. 193-223.

[18] Tariboon J., Ntouyas S.K., Quantum integral inequalities on finite intervals, *Journal of Inequalities and Applications* (2014), 2014:121.

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